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NON-NEGATIVE SOLUTIONS FOR NON-POSITONE PROBLEMS

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1. INTRODUCTION

In these lectures we present various techniques proven useful in the study of the existence of non-negative solutions to the elliptic boundary value problem

$$(1.1)\qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the Laplacean, $\lambda > 0$, $f:[0,\infty) \to \mathbb{R}$ is a locally Lipschitzian nondecreasing function, Ω is a smooth bounded region in $\mathbb{R}^n$, and $f(0) < 0$ (non-positone).

The problem (1.1) under the assumption $f(0) \geq 0$ has been extensively studied. The reader is refered to [F], [G], [C-R] for information. Among the methods used in this case we have monotone iteration techniques, ordered maps in cones, continuation arguments, and supper-sub solutions. In all these methods plays an important role the fact that $f(0) \geq 0$ implies that $u = 0$ is a subsolution. This is not the case in this study.

Here we present the so called "quadrature method" to study the one-dimensional case ($n = 1$), phase-plane analysis combined with Pohozaev identity in order to study the radially symmetric case (Ω a ball), and mountain pass type of arguments in order to study the general case.

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Before we stablish our existence results, let us point out that (1.1) has no non-negative solution if f grows too fast. For example, if $n > 1$, $f(u) = |u|^{(n+2)/(n-2)} - 1$, and Ω the unit ball in $\mathbb{R}^n$ centered at the origin, then by Pohozaev's identity (see [P]) we have

$$-\int_{\Omega} u\left[(n+2)/2\right] dx = \int_{\partial\Omega} \|\nabla u\|^2 dx$$

which is impossible for $u \geq 0$ $(u \not\equiv 0)$.

2. QUADRATURE AND THE ONE-DIMENSIONAL CASE

For $n = 1$ we can assume, without loss of generality, that $\Omega = (0,1)$. That is we consider the equation

(2.1) $\quad -u''(x) = \lambda f(u(x)) \qquad x \in (0,1),$

(2.2) $\quad u(0) = u(1) = 0.$

Remark 2.1. First we observe that the solution to (2.1) are symmetric about its critical points. Indeed, it is easily seen that if $u'(a) = 0$ then $v(x) = u(2a - x)$ is also a solution. Thus by the uniqueness of the solution to the initial value problem (2.1), $u(a) = u(a)$, $u'(a) = 0$ (only uses that f is locally Lipstchitzian) we have $v \equiv u$, or $u(a + x) = u(a - x)$.

Remark 2.2. From the above remark it follows that if u is a solution to (2.1)-(2.2) then $u'(1/2) = 0$.

From Remark 2.1 it follows that in order to obtain a solution to (2.1)-(2.2) positive in $(0,1)$ it is sufficient to find a solution to (2.1) such that

(2.3) $\quad u(0) = 0, \quad u'(1/2) = 0, \quad u' > 0 \text{ on } (0,1/2).$

Multiplying (2.1) by u' we see that (2.1) is equivalent to $-(u')^2 = \lambda F(u) + C$ where $F(r) = \int_0^r f(s)ds$. Thus if in addition u satisfies (2.3) we have $C = -2\lambda F(u(1/2))$. Hence

(2.4) $\quad u'(x) = (2\lambda[F(\rho) - F(u(x))])^{1/2} \qquad x \in [0,1/2].$

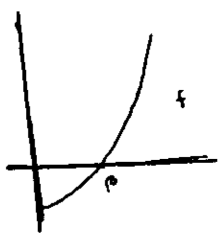
where $\rho = u(1/2)$. Integrating (2.4) we have now

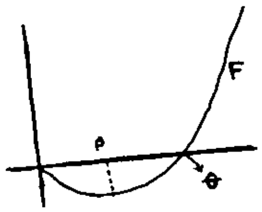
(2.5) $\quad \int_0^{u(x)} (F(\rho) - F(u))^{1/2} du = (2\lambda)^{1/2} x \qquad x \in [0,1/2].$

From (2.3) and (2.5) we see that

(2.6) $\quad (\lambda)^{1/2} = 2^{1/2} \cdot \int_0^{\rho} (F(\rho) - F(u))^{1/2} du := G(\rho).$

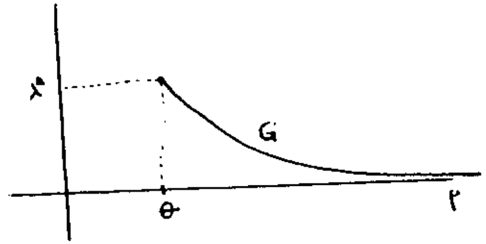
where $\rho \in [\theta,\infty)$, and θ is the positive zero of F.



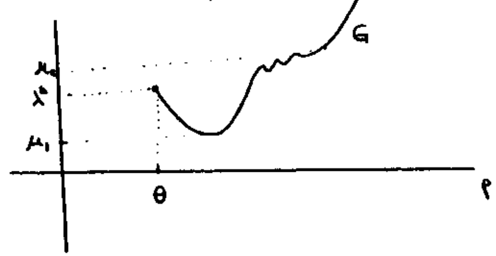


Thus, from (2.6) we see that (2.1)-(2.2) has a positive solution iff $(\lambda)^{1/2}$ is in the range of G. The following pictures express the relationship between f and G.

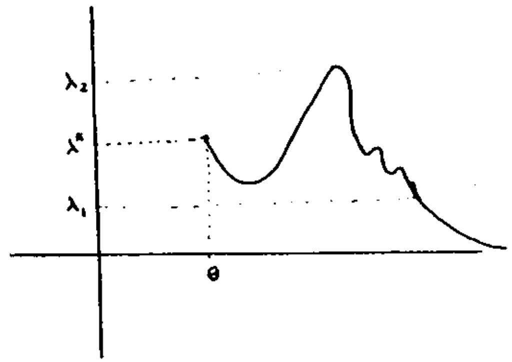
(A) $f''(s) > 0$ for $s \geq 0$, and $\lim_{s\to\infty} (f(s)/s) = \infty$.



(B) $f''(s) \leq 0$ for $s \geq 0$, $\lim_{s\to\infty} f(s) = M \quad 0 < M \leq \infty$, $\lim_{s\to\infty} (f(s) - sf'(s)) > 0$, and $(f(\theta)/\theta) < f'(\theta)$



(C) $f''(s) < 0$ for $s \in [0, s_0)$ and $s_0 > \theta$, $f''(s) > 0$ for $s > s_0$, $f'(\theta) > (f(\theta)/\theta)$. There exists $\sigma > \theta$ with $F(\sigma) - (\sigma f(\sigma)/2) > 0$.



For justification of the above diagrams we refer the reader to [C-S,2]. From (2.6) and the above diagrams we have:

THEOREM 2.1. If (A) holds then there exists λ^* such that if $\lambda \in (0,\lambda^*]$ then (2.1)-(2.2) has a unique positive solution.

THEOREM 2.2. If (B) holds then there exists λ^*, μ_1, and μ_2 such that $\mu_1 < \lambda^*$, and (2.1)-(2.2) has a) no positive solution for $\lambda < \mu_1$

b) at least two positive solutions for $\mu_1 < \lambda \leq \lambda^*$

c) a unique positive solution for $\lambda > \mu_2$.

THEOREM 2.3. If (C) holds then there exist $\lambda_1 < \lambda^* \leq \lambda_2$ such that (2.1)-(2.2) has a unique positive solution if $\lambda < \lambda_1$ and no positive solution if $\lambda > \lambda_2$. Furthermore, there is a range for λ between λ_1 and λ^* in which (2.1)-(2.2) has at least three positive solutions. Also if $\lambda^* < \lambda_2$ then (2.1)-(2.2) has at least two positive solutions for $\lambda \in [\lambda^*,\lambda_2)$.

In order to obtain nonnegative solutions to (2.1)-(2.2) with interior zeroes we observe that when $\rho = \theta$ (see (2.4)) we have $u'(0) = 0$. Thus if, in addition, in (2.3) 1/2 is replaced by $1/2(n+1)$ $(n = 2,3,\ldots)$ then we see that (2.1)-(2.2) has a nonnegative solution with n interior zeroes when $\lambda = (G(\theta))^2 \cdot (n+1)^2$. That is we have

THEOREM 2.4. In each of the cases (A), (B), or (C) of the above theorems, the problem (2.1)-(2.2) has a nonnegative solutions with n interior zeroes iff $\lambda = (n+1)\lambda^*$

3. PHASE-PLANE ANALYSIS AND RADIALLY SYMMETRIC SOLUTIONS.

In this section we consider the case in which Ω is a ball in $\mathbb{R}^n$. Without loss of generality we may assume that $\Omega = B$ is the unit ball in $\mathbb{R}^n$ centered at the origin.

Arguments based on the ideas of [G-N-N] show that if $n \geq 2$, $\Omega = B$ then all nonnegative solutions to (1.1) are actually positive and radially symmetric. That is (1.1) is equivalent to (use polar coordinates)

(3.1) $$-u'' - \frac{mu'}{r} = \lambda f(u(r)) \qquad r \in [0,1)$$

(3.2) $$u'(0) = 0, \quad u(1) = 0,$$

where $m = n - 1$. The main difference between (3.1) and (2.1) is, of course, the singular term m/r. As pointed out in the introduction, in general the problem (3.1)-(3.2) has no positive solution. Restrictions on the growth of f are necessary.

First of all we extend f to $(-\infty,\infty)$ by defining $f(x) = f(0)$ for $x < 0$, and we consider (3.1) subject to the initial condition

(3.3) $u(0) = d \in \mathbb{R}$, $u'(0) = 0$.

Because we are assuming f to be locally Lipschitzian it follows that the problem (3.1),(3.3) has a unique solution $u(\cdot,d,\lambda)$ which depends continuosly on the parameter (d,λ). That is, if $((d_n,\lambda_n)) \to (d,\lambda)$ then $(u(\cdot,d_n,\lambda_n))$ converges uniformly on [0,1] to $u(\cdot,d,\lambda)$.

In this section we will assume

(3.4) $\lim_{d\to\infty} (f(d)/d) = \infty$, and

(3.5) There exists $k \in (0,1)$ such that

$$\Lambda(d) := (d/f(d))^{n/2}(F(kd) - [(n-2)/2n]df(d)) \to \infty, \quad \text{as } d \to \infty.$$

where $F(x) = \int_0^x f(y)dy$. We observe that condition (3.5) is a growth condition. Indeed, if $f(u)$ has the form $u^q - c$ the (3.5) is satisfied if and only if $q < (n+2)/(n-2)$.

Let $H(r,d,\lambda) = r((u'(r,d,\lambda))^2/2) + \lambda r F(u(r,d,\lambda)) + (n-2)u(r,d,\lambda)u'(r,d,\lambda)/2$.

Hypothesis (3.4) and (3.5) together with the identity (Pohozaev's identity)

(3.6) $$t^m H(t,d,\lambda) = s^m H(s,d,\lambda) + \lambda\int_s^t \Lambda(r) r^m dr$$

show that there exists $\lambda_2 > 0$ and $\gamma > 0$ such that if $\lambda \in (0,\lambda_2)$ and $d > \gamma$ then $u^2 + (u')^2 > 0$ on $[0,1]$. Thus in order to show that (3.1)-(3.2) has a solution it is sufficient to show that, for a given $\lambda \in (0,\lambda_2)$, there exists $d_1 > \gamma$ such that $u(\cdot,d_1,\lambda) > 0$ on $[0,1]$ and $d_2 > d_1$ such that $u(1,d_2,\lambda) < 0$. The existence of such d's is based on hypothesis (3.4). Thus we have

THEOREM 3.1 <u>There exists</u> $\lambda_1 > 0$ <u>such that for</u> $\lambda \in (0,\lambda_1)$ <u>the problem</u> (3.1)-(3.2) <u>has a positive solution</u>.

For details see [C-S,3].

Unlike the onedimensional case, when $\Omega = B$ the problem (1.1) has no nonnegative solutions with interior zeroes. Indeed, it is easily seen that if we define $E = (u')^2 + 2\lambda F(u(\cdot))$ that then E is strictly decreasing, which makes it impossible for a nonnegative solution to (3.1)-(3.2) to have more than one zero in $(0,1]$.

4. THE MOUNTAIN PASS LEMMA AND GENERAL REGIONS

In order to minimize technicalities we will assume in this section $f(u) = u^q - 1$, with $q \in (1,(n+2)/(n-2))$. The more general case can be found in [C-U]. We extend f to $(-\infty,\infty)$ by $f(x) = f(0)$ for $x < 0$. As is well documented (see [A-R]) the solutions to (1.1) are the critical points of

(4.1) $$J(u) = \int_\Omega (|\nabla u|^2 - \lambda 2F(u))dx \qquad u \in \mathring{H}^1(\Omega) := H,$$

where H is the usual Sobolev space of square integrable functions in Ω having square integrable first order partial derivatives in Ω and vanishing on $\partial\Omega$ (see [A].)

Because $q \in (1,(n+2)/(n-2))$ it follows that the functional J satisifies the Palais-Smale condition. By the Sobolev imbedding theorem there exists a constant $m > 0$ such that for all $u \in H$

(4.3) $$\int_\Omega |u|^{q+1}dx \le m\left(\int_\Omega |\nabla u|^2 dx\right)^{(q+1)/2}.$$

Also, because of the definition of f, we see that there exists a constant D such that for all $x \in \mathbb{R}$

(4.4) $$F(x) \le (|x|^{q+1} + D)/(q+1).$$

Because $q > 1$ there exists $C_1 > 0$ such that for $\lambda > 0$ small enough

(4.5) $$2C_2 := ((C_1^2/2) - m(C_1)^{q+1} - \lambda^{(q+1)/(q-1)} D|\Omega|) > 0.$$

Now a simple calculation shows that if $\|u\|_H := \left(\int_\Omega |\nabla u|^2\right)^{1/2} = C_1\lambda^{-1/(q-1)} := \rho$ then

(4.6) $$J(u) \ge C_2\lambda^{-2/(q-1)}.$$

We emphasize that C_2 is independent of λ, for $\lambda > 0$ small enough.

A straight forward calculation shows that

$$(4.7) \qquad \max_{a\in[0,\infty)} J(a\varphi_1) \le C_3 \lambda^{-2/(q-1)},$$

where $\varphi_1 > 0$ is the principal eigenfunction of $-\Delta\varphi = \lambda\varphi$ in Ω, $\varphi = 0$ on $\partial\Omega$, and C_3 is independent of λ for $\lambda > 0$ small enough.

Since J is class C^1, satisfies the Palais-Smale condition, $J(0) = 0$, (4.6), and (4.7), by the mountain pass lemma (see [A-R]) there exists $u_\lambda = u \in H$ which is a critical point of H and

$$(4.8) \qquad C_2 \lambda^{-2/(q-1)} \le J(u_\lambda) \le C_3 \lambda^{-2/(q-1)}.$$

So far there is nothing to indicate that $u_\lambda \ge 0$ on Ω. In order to prove that for λ small enough $u_\lambda \ge 0$ we proceed as follows. We let f_+ and f_- be defined by

$$f_+(x) = \begin{cases} f(x) & \text{if } x \ge 1 \\ 0 & \text{if } x < 1 \end{cases}$$

$$f_-(x) = \begin{cases} 0 & \text{if } x \ge 1 \\ f(x) & \text{if } x \le 1. \end{cases}$$

Clearly $f = f_+ + f_-$, $f_+ \ge 0$, $f_- \le 0$. We also define z_λ, w_λ by

$$\begin{cases} -\Delta z_\lambda = \lambda f_-(u_\lambda) & \text{in } \Omega \\ z_\lambda = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta w_\lambda = \lambda f_+(u_\lambda) & \text{in } \Omega, \\ w_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

Of course $u_\lambda = z_\lambda + w_\lambda$. Because f_- is bounded, by regularity of elliptic equations, it follows that there exists a constant C_4 such that

$$(4.9) \qquad \|z_\lambda\|_{C^1(\bar\Omega)} \le C_4 \lambda.$$

From (4.8) it is proven that there exists $C_5 > 0$ such that for $\lambda > 0$ small enough

$$(4.10) \qquad \lambda \int_\Omega F(u_\lambda) \le C_5 \lambda^{-2/(q-1)}.$$

Also by (4.8) and the fact that u_λ is a critical point of J it is shown that

$$(4.11) \qquad \|u_\lambda\| \le C_6 \lambda^{-1/(q-1)},$$

for λ small enough. Now from (4.10) and (4.11) it shown that there exists $\gamma > 0$ such that

(4.12) $$m(S_\lambda) \geq C_7$$

where $S_\lambda = \{x \in \Omega;\ u(x) \geq \gamma \lambda^{-1/(q-1)}\}$, for $\lambda > 0$ small enough. From (4.12) and the Hopf maximum principle it is show that there exists a neighborhood of $\partial\Omega$ such that

(4.13) $$\frac{\partial w_\lambda}{\partial \eta} \geq C_8$$

where $\frac{\partial}{\partial \eta}$ denotes the inward unit normal. From (4.13) and (4.9) it follows that $w_\lambda + z_\lambda = u_\lambda \geq 0$ in Ω, which sketches the proof of

THEOREM 4.1.: <u>There exists $\lambda_1 > 0$ such that for $\lambda \in (0, \lambda_1)$ the problem (1.1) has a nonnegative solution.</u>

For further details see [C-U]

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ABSTRACT

We consider the existence of radially symmetric non-negative solutions for the boundary value problem

$$-\Delta u(x) = \lambda f(u(x)) \; ; \; \|x\| \le 1 \; , \; x \in R^N \; (N \ge 2)$$
$$u(x) = 0 \qquad ; \; \|x\| = 1$$

where $\lambda > 0$, $f(0) < 0$ (non-positone), $f' \ge 0$ and f is superlinear. We establish existence of non-negative solutions for λ small which extends some work of our previous paper on non-positone problems, where we considered the case $N = 1$. Our work also proves a recent conjecture by Joel Smoller & Arthur Wasserman.

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1. INTRODUCTION

Here we consider the existence of radially symmetric non-negative solutions for the boundary value problem

$$-\Delta u(x) = \lambda f(u(x)) \; ; \; \|x\| < 1, \; x \in \mathbb{R}^N, \; N \geq 2 \tag{1.1}$$

$$u(x) = 0 \qquad ; \; \|x\| = 1 \tag{1.2}$$

where $\lambda > 0$ and $f:[0,\infty) \to \mathbb{R}$ is such that $f' \geq 0$. As is well documented, the study of (1.1) - (1.2) is equivalent to the problem

$$-u'' - (n/r)\, u' = \lambda f(u) \; ; \; r \in (0,1) \tag{1.3}$$

$$u'(0) = 0 \tag{1.4}$$

$$u(1) = 0, \tag{1.5}$$

where $n = N-1$. We will assume that

$$\lim_{u \to +\infty} (f(u))/u = +\infty, \text{ i.e., } f \text{ is superlinear,} \tag{1.6}$$

$$f(0) < 0 \quad \text{(non-positone),} \tag{1.7}$$

and for some $k \in (0,1)$,

$$A = \lim_{d \to +\infty} (d/f(d))^{N/2} (F(kd) - [(N-2)/(2N)]df(d)) = +\infty \tag{1.8}$$

where $F(x) = \int_0^x f(r)dr$.

If $f(0) > 0$ (positone) and $\lambda > 0$ small, it is known that (1.1)-(1.2) has two solutions: one near zero, the other bifurcating from infinity. However, the popular method of sub-super solutions used in positone problems seems rather difficult to apply when $f(0) < 0$, since $v \equiv 0$ is no longer a sub-solution. In fact, it is a super solution. This is why we have been motivated to undertake this study. Our main result is given in Theorem 1.1.

THEOREM 1.1. Under the above assumptions, there exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, then (1.1)-(1.2) has a non-negative solution u_λ such that $u_\lambda > 0$ and decreasing on [0,1] and $u_\lambda'(1) < 0$.

Castro and Shivaji [1987] have made an extensive study of the one-dimensional problem ($N = 1$). Our proof of Theorem 1.1 is based on the shooting method. That is, to prove that (1.3)-(1.5) has a solution, we consider the problem (1.3)-(1.4) subject to $u(0) = d$. By analyzing this problem depending on the parameter d, we show that for an adequate value of d, u satisfies also (1.5). To prove Lemma 3.2 we use an identity of Pohozaev type (see Section 2) used by Castro and Kurepa [1987a and 1987b] to study oscillatory solutions of other radially symmetric problems. For other applications and extensions of this type of identity, see Ni and Serrin [1986].

Our work also proves a recent conjecture by Smoller and Wasserman [1987]. In their paper they proved an existence result applicable to functions of the type $f(u) = u^q - \epsilon$ where $\epsilon > 0$, $1 < q < N/(N-2)$ and conjectured that an optimal result would be to extend it to $1 < q < (N+2)/(N-2)$. In fact, our work includes this optimal result since if $f(u) = u^q - \epsilon$ where $\epsilon > 0$, $1 < q < (N+2)/(N-2)$ then (1.8) is satisfied with k chosen larger than $[(q+1)(N-2)/(2N)]^{(1/(q+1))}$. Note here that if $q < (N+2)/(N-2)$ then $(q+1)(N-2)/(2N) < ((N+2)/(N-2) + 1)(N-2)/(2N) = 1$.

We will restrict our proofs in this paper to the case $N > 2$. In fact when $N = 2$, the proof is easier along the same lines as in the case $N > 2$.

2. PRELIMINARIES AND NOTATIONS.

First of all we extend f to $(-\infty,\infty)$ by defining $f(x) = f(0)$ for $x < 0$. By (1.6) we see that $\lim_{d\to\infty} F(d) = \infty$. Hence (see(1.7)) there exist positive real numbers $\beta < \theta$ such that

$$0 = f(\beta) = F(\theta). \tag{2.1}$$

Since (see (1.8)) $A = \infty$, we see that there exists $\gamma > (\theta/k)$ such that

$$2NF(kd) - (N-2)df(d) \geq 0 \text{ for } d \geq \gamma. \tag{2.2}$$

Now for each real number d, the initial value (1.3), (1.4), $u(0) = d$ has a unique solution $u(t,d,\lambda)$. This solution depends continuously on (d,λ) in the sense that if $((d_n,\lambda_n)) \to (d,\lambda)$, then $(u(\ ,d_n,\lambda_n))$ converges uniformly to $u(\ ,d,\lambda)$ on $[0,1]$. To see this, we observe that for each (d,λ) the map

$$u(s) \to d + \lambda\int_0^s t^{-n}\int_0^t r^n(-f(u(r)))dr \tag{2.3}$$

defines a contraction on $C([0,\epsilon],R)$ for ϵ small enough.

Next given $d \in R$, $\lambda \in R$, we define

$$E(t,d,\lambda) = \frac{(u'(t,d,\lambda))^2}{2} + \lambda F(u(t,d,\lambda)), \tag{2.4}$$

$$H(t,d,\lambda) = tE(t,d,\lambda) + \frac{N-2}{2}u(t,d,\lambda)u'(t,d,\lambda). \tag{2.5}$$

Multiplying (1.3) by $r^N u'$ and integrating over $[\hat{t},t]$, and then multiplying (1.3) by $r^n u$ and integrating over $[\hat{t},t]$, we obtain

$$t^{N-1}H(t,d,\lambda) = \hat{t}^{N-1}H(\hat{t},d,\lambda) + \int_{\hat{t}}^t r^n\lambda[NF(u(r,d,\lambda)) - \frac{N-2}{2}f(u(r,d,\lambda))u(r,d,\lambda)]dr. \tag{2.6}$$

This identity is a form of "Pohozaev identity." For more details see Castro and Kurepa [1987a] and Pucci and Serrin [1986].

Further, for $d \geq \gamma$ let $t_0 := t_0(d,\lambda)$ be such that $d \geq u(t_0,d,\lambda) \geq kd$ for all $t \in [0,t_0)$ and $u(t_0,d,\lambda) = kd$. Multiplying by r^n (1.3)-(1.4) and $u(0) = d$ gives $u'(t,d,\lambda) = -\lambda t^{-n}\int_0^t r^n f(u(r,d,\lambda))dr$. Hence $-\lambda tf(kd) \geq Nu'(t,d,\lambda) \geq -\lambda tf(d)$, and integrating on $[0,t_0]$ we have

$$C_1(d/(\lambda f(kd)) \geq t_0 \geq C_1(d/(\lambda f(d)))^{1/2} \tag{2.7}$$

where $C_1 = ((1 - k)2N)^{1/2} > 0$. Also choosing $\hat{t} = 0$, $t = t_0$, (2.6) gives

$$\begin{aligned} t_0^n H(t_0,d,\lambda) &= \lambda\int_0^{t_0} r^n(NF(u(r,d,\lambda)) - [(N-2)/2]f(u(r,d,\lambda))u(r,d,\lambda))dr \\ &\geq \lambda\int_0^{t_0} r^n(NF(kd) - [(N-2)/2]f(d)d)dr \\ &\geq \lambda(NF(kd) - [(N-2)/2]f(d)d)t_0^N/N \\ &\geq \lambda(NF(kd) - [(N-2)/2]f(d)d)\cdot(C_1^N/N)\cdot(d/(\lambda f(d)))^{N/2} \\ &= C_2\lambda^{(1-N/2)}\{F(kd) - [(N-2)/(2N)]df(d)\}\cdot(d/f(d))^{N/2} \end{aligned} \tag{2.8}$$

where $C_2 = (C_1)^N > 0$.

3. MAIN LEMMAS AND PROOF OF THEOREM 1.1.

LEMMA 3.1. *If* $\lambda \in (0, \lambda_1 := N(\gamma-\beta)/f(\gamma))$, *then* $u(t,\gamma,\lambda) \geq \beta$ *for all* $t \in [0,1]$.

PROOF: Let $t_1 := \sup\{t \leq 1;\ u(r,\gamma,\lambda) \geq \beta, \text{for all } r \in (0,t)\}$. Since $u'(t,\gamma,\lambda) = -\lambda t^{-n}\int_0^t s^n f(u(s,\gamma,\lambda))ds$, u is decreasing on $[0,t_1]$. Also if $\lambda \in (0,\lambda_1)$, $t \in [0,t_1]$, we have

$$|u'(t,\gamma,\lambda)| \leq \lambda t f(\gamma)/N < \gamma - \beta.$$

Hence $u(t_1,\gamma,\lambda) > \gamma - (\gamma-\beta)t_1$. In particular, if $t_1 < 1$ this gives $u(t_1,\gamma,\lambda) > \beta$ contradicting the definition of t_1. Thus $t_1 = 1$ and the lemma is proven.

LEMMA 3.2. __There exists__ $\lambda_2 > 0$ __such that if__ $\lambda \in (0,\lambda_2)$, __then__ $(u(t,d,\lambda))^2 + (u'(t,d,\lambda))^2 > 0$ __for__ $t \in [0,1]$, $d \in [\gamma, +\infty)$.

PROOF: Now for $t \geq t_0$, (2.6) and (2.8) gives

$$t^n H(t) \geq C_2\lambda^{(1-N/2)}(Fkd) - [(N-2)/(2N)]df(d))(d/f(d))^{n/2} \quad (3.1)$$
$$+ \lambda\int_{t_0}^{t} r^n(NF(u(r,d,\lambda)) - [(N-2)/2]f(u(r,d,\lambda))u(r,d,\lambda))dr.$$

Now by (1.8), our definition of $f(x)$ for $x < 0$ and the fact that $f(0) < 0$, there exists a constant $B < 0$ such that $G(s) = NF(s) - [(N-2)/2]f(s)s \geq B$ for all s. Further using (1.8), we may assume without loss of generality that γ is large enough so that $(F(kd) - [(N-2)/(2N)]df(d))\ (d/f(d))^{N/2} \geq 1$ for $d \geq \gamma$. Hence by (3.1) we have, for $t \in [t_0,1]$,

$$t^n H(t) \geq C_2\lambda^{(1-N/2)}(F(kd) - [(N-2)/(2N)]df(d))\cdot(d/f(d))^{N/2} \quad (3.2)$$
$$+ \lambda B(t^N - t_0^N)/N$$
$$\geq C_2\lambda^{(1-N/2)} + \lambda B/N$$
$$= \lambda(C_2\lambda^{-N/2} + B/N). \quad (3.3)$$

That is, there exists λ_2 such that for $\lambda \in (0,\lambda_2)$, $H(t)$ (and hence $[u(t,\lambda,d)]^2 + [u'(t,\lambda,d)]^2$) is positive for every $t \in [0,1]$ and every $d \in [\gamma,+\infty)$ and the lemma is proven.

LEMMA 3.3. __Given any__ $\lambda > 0$, __there exists__ $d > \gamma$ __such that__ $u(t,d,\lambda) < 0$ __for some__ $t \in [0,1]$.

PROOF: Let $\rho > 0$ and ω be such that $\omega'' + (n/\gamma)\omega' + \rho\omega = 0$, $\omega(0) = 1$, $\omega'(0) = 0$ and the first zero of ω is 1/4. By (1.6), there exists $d_0(\lambda) \geq \theta/k$ such that if $x \geq d_0$ then

$$(f(x)/x) \geq (\rho/\lambda). \quad (3.4)$$

Suppose now that for every $d > \gamma$, $u(t,d,\lambda) \geq 0$ for all $t \in [0,1]$. First we show that there exists $d_1(\lambda) \geq d_0(\lambda)$ such that for $d > d_1(\lambda)$ and $t_1 \in (0,1]$

$$\text{if } u'(t_1,d,\lambda) = 0 \text{ then } u''(t_1,d,\lambda) < 0. \quad (3.5)$$

In fact, let $d_1(\lambda) > d_0(\lambda)$ be such that if $d > d_1(\lambda)$ then $t^n H(t) > 0$ for all $t \in [t_0,1]$ (see (3.2).) Suppose there exists $t_1 \in [0,1]$ with $u'(t_1,d,\lambda) = 0$, $u''(t_1,d,\lambda) \geq 0$. By (1.3) we have $u(t_1,d,\lambda) \leq \beta$. Since $kd > \beta$ we have $t_1 > t_0$. Thus $t_1^n H(t_1) = t_1^N \lambda F(u(t_1,d,\lambda)) < 0$, which contradicts the definition of $d_1(\lambda)$. Thus (3.5) holds.

By (3.5) we have that if $d > d_1(\lambda)$ and $t \in [0,1]$ then $u(t,d,\lambda)\cdot u'(t,d,\lambda) \leq 0$. Hence, by (3.2) and the fact that $E(t,d,\lambda) \geq \lambda F(kd)$ for $t \in [0,t_0]$, there exists $d_2(\lambda) > d_1(\lambda)$ such that for $d > d_2$

$$E(t,d,\lambda) \geq \lambda F(d_0) + 2d_0^{\,2} \quad (3.6)$$

for all $t \in [0,1]$. Now let $d > d_2$. Since $(d\omega)'' + (n/r)(d\omega)' + \rho(dw) = 0$ and $u'' + (n/r)u' + \lambda(f(u)/u)u = 0$, we get

$$u(t)(t^n v'(t)) - v(t)(t^n u'(t)) = \int_0^t s^n\left(\frac{\lambda f(u/s))}{u(s)} - \rho\right)ds \quad (3.7)$$

where $v = d\omega$. Hence if $u(t,d,\lambda) \geq d_0$ for all $t \in [0,1/4]$ then by (3.4) and facts that $v(1/4) = 0$, $v'(1/4) < 0$ we obtain a contradiction to (3.6). Thus there exists $t^* \in (0,1/4)$ such that

$$u(t^*, d, \lambda) = d_0. \tag{3.8}$$

Also u is decreasing on $(0, t^*)$ and (3.6) implies $u'(t^*, d, \lambda) \leq -2d_0$. Since we are assuming $u(t, d, \lambda) \geq 0$ for all $t \in (t^*, 1]$, we have $u' \leq 0$ for $t \in [t^*, 1]$. Therefore $0 \leq u(t, d, \lambda) \leq d_0$ for $t \in (t^*, 1]$ and by (3.6) we have $u'(t, d, \lambda) \leq -2d_0$ for all $t \in (t^*, 1]$. Hence integrating we have

$$u(t^* + 1/2, d, \lambda) - d_0 \leq -2d_0 \cdot (1/2),$$

that is, $u(t^* + 1/2, d, \lambda) \leq 0$ where $t^* + 1/2 < 1$, with $u'(t^* + 1/2, d, \lambda) \leq -2d_0$. Thus there exists $T \in (0,1)$ such that $u(T, d, \lambda) < 0$ which is a contradiction, hence the lemma is proven.

PROOF OF THEOREM 1.1. Let $\lambda_0 = \min. (\lambda_1, \lambda_2)$ and $\lambda \in (0, \lambda_0)$. Let $\hat{d}(\lambda) := \hat{d} = \sup\{d \in [\gamma, +\infty);\ u(t, d, \lambda) \geq 0 \text{ for all } t \in [0,1]\}$. By Lemma 3.3 we have that $\hat{d} < +\infty$. Now we claim that:

(A) $u(1, \hat{d}, \lambda) = 0$,

(B) $u(t, \hat{d}, \lambda) > 0 \ \forall\, t \in [0,1)$,

(C) $u'(1, \hat{d}, \lambda) < 0$ and

(D) u is decreasing on $[0,1]$.

Suppose there exists $T_1 < 1$ such that $u(T_1, \hat{d}, \lambda) = 0$. Then Lemma 3.2 gives $u'(T_1 \hat{d}, \lambda) \neq 0$ and without loss of generality we can assume $u'(T_1, \hat{d}, \lambda) < 0$. Thus there exists $T_2 \in (T_1, 1)$ such that $u(T_2, \hat{d}, \lambda) < 0$, a contradiction to the definition of $\hat{d}$. This proves (B). That is, $u(1, \hat{d}, \lambda) \geq 0$. Suppose $u(1, \hat{d}, \lambda) > 0$. Then there exists $\eta > 0$ such that $u(t, \hat{d}, \lambda) \geq \eta$ for all $t \in [0,1]$. Thus there

exists $\delta > 0$ such that $u(t, \hat{d} + \delta, \lambda) \geq \eta/2$ for all $t \in [0,1]$, which contradicts the definition of $\hat{d}$. Hence $u(1, \hat{d}, \lambda) = 0$ and (A) is proven. Finally (C) follows from Lemma 3.2 and (D) follows by Gidas, Ni & Nirenberg [1979].

4. REMARK.

Unlike the case N=1 (see Castro and Shivaji [1987, Theorem 1.2]), for $N \geq 2$ the problem (1.1)-(1.2) does not have non-negative solutions with interior zeros. This follows because if there exists $t_0 \in (0,1)$ for which $u'(t_0) = u(t_0) = 0$ then $E(t_0) = 0$. By (1.3) we obtain $dE/dt = -n(u')^2/t \leq 0$. But $E(1) \geq 0$. Thus $E = 0$ for all $t \in [t_0, 1]$ which is possible only if $u' = 0$ and hence $u = 0$ for all $t \in [t_0, 1]$. But from (1.3) we see that this is impossible with $f(0) < 0$.

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NON-NEGATIVE SOLUTIONS FOR A CLASS OF NON-POSITONE PROBLEMS

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Abstract

In the recent past many results on non-negative solutions to boundary value problems of the form

$$-u''(x) = \lambda f(u(x)) \quad ; \quad 0 < x < 1$$

$$u(0) = 0 = u(1)$$

where $\lambda > 0$, $f(0) > 0$ (positone problems) have been established. In this paper we consider the impact on the non-negative solutions when $f(0) < 0$.

We find that, we need $f(u)$ to be convex to guarantee uniqueness of positive solutions and $f(u)$ to be appropriately concave for multiple positive solutions. This is in contrast to the case of positone problems, where the roles of convexity and concavity were interchanged to obtain similar results.

We further establish the existence of non-negative solutions with interior zeroes, which did not exist in positone problems.

1. INTRODUCTION

Consider the two point boundary value problem

$$-u''(x) = \lambda f(u(x)) \quad ; \quad x \in (0,1) \tag{1.1}$$

$$u(0) = 0 = u(1) \tag{1.2}$$

where $\lambda > 0$ is a constant and $f \in C^2$ satisfies

$$f(0) < 0. \tag{1.3}$$

First note that any solution $u(x)$ of (1.1)-(1.2) is symmetric about any point $x_0 \in (0,1)$ such that $u'(x_0) = 0$. Then by (1.3) it is possible that a non-negative solution $u(x)$ of (1.1)-(1.2) may have interior zeroes in (0,1). (This was not the case when $f(0) > 0$ where if $u(x)$ is a non-negative solution of (1.1)-(1.2) then $u(x) > 0$ on (0,1).) So we will distinguish those solutions which do not have interior zeroes by refering to them as positive solutions. We shall consider three distinct cases:

(A) $f''(s) > 0$ for $s \geq 0$,

(b) $f''(s) < 0$ for $s \geq 0$ and

(c) There exists some s_0 such that $f''(s) < 0$ for $0 \leq s < s_0$ and $f''(s) > 0$ for $s > s_0$.

Our main results are:

THEOREM 1.1

Let $f(0) < 0$, $f'(s) > 0$ for $s \geq 0$ and β, θ be positive real numbers that satisfy $f(\beta) = 0$, $F(\theta) = 0$ respectively, where $F(s) = \int_0^s f(t)dt$.

(A) If $f''(s) > 0$ for $s \geq 0$ and $\lim_{s\to+\infty} f(s)/s = +\infty$ then there exist $\lambda^* > 0$ such that (1.1)-(1.2) has a unique positive solution for $0 < \lambda \leq \lambda^*$ and has no positive solutions for $\lambda > \lambda^*$. Also denoting by ρ_λ the supremum norm of the positive solution, ρ_λ increases as λ decreases, and in particular, $\rho_{\lambda^*} = \theta$, $\lim_{\lambda\to 0} \rho_\lambda = +\infty$.

(B) If $f''(s) < 0$ for $s \geq 0$, $\lim f(s) = M$ where $0 < M \leq +\infty$, $\lim_{s\to\infty} (f(s) - sf'(s)) = 2a > 0$ and $(f(\theta)/\theta) < f'(\theta)$ then there exist λ^*, μ_1 such that $0 < \mu_1 < \lambda^*$ and (1.1)-(1.2) has no positive solutions for $0 < \lambda < \mu_1$ and at least one positive solution for $\lambda \geq \mu_1$. Further (1.1)-(1.2) has at least two positive solutions for $\mu_1 < \lambda \leq \lambda^*$ and there exists $\mu_2 \geq \lambda^*$ such that (1.1)-(1.2) has a unique positive solution for $\lambda > \mu_2$. Also, $\rho_{\lambda^*} = \theta$ and $\lim_{\lambda\to+\infty} \rho_\lambda = +\infty$.

(C) If $f''(s) < 0$ for $s \in [0,s_0)$ with $s_0 > \theta$, $f''(s) > 0$ for $s > s_0$, $(f(\theta)/\theta) < f'(\theta)$, there exists $\sigma > \theta$ such that $H(\sigma) = F(\sigma) - (\sigma/2)f(\sigma) > 0$, $\lim_{s\to+\infty} (f(s)/s) = +\infty$ and $\lim_{s\to+\infty} f(s) - \lim_{s\to+\infty} sf'(s) < 0$ then there exists λ^*, λ_1, λ_2 such that $0 < \lambda_1 < \lambda^* \leq \lambda_2$ and (1.1)-(1.2) has a unique positive solution for $0 < \lambda < \lambda_1$ and no positive solutions for $\lambda > \lambda_2$. Further there exists a range for λ in (λ_1,λ^*) in which (1.1)-(1.2) has at least three positive solutions and if $\lambda_2 > \lambda^*$ then (1.1)-(1.2) has at least two positive solutions for $\lambda \in [\lambda^*,\lambda_2)$. Also $\rho_{\lambda^*} = \theta$ and $\lim \rho_\lambda = +\infty$.

BIFURCATION DIAGRAM (Theorem 1.1)

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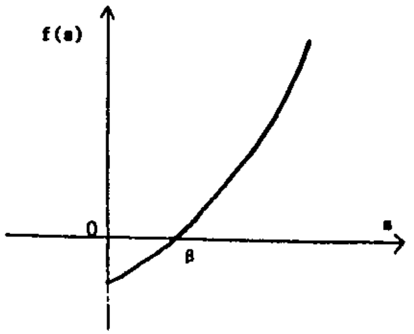

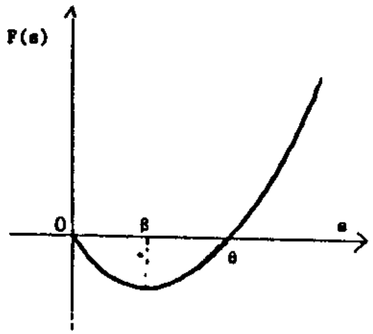

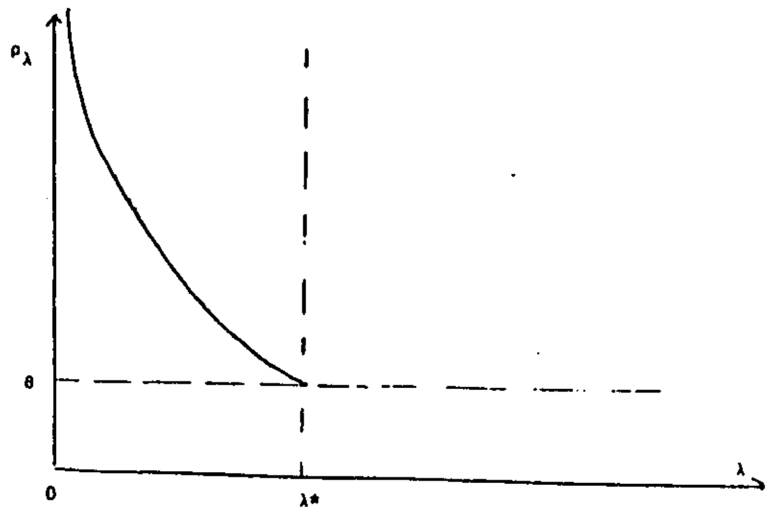

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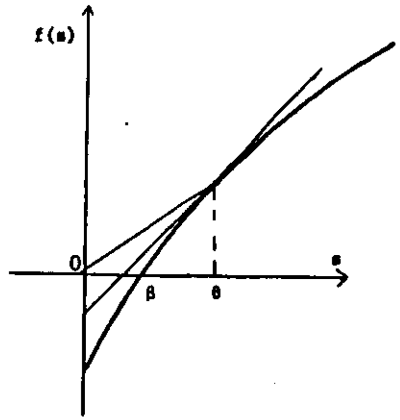

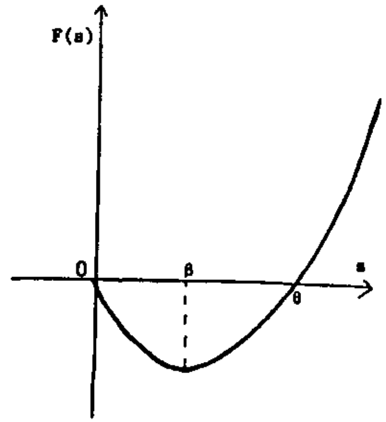

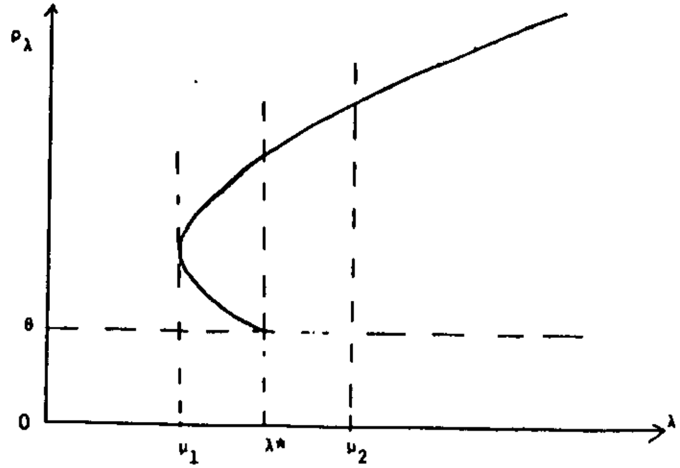

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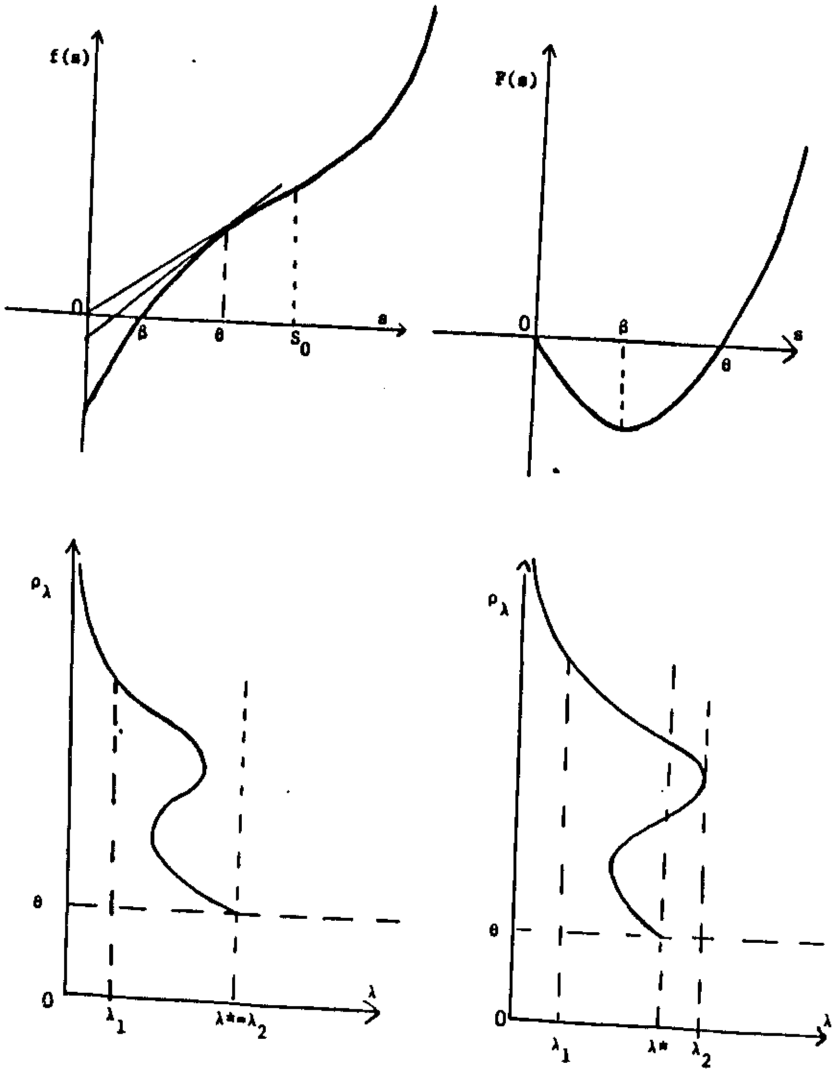


LEMMA 1.1

Let $\alpha = \theta^2/[-F(\beta)]$. Then $2\alpha \le \lambda^* \le 8\alpha$.

LEMMA 1.2

If $u_1(x)$ and $u_2(x)$ are distinct positive solutions of (1.1)-(1.3) then $\{x \in (0,1)/u_1(x) = u_2(x)\}$ is the empty set.

THEOREM 1.2

Let the hypotheses of Theorem 1.1 hold. Then in each of the cases (A), (B), or (C), given n a positive integer, (1.1)-(1.2) has a non-negative solution with n interior zeroes if and only if $\lambda = (n+1)^2\lambda^*$. Further, such a solution is unique.

REMARK 1.1

These above results on positive solutions are in contrast to the case of positone problems (see [1]-[3]) where concavity (not convexity) guaranteed uniqueness while convexity (not concavity) allowed multiplicity to be a possibility.

REMARK 1.2

Our results are related to the fact that the Hessian of the variational functional corresponding to

$$-u''(x) = f(u(x)) - f(0) \quad ; \quad 0 < x < 1$$

s non-singular at a positive solution. However, our assumptions are quite general and as such are not enough to warranty the above.

REMARK 1.3

Unlike in the case of positone problems where $f(0) > 0$, here, in general, it seems rather difficult to apply the method of sub-super solutions to track down the positive solutions. This is because, $f(0) < 0$ makes $\phi \equiv 0$ a super solution and not a sub solution as in the case of positone problems. In fact, for many of these solutions we suspect that it will not be possible to use this method. However, see [4] where boundary value problems of the type

$$-\Delta u = \lambda(u - u^3) - \epsilon \quad ; \quad x \in \Omega \subset R^n$$
$$u = 0 \quad ; \quad x \in \partial\Omega$$

were considered and existence of positive solutions for certain range of λ and $\epsilon > 0$ small enough were derived, using sub-super solutions via the anti-maximum principle due P. Clement and L. A. Peletier.

REMARK 1.4

See [5] where (1.1)-(1.3) is considered when $f''(u) < 0$ and when f satisfies certain additional hypotheses which are not easy to verify in general. They prove that, at most there are only two positive solutions for any λ.

We shall prove our results in Section 2. In Section 3 we give an example satisfying the hypotheses of Theorem 1.1. In particular, we choose the (C) part as construction of examples for part (A) and part (B) are much easier.

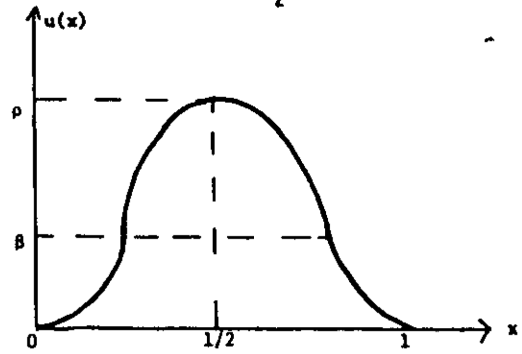
2. PROOFS OF RESULTS

First we will prove Theorem 1.1.

Multiplying (1.1) by $u'(x)$ and integrating we obtain

$$-[u'(x)]^2/2 = \lambda F(u(x)) + C. \tag{2.1}$$

Since we are dealing with positive solutions, $u(x)$ has to be symmetric with respect to $x = 1/2$ and $u'(x) > 0$ for $x \in (0, \frac{1}{2})$.



In fact, if $\rho = \sup_{x \in (0,1)} u(x)$ then $u(\frac{1}{2}) = \rho$, $\rho \geq \theta$ and substituting $x = \frac{1}{2}$ in (2.1) we obtain

$$u'(x) = \sqrt{2\lambda[F(\rho)-F(u)]} \quad ; \quad x \in [0, \tfrac{1}{2}]. \tag{2.2}$$

Now integrating (2.2) on $[0,x]$ and using (1.2) we obtain

$$\int_0^{u(x)} \frac{du}{\sqrt{F(\rho)-F(u)}} = \sqrt{2\lambda}\, x \quad ; \quad x \in [0,\tfrac{1}{2}] \tag{2.3}$$

and hence substituting $x = \frac{1}{2}$ in (2.3) we get

$$\sqrt{\lambda} = \sqrt{2}\int_0^\rho \frac{du}{\sqrt{F(\rho)-F(u)}} := G(\rho)\ . \qquad (2.4)$$

Now for positive solutions ρ must be in $[\theta,\infty)$. In fact, if ~~give~~ λ is such that there exists a $\rho \in [\theta,\infty)$ for which $G(\rho) = \sqrt{\lambda}$, it follows that (1.1)-(1.2) has a positive solution $u(x)$ given by (2.3) such that $\sup\limits_{x\in(0,1)} u(x) = u(\frac{1}{2}) = \rho$.

Further it follows that, $G(\rho)$ is a continuous function and differentiable for $\rho \in (\theta,\infty)$ with

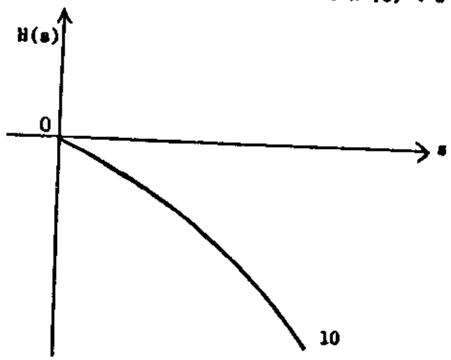
$$\frac{d}{d\rho}G(\rho) = \sqrt{2}\int_0^1 \frac{H(\rho)-H(\rho v)}{[F(\rho)-F(\rho v)]^{3/2}}dv \qquad (2.5)$$

where

$$H(s) = F(s) - (s/2)f(s). \qquad (2.6)$$

<u>Hence in order to prove part (A)</u> of Theorem 1.1 we will prove that $G'(\rho) < 0$ and $\lim\limits_{\rho\to+\infty} G(\rho) = 0$. Now $H'(s) = \frac{1}{2}[f(s)-sf'(s)]$ and $H''(s) = -(\frac{1}{2})sf''(s)$.

But $f(0) < 0$, $f''(s) > 0$ for $s > 0$. Thus $H'(s) < 0$ for $s > 0$.

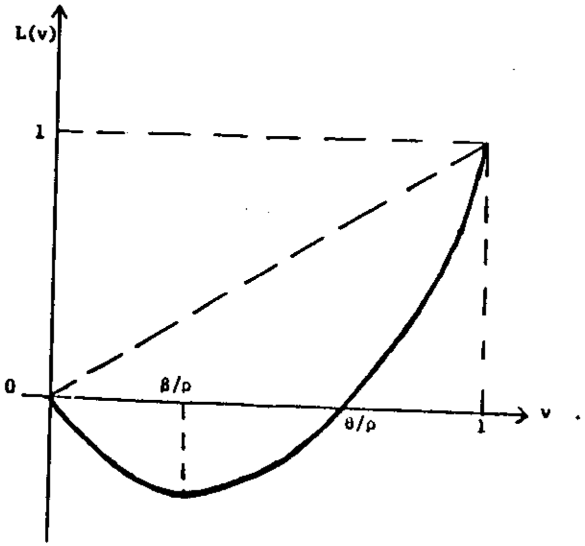


Hence $H(\rho) - H(\rho v) < 0$ for $v \in [0,1)$ and consequently $G'(\rho) < 0$.

Next note that for $\rho \in (\theta,\infty)$

$$G(\rho) = \sqrt{2}\frac{\rho}{\sqrt{F(\rho)}}\int_0^1 \frac{dv}{\sqrt{1-[F(\rho v)/F(\rho)]}}\ . \qquad (2.7)$$

Let $L(v) := F(\rho v)/F(\rho)$. Then $L(0) = 0$, $L'(v) = (f(\rho v)\rho)/F(\rho)$ and $L''(v) = (f'(\rho v)\rho^2)/F(\rho)$. But $f(s) < 0$ for $s \in (0,\beta)$, $f(s) > 0$ for $s > \beta$ and $f'(s) > 0$ for $s \geq 0$. Hence for a given $\rho \in (\theta,\infty)$, $L(v)$ takes the shape



Hence clearly $L(v) \leq v$ for $v \in [0,1]$. Consequently from (2.7)

$$G(\rho) \leq \sqrt{2}\frac{\rho}{\sqrt{F(\rho)}}\int_0^1 \frac{dv}{\sqrt{1-v}} = 2\sqrt{2}\,\rho/\sqrt{F(\rho)}\ .$$

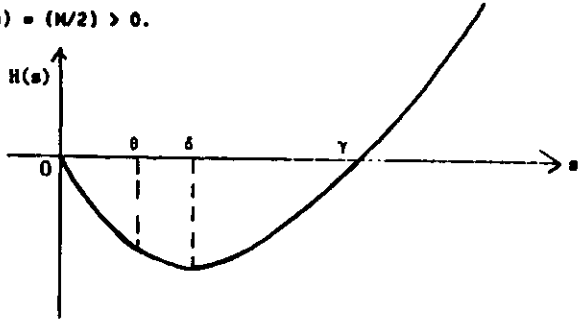
But since $\lim\limits_{s\to+\infty} f(s)/s = +\infty$, we have $\lim\limits_{\rho\to+\infty} \rho^2/F(\rho) = \lim\limits_{\rho\to+\infty} 2\rho/f(\rho) = 0$

and hence $\lim\limits_{\rho\to+\infty} G(\rho) = 0$. This completes the proof of part (A) in Theorem 1.1.

In order to prove part (B), note that $f(0) < 0$, $f''(s) < 0$ for $s > 0$,

$f(\theta) < \theta f'(\theta)$ and $\lim\limits_{s\to+\infty} f(s) = M$. Thus $H''(s) > 0$ for $s > 0$, $H'(s) < 0$ for

$s \in [0,\theta]$ and $\lim\limits_{s\to+\infty} H'(s) = (M/2) > 0$.



Hence there exists δ, γ such that $\theta < \delta < \gamma$, $H'(\delta) = 0$ and $H(\gamma) = 0$.

Consequently, $G'(\rho) < 0$ for $\rho \in [\theta,\delta]$ and $G'(\rho) > 0$ for $\rho \geq \gamma$ and, in order

to complete the proof of part (B) in Theorem 1.1 we are left with to prove

that $\lim\limits_{\rho\to+\infty} G(\rho) = +\infty$.

Now $L(v) = F(\rho v)/F(\rho) \geq [f(0)\rho v]/F(\rho)$ since $F'' = f' > 0$. Hence from (2.7) we have

$$G(\rho) \geq \sqrt{2}(\rho/\sqrt{F(\rho)}) \int_0^1 \frac{dv}{\sqrt{1-[(f(0)\rho v)/F(\rho)]}}\ ,$$

$$= -\sqrt{2}(\sqrt{F(\rho)}/f(0)) \int_0^{-(f(0)\rho)/F(\rho)} \frac{d\omega}{\sqrt{1+\omega}}$$

where $\omega = -(f(0)\rho v)/F(\rho)$. Hence

$$G(\rho) \geq -2\sqrt{2}\,(\sqrt{F(\rho)}/f(0))\,[[1 - (f(0)\rho)/F(\rho)]^{1/2} - 1]$$

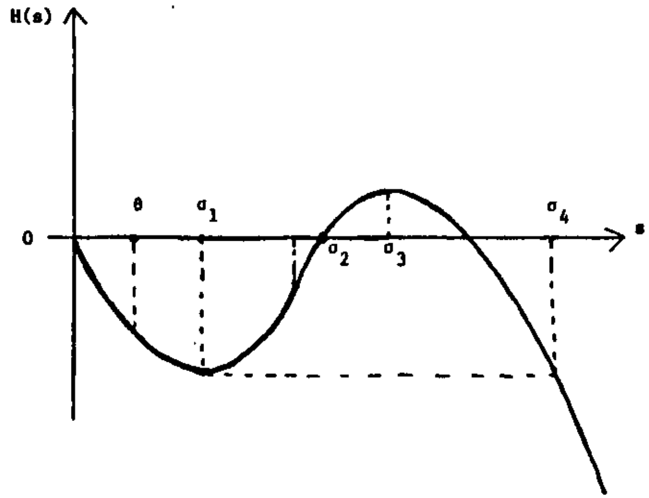
$$= 2\sqrt{2}\,(\rho/\sqrt{F(\rho)})[[1-(f(0)\rho)/F(\rho)]^{1/2} + 1]^{-1/2}.$$

But $\lim\limits_{\rho\to+\infty} \rho/F(\rho) = \frac{1}{M}$ while $\lim\limits_{\rho\to+\infty} \rho^2/F(\rho) = \lim\limits_{\rho\to+\infty} 2\rho/f(\rho) = +\infty$. Hence

$\lim\limits_{\rho\to+\infty} G(\rho) = +\infty$ and part (B) of Theorem 1.1 is proven.

Next we consider the part (C) of Theorem 1.1. The hypotheses imply that $H(s)$

takes the form



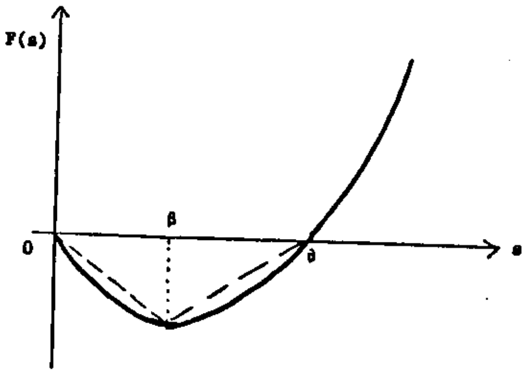
where σ_1, σ_2, σ_3 and σ_4 are as described in the above figure. Hence

$H(\rho) - H(\rho v) < 0$ for $v \in [0,1)$ when $\rho \in [\theta,\sigma_1]$ and when $\rho > \sigma_3$ while

$H(\rho) - H(\rho v) > 0$ for $v \in [0,1)$ when $\rho \in (\sigma_2, \sigma_3]$. Consequently $G'(\rho) < 0$ for $\rho \in [\theta, \sigma_1]$ and for $\rho \geq \sigma_4$ while $G'(\rho) > 0$ for $\rho \in [\sigma_2, \sigma_3]$. Now to complete the proof of Theorem 1.1 we are left with to prove that $\lim_{\rho \to +\infty} G(\rho) = 0$ which follows by identical arguments as in the proof of part (A).

Next we prove Lemma 1.1. First we note that $(1/\sqrt{-F(s)}) \geq (1/\sqrt{-F(\beta)})$ for $s \in (0,\theta)$, and since $F''(s) = f'(s) > 0$ for $s \geq 0$ we have

$$-F(s) \geq \begin{cases} -(F(\beta)/\beta)s & \text{for } 0 \leq s \leq \beta \\ -[F(\beta)/(\theta-\beta)](\theta-s) & \text{for } \beta < s \leq \theta. \end{cases}$$



Thus

$$(1/\sqrt{-F(s)}) \leq \begin{cases} \sqrt{\beta/(-F(\beta)s)} & \text{for } 0 \leq s \leq \beta \\ \sqrt{(\theta-\beta)/[-F(\beta)(\theta-s)]} & \text{for } \beta < s \leq \theta. \end{cases}$$

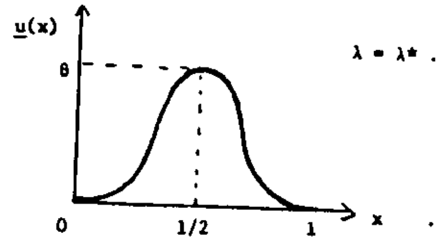
Hence

$$G(\theta) = \sqrt{2} \int_0^\theta \frac{ds}{\sqrt{-F(s)}} \geq \sqrt{2}\ \theta/\sqrt{-F(\beta)}\ , \tag{2.8}$$

and

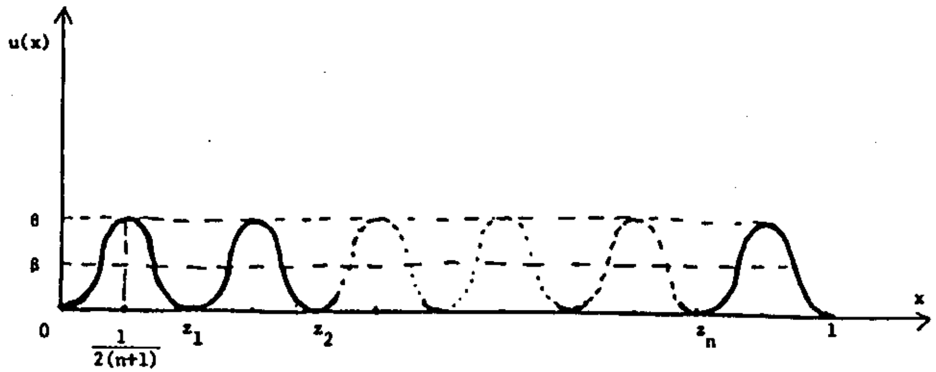
$$\begin{aligned} G(\theta) &= \sqrt{2} \int_0^\beta \frac{ds}{\sqrt{-F(s)}} + \sqrt{2} \int_\beta^\theta \frac{ds}{\sqrt{-F(s)}} \\ &\leq \sqrt{2}\sqrt{\beta/(-F(\beta))}\ 2\sqrt{\beta} + \sqrt{2}\sqrt{(\theta-\beta)/(-F(\beta))}\ 2\sqrt{\theta-\beta} \\ &= 2\sqrt{2}\ \theta/\sqrt{-F(\beta)} \end{aligned} \tag{2.9}$$

From (2.8) and (2.9) it easily follows that $2\alpha \leq \lambda^* = [G(\theta)]^2 \leq 8\alpha$ where $\alpha = \theta^2/[-F(\beta)]$ and Lemma 1.1 is proven. Note here that for $\lambda = \lambda^*$ the minimal solution $\underline{u}(x)$ has supremum norm $\rho = \theta$ and hence $\underline{u}'(0) = 0$.



Finally, the proof of Lemma 1.2 follows easily from (2.2) and we will conclude this section with the proof of Theorem 1.2.

First we note that in order to study a non-negative solution with n interior zeroes, due to the symmetry, we need to study the solution only in the interval $[0, 1/(2(n+1))]$.



From (2.1) we have for $x \in [0, 1/(2(n+1))]$,

$$u'(x) = \sqrt{2\lambda[F(\rho)-F(u)]} \qquad (2.10)$$

where $\rho = \sup_{x\in[0,1]} u(x) = u(1/(2(n+1)))$. But since we must have $u'(0) = 0$,

$F(\rho) = 0$ and hence $\rho = \theta$. Now integrating (2.10) on $[0,x]$ we obtain

$$\int_0^{u(x)} \frac{du}{\sqrt{-F(u)}} = \sqrt{2\lambda}\, x \quad ; \quad x \in [0, 1/(2(n+1))] \qquad (2.11)$$

and hence substituting $x = 1/(2(n+1))$ in (2.11) we must have

$$\sqrt{\lambda} = (n+1)\sqrt{2} \int_0^{\theta} \frac{du}{\sqrt{-F(u)}} = (n+1)\sqrt{\lambda^*}. \qquad (2.12)$$

Thus in order to have a non-negative solution $u(x)$ to (1.1)-(1.2) with n interior zeroes λ must equal $(n+1)^2\lambda^*$ and $\sup_{x\in[0,1]} u(x) = \theta$. In fact, given $\lambda = (n+1)^2\lambda^*$ it follows that (1.1)-(1.2) has a unique non-negative solution

$u(x)$ with n interior zeroes given by (2.11) with $\sup_{x\in[0,1]} u(x) = u(1/(2(n+1))) = \theta$

Hence theorem 1.2 is proven.

3. EXAMPLE (THEOREM 1.1 - (C) part)

Consider

$$f(s) = s^3 - as^2 + bs - c \tag{3.1}$$

where $a > 0$, $b > 0$, $c > 0$ and satisfy

$$b > (32/81)a^2 \ , \tag{3.2}$$

$$a^3 > 54c \ . \tag{3.3}$$

Clearly $f(0) < 0$. Further

$$f'(s) = 3s^2 - 2as + b$$

$$= 3\{[s-(a/3)]^2 + (b/3) - (a^2/9)\} \ . \tag{3.4}$$

But (3.2) implies that $(b/3) > (a^2/9)$. Hence $f'(s) > 0$ for $s \geq 0$ and since $\lim_{s \to +\infty} f(s) = +\infty$, there exists a unique $\beta > 0$ such that $f(\beta) = 0$.

Next $F(s) = sg(s)$ where

$$g(s) = (s^3/4) - (as^2/3) + (bs/2) - c \tag{3.5}$$

and

$$g'(s) = (3s^2/4) - (2as/3) + (b/2)$$

$$= (3/4)\ \{[s-(4a/9)]^2 + (2b/3) - (16a^2/81)\}. \tag{3.6}$$

Once again (3.2) implies that $(2b/3) > (16a^2/81)$ and hence $g'(s) > 0$ for $s \geq 0$. But $g(0) < 0$ and $\lim_{s \to +\infty} g(s) = +\infty$. Hence there exists a unique $\theta > \beta$ such that $F(\theta) = g(\theta) = 0$.

Now note that

$$f''(s) = 6s - 2a \ . \tag{3.7}$$

Clearly there exists $s_0 = (a/3)$ such that $f''(s) < 0$ for $s \in [0,s_0)$ and $f''(s) > 0$ for $s > s_0$. Also (3.2), (3.3) imply that

$$F(a/3)=(a/3)\{(a^3/108) - (a^3/27) + (ab/6) - c\}$$

$$= (a/3)\{(ab/6) - c - (a^3/36)\}$$

$$\geq (a/3)\{(32a^3/486) - (a^3/54) - (a^3/36)\}$$

$$= (19a^4/2916) > 0 \ . \tag{3.8}$$

Hence $\theta < s_0 = (a/3)$.

Next consider

$$f(\theta) - \theta f'(\theta) = -2\theta^3 + a\theta^2 - c.$$

But $g(\theta) = 0$ and thus substituting for c we get

$$f(\theta) - \theta f'(\theta) = (-9\theta^3/4) + (4a\theta^2/3) - (b\theta/2)$$

$$= (-9\theta/4)\{[\theta - (8a/27)]^2 + (6b/27) - [64a^2/(27)^2]\} \ .$$

Hence using (3.2) we have $f(\theta) - \theta f'(\theta) < 0$.

Finally, the hypotheses $\lim_{s \to +\infty} (f(s)/s) = +\infty$ and $\lim_{s \to +\infty} f(s) - \lim_{s \to +\infty} sf'(s) < 0$ are easily seen to be satisfied, and hence we are left with to prove the existence of a number $\sigma > \theta$ such that $H(\sigma) > 0$. But (3.3) implies that

$$H(a/3) = (-a^4/324) + (a^4/162) - (ac/6)$$

$$= \{[a^4-54ac]/324\} > 0,$$

and $(a/3) > \theta$. Hence the result.

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