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SMR 281/17

COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS
(11 January - 5 February 1988)

NON-RESONANCE CONDITIONS IN SEMILINEAR ELLIPTIC PROBLEMS

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NONRESONANCE BELOW THE FIRST EIGENVALUE FOR A SEMILINEAR ELLIPTIC PROBLEM(*)

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(*) Part of this work was carried out with the support of the CNPq and the FNRS.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let Ω be a bounded open subset in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and let $f(x,s)$ be a real-valued Caratheodory function on $\Omega \times \mathbb{R}$. We consider the semilinear Dirichlet problem

$$(1.1)\qquad \begin{cases} -\Delta u = f(x,u)+h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The nonlinearity f generates a potential

$$F(x,s) = \int_0^s f(x,t)dt$$

which, in this paper, will always be assumed to lie asymptotically as $s \to \pm\infty$ at the left side of the first eigenvalue λ_1 of $-\Delta$ on $H_0^1(\Omega)$, i.e.

$$(1.2)\qquad \limsup_{s\to\pm\infty} \frac{2F(x,s)}{s^2} \le \lambda_1,$$

with some uniformity with respect to x. We are interested in the additional conditions to be imposed on f or F in order that (1.1) be solvable <u>for any</u> h. Such conditions are sometimes referred to as <u>nonresonance conditions</u>.

A classical result in that direction, which goes back to Hammerstein [7], says that if f is continuous, satisfies a linear growth condition and if for some number $\mu < \lambda_1$,

$$(1.3)\qquad \limsup_{s\to\pm\infty} \frac{2F(x,s)}{s^2} \le \mu$$

uniformly for $x \in \Omega$, then (1.1) is solvable for any h. This was extended by Mawhin-Ward-Willem [9] who assume that f grows at most as $|s|^\sigma$ for some $\sigma < (N+2)/(N-2)$ ($\sigma < \infty$ if N=1 or 2) and that for some function $\alpha \in L^\infty(\Omega)$ with $\alpha(x) \le \lambda_1$ a.e. in Ω and $\alpha(x) < \lambda_1$ on a subset of positive measure,

$$(1.4)\qquad \limsup_{s\to\pm\infty} \frac{2F(x,s)}{s^2} \le \alpha(x)$$

uniformly for a.e.$x \in \Omega$.

We are mainly interested in this paper in the situation wher(e) (1.4) does not hold, e.g. when

$$(1.5)\qquad \limsup_{s\to\pm\infty} \frac{2F(x,s)}{s^2} = \lambda_1$$

for a.e. $x \in \Omega$. The new assumption which will eventually yield th(e) solvability of (1.1) bears on the way the limits in (1.5) are approache(d). Roughly speaking, it consists in requiring that the quotient $2F(x,s)/s^2$ stays below λ_1 for sufficiently many values of s as $s \to \pm\infty$. This i(s) expressed in terms of the density of certain sets at infinity. Densit(y) conditions of the same type but bearing on the quotient $f(x,s)/s$ wer(e) introduced in [5] (see in this respect proposition 4.10 below).

Let us state our results. Take an exponent p with $p > 2N/(N+2)$ i(f) $N \ge 2$, $p \ge 1$ if N=1. We assume that the Caratheodory function $f(x,s)$ and its potential $F(x,s)$ satisfy:

(f_1) there exist a constant A and a function B(x) in $L^p(\Omega)$ such that

$$|f(x,s)| \le A|s|+B(x)$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$;

(F_1) inequality (1.2) holds uniformly for a.e. $x \in \Omega$, i.e. for any $\varepsilon >$ (0) there exists $b_\varepsilon(x)$ in $L^1(\Omega)$ such that

$$F(x,s) \le \frac{\lambda_1}{2} s^2+\varepsilon s^2+b_\varepsilon(x)$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$.

To state our last assumption on F, we need the following

DEFINITION 1.1. Let E be a measurable subset of $\mathbb{R}$. We say that E has <u>positive density at</u> $+\infty$ if

$$\liminf_{r\to+\infty} \frac{\mu_1(E\cap[0,r])}{\mu_1([0,r])} > 0.$$

Here μ_1 denotes the Lebesgue measure on $\mathbb{R}$. Of course one can give a similar definition at $-\infty$ by considering the intersection of E with the interval $[r,o]$ and letting $r \to -\infty$.

It will also be convenient to say that a measurable subset A of a measurable set B is a full subset of B if $B \setminus A$ has measure zero.

The last assumption imposed on F in our first result is now the following, where we denote $\mathbb{R} \setminus \{o\}$ by $\mathbb{R}_0$:

(F_2) there exist a full subset $\Omega' \subset \Omega$ and $\eta > o$ such that

$$\bigcap_{x \in \Omega'} \{s \in \mathbb{R}_0 ; \frac{2F(x,s)}{s^2} \leq \lambda_1 - \eta\}$$

has positive density at both $+\infty$ and $-\infty$.

THEOREM 1.2. *Assume* (f_1), (F_1) *and* (F_2). *Then, for any* $h \in L^p(\Omega)$, *there exists* $u \in H_o^1(\Omega) \cap W^{2,p}(\Omega)$ *solution of* (1,1), *which minimizes the associated functional*

$$\phi(u) = 1/2 \int_\Omega |\nabla u|^2 - \int_\Omega F(x,u) - \int_\Omega hu \tag{1.6}$$

on $H_o^1(\Omega)$.

Theorem 1.2. applies in particular to the case where $f(x,u)$ does not depend on x, i.e. $f(x,u)=f(u)$.

Assumption (F_2) is global with respect to Ω. It can be replaced by a local one, at the expense of strengthenning a little bit the density condition at infinity. Let us introduce the following variant of (F_2):

$(F_3)_+$ there exist an open set $\omega_+ \subset \Omega$ and a corresponding full subset $\omega'_+ \subset \omega_+$ with the following property: for any $o \leq \upsilon < 1$ there exists $\eta > o$ such that

$$\liminf_{r \to \infty} \frac{\mu_1(E(\omega'_+, \lambda_1 - \eta) \cap [\upsilon r, r])}{\mu_1([\upsilon r, r])} > o \tag{1.7}$$

where

$$E(\omega'_+, \lambda_1 - \eta) = \bigcap_{x \in \omega'_+} \{s \in \mathbb{R}_0; \frac{2F(x,s)}{s^2} \leq \lambda_1 - \eta\};$$

$(F_3)_-$ there exist an open set $\omega_- \subset \Omega$ and a corresponding full subset $\omega'_- \subset \omega$ with the following property: for any $o \leq \upsilon < 1$ there exists $\eta > o$ such that

$$\liminf_{r \to -\infty} \frac{\mu_1(E(\omega'_-, \lambda_1 - \eta) \cap [r, \upsilon r])}{\mu_1([r, \upsilon r])} > o.$$

Condition (F_2) at $+\infty$ corresponds to $(F_3)_+$ with $\omega_+ = \Omega$ and the single choice $\upsilon = o$. Observe also that if for some open set $\omega_+ \subset \Omega$ and some number $\mu < \lambda_1$,

$$\limsup_{s \to +\infty} \frac{2F(x,s)}{s^2} \leq \mu$$

uniformly for a.e. $x \in \omega_+$, then $(F_3)_+$ holds. Indeed, for a full subset $\omega'_+ \subset \omega_+$, the set $E(\omega'_+, (\mu+\lambda_1)/2)$ then contains a half line. Similar observations can of course be made for $(F_3)_-$.

THEOREM 1.3. *Assume* (f_1), (F_1), $(F_3)_+$ *and* $(F_3)_-$. *Then for any* $h \in L^p(\Omega)$, *there exists* $u \in H_o^1(\Omega) \cap W^{2,p}(\Omega)$ *solution of* (1.1.), *which minimizes* ϕ *on* $H_o^1(\Omega)$.

The main part of the proof of these two theorems consists in showing that the functional ϕ is coercive on $H_o^1(\Omega)$. Conditions (F_2) and $(F_3)_\pm$ are applied in direct connection with two propositions relative to inverse images. Given a Lipschitz function u and a Borel set B in its range, these propositions provide an uniform estimate from below on the measure of $u^{-1}(B)$ in terms of the measure of B and the Lipschitz constant of u. These propositions are proved by rearrangement methods. In this respect, the main (technical) difficulty to go from the global assumption (F_2) to the local one $(F_3)_\pm$ comes from the fact that the classical Polya-Szegö theorem about the Schwarz rearrangement of a nonnegative Lipschitz function is no longer true if the function does not vanish on the boundary. We use instead the Steiner rearrangement when dealing with such functions.

Section 2 is devoted to these two propositions. Since they may have some independant interest, we have tried to present them in a rather sharp form, which is stronger than what is really needed for their application in this paper. Section 3 contains the proof of theorems 1.2 and 1.3 as well as various remarks. Several examples are given in section 4, in particular when equality (1.5) holds. We also show there that if a nonlinearity satisfies our assumptions in a nontrivial way, i.e. with (1.5), then a certain crossing of the eigenvalue λ_1 must necessarily occur (cf.proposition 4.2).

We wish to thank P.L.LIONS for suggesting the use of the Steiner rearrangement and providing us with a variant of example 2.5 and A.ANANE for some interesting comments about the density condition at infinity, in particular lemma 4.5.

2 TWO PROPOSITIONS ABOUT INVERSE IMAGES.

The following proposition enters the proof of theorem 1.2.

PROPOSITION 2.1. *There exists a constant* $c=c(N) > o$ *such that*

(2.1) $$\mu_N(u^{-1}(B)) \geq c\,\frac{\mu_1(B)^N}{\mathrm{Lip}(u)^N}$$

for any open set $\Omega \subset \mathbb{R}^N$ *with finite measure, any nonconstant Lipschitz function* u *on* $\overline{\Omega}$ *which vanishes on* $\partial\Omega$ *and any Borelian set* B *in the range of* u.

Here μ_1 and μ_N denote the Lebesgue measures on $\mathbb{R}$ and $\mathbb{R}^N$ respectively and Lip(u) is the Lipschitz constant of u on Ω.

PROOF OF PROPOSITION 2.1. Let Ω,u and B be as in the proposition. We first consider the case where the function u is nonnegative. Let then u^* be the Schwarz rearrangement of u. It is defined on the corresponding rearranged domain Ω^*, which is the ball in $\mathbb{R}^N$ centered at zero with measure $\mu_N(\Omega)$. It is known that the range of u^* coincides with that of u and that

(2.2) $$\mu_N(u^{*-1}(B))=\mu_N(u^{-1}(B))\ ;$$

moreover u^* is Lipchitzian on $\overline{\Omega}^*$ and one has the Polya-Szegö estimate

(2.3) $$\mathrm{Lip}(u^*) \leq \mathrm{Lip}(u).$$

The proof of (2.3) is given in [3,10] when Ω is bounded. It carries over with little change to the present situation if one observes that our conditions imply that $u(x) \to o$ as $|x| \to \infty$, $x \in \Omega$.

We claim that for any radius R of the ball Ω^*,

(2.4) $$\mu_1(B) = \int_{u^{*-1}(B)\cap R} \left|\frac{\partial u^*}{\partial r}\right| \, dr,$$

where r denotes the distance to the origin. This is clear when B is an interval since, along R, u^* is nonincreasing and is the integral of its derivative. The general case of a Borelian set B then follows since both sides of (2.4) are measures on the range of u. Relation (2.4) implies that for any radius R of Ω^*,

$$\mu_1(u^{*-1}(B)\cap R) \geq \frac{\mu_1(B)}{\mathrm{Lip}(u^*)} \quad (2.5)$$

We now consider the radially symmetric set $u^{*-1}(B)$ and let A denote the ball centered at zero of radius $\mu_1(u^{*-1}(B)\cap R)$. To compare the N-measures of $u^{*-1}(B)$ and of A, we write

$$\mu_N(u^{*-1}(B))=\int\left(\int_{u^{*-1}(B)\cap R_\theta} r^{N-1}dr\right)d\theta$$

where R_θ is the radius through the point θ of the unit sphere. An application of lemma 2.2 below to the inner integral gives

$$\mu_N(u^{*-1}(B)) \geq \mu_N(A).$$

Consequently, using (2.5), we obtain

$$\mu_N(u^{*-1}(B)) \geq \omega_N \frac{\mu_1(B)^N}{\mathrm{Lip}(u^*)^N}$$

where ω_N denotes the N-measure of the unit ball in $\mathbb{R}^N$. The conclusion (2.1) with $c=\omega_N$ then follows by combining the previous inequality with (2.2) and (2.3).

To deal with the case of an arbitrary function u, we write

$$u=u_+-u_-.$$

This provides a decomposition of u into two nonnegative Lipschitz functions u_+ and u_- which vanish on $\partial\Omega$ and satisfy $\mathrm{Lip}(u_+) \leq \mathrm{Lip}(u)$ and $\mathrm{Lip}(u_-) \leq \mathrm{Lip}(u)$. Moreover, writing

$$B=B^- \cup B^\circ \cup B^+$$

where $B^-=B \cap \mathbb{R}^-_o$, $B^\circ=B \cap \{o\}$ and $B^+=B \cap \mathbb{R}^+_o$, we have

$$\mu_N(u^{-1}(B)) \geq \mu_N(u_-^{-1}(-B^-))+\mu_N(u_+^{-1}(B^+)).$$

Applying (2.1) to u_- and to u_+, we deduce

$$\mu_N(u^{-1}(B))^{1/N} \geq \frac{\omega_N{}^{1/N}}{\mathrm{Lip}(u)} (\mu_1(B^-)^N+\mu_1(B^+)^N)^{1/N}.$$

Consequently

$$\mu_N(u^{-1}(B))^{1/N} \geq \frac{\omega_N{}^{1/N}c_0}{\mathrm{Lip}(u)} \quad \mu_1(B)$$

where the constant $c_0 > o$ comes from the equivalence between the N-norm and the 1-norm on $\mathbb{R}^2$. Q.E.D.

LEMMA 2.2. <u>Let</u> $f:[a,b] \to \mathbb{R}$ <u>be nondecreasing. Take</u> $D \subset [a,b]$ <u>measurable and let</u> D' <u>be the interval with left point</u> a <u>and length</u> $\mu_1(D)$. <u>Then</u>

$$\int_{D'} f \leq \int_D f. \quad (2.6)$$

PROOF. Letting $A=D \setminus (D\cap D')$ and $B=D' \setminus (D\cap D')$, we have the disjoint unions

$$D=(D\cap D') \cup A ,$$
$$D'=(D\cap D') \cup B .$$

Introducing this in (2.6), we are reduced to proving

$$\int_B f \leq \int_A f .$$

Since $\mu_1(A)=\mu_1(B)$ and $\sup B \leq \inf A$, we have

$$\int_B f \leq \mu_1(B)\, f(\sup B) \leq \mu_1(A)\, f(\inf A) \leq \int_A f .$$

Q.E.D.

REMARK 2.3. It follows from (2.1) that for any $q > N$ and any open set Ω with finite measure, there exists $c_1=c_1(N,q,\mu_N(\Omega))$ such that

$$\mu_N(u^{-1}(B)) \geq c_1 \frac{\mu_1(B)^q}{\mathrm{Lip}(u)^q} \quad (2.7)$$

for any u and B as in proposition 2.1. Indeed (2.1) implies

$$c_2 \frac{\mu_1(B)}{\mathrm{Lip}(u)} \leq 1$$

where $c_2 = c^{1/N}\, \mu_N(\Omega)^{-1/N}$. Rewriting (2.1) in the form

$$\mu_N(u^{-1}(B)) \geq \frac{c}{c_2^N} \left(c_2 \frac{\mu_1(B)}{\mathrm{Lip}(u)}\right)^N$$

and observing that the N^{th} power of a number between o and 1 is greater than its q^{th} power, we deduce (2.7). We also remark that an estimate like (2.7) cannot hold with an exponent $q < N$. This can be seen by considering the functions

$$u_k(x) = \begin{cases} 1-k|x| & \text{for } |x| \leq 1/k\,, \\ 0 & \text{for } 1/k \leq |x| < 1 \end{cases}$$

on Ω = the unit ball in $\mathbb{R}^N$ and the Borelian set $]0,1[$.

Estimate (2.7) was announced in [5]. The preceding remark shows that the exponent N in (2.1) is sharp.

REMARK 2.4. The result of proposition 2.1 is valid for an arbitrary open set Ω provided the vanishing of u on $\partial\Omega$ is understood to hold also at infinity, i.e. $u(x) \to 0$ as $|x| \to \infty$, $x \in \Omega$. Estimate (2.3) remains true under these conditions, with a proof easily adapted from [3,10].

In the proof of theorem 1.3, we will need a result analogous to proposition 2.1 for functions which do not necessarily vanish on the boundary. For such functions, even nonnegative, the Lipschitz character is not necessarily preserved by taking the Schwarz rearrangement (cf.example 2.5 below; cf.also remark 2.9). We will see however that the conclusion of proposition 2.1 essentially remains valid.

EXAMPLE 2.5. Let $\Omega = B(0,1) \subset \mathbb{R}^2$ and consider $u(x,y) = (x^2+y^2)^{1/2}$. Then $u^* = 0$ on $\partial B(0,1)$ and, for $0 < t < 1$, $u^* > t$ in $B(o,r)$ where $\pi r^2 = \pi - \pi t^2$. Letting $t \to 0$, one easily concludes that u^* is not Lipschitzian on $\overline{\Omega}^*$.

PROPOSITION 2.6. <u>Let</u> $\Omega \subset \mathbb{R}^N$ <u>be an open parallelipiped. Then ther</u> <u>exists a constant</u> $c = c(N,\Omega) > 0$ <u>such that</u>

$$\mu_N(u^{-1}(B)) \geq c\, \frac{\mu_1(B)^N}{\mathrm{Lip}(u)^N} \tag{2.8}$$

<u>for any nonconstant Lipschitz function</u> u <u>on</u> $\overline{\Omega}$ <u>and any Borelian set</u> <u>in the range of</u> u.

Observe that without loss of generality, we can suppose the face of the parallelipiped Ω to be parallel to the coordinate hyperplane This will be assumed from now on, without further notice.

The proof of proposition 2.6 uses the Steiner rearrangement of It will be denoted by $\tilde{u}$. It is obtained from u by N successiv one-dimensional Schwarz rearrangements with respect to each variabl $x_1,\dots,x_N$ (cf.[11,8]). The function $\tilde{u}$ is defined on the correspondir rearranged domain $\tilde{\Omega}$, which here is simply the parallelipiped centered at zero. It enjoys properties similar to those of u^*: th range of $\tilde{u}$ coincides with that of u,

$$\mu_N(\tilde{u}^{-1}(B)) = \mu_N(u^{-1}(B))\,, \tag{2.9}$$

$\tilde{u}$ is Lipschitzian on $\tilde{\Omega}$ and one has a Polya-Szegö type estimate :

$$\mathrm{Lip}(\tilde{u}) \leq 3^N\, \mathrm{Lip}(u)\,. \tag{2.10}$$

The proof of (2.10) can be adapted from that of a similar estimat sketched in [8;p.82] for periodic functions.

The following preliminary result will be used in the proof c proposition 2.6.

LEMMA 2.7. <u>For any measurable function</u> u <u>on</u> Ω <u>there exists</u> α <u>such that</u>

$$\mu_N\{x \in \Omega\ ;\ u(x) \leq \alpha_u\} \geq \mu_N(\Omega)/2 \tag{2.11}$$

<u>and</u>

$$\mu_N\{x \in \Omega\ ;\ u(x) \geq \alpha_u\} \geq \mu_N(\Omega)/2. \tag{2.12}$$

PROOF. Consider the distribution functions of u :

$$d_u(t) = \mu_N\{x \in \Omega \; ; \; u(x) > t\} \; ,$$
$$d^*_u(t) = \mu_N\{x \in \Omega \; ; \; u(x) \geq t\} \; ,$$

and define

$$\alpha_u = \inf\{t \; ; \; d_u(t) \leq \mu_N(\Omega)/2\}.$$

Since d_u is right continuous, one has

$$d_u(\alpha_u) \leq \mu_N(\Omega)/2 \; ,$$

which gives (2.11). We claim that (2.12) holds, i.e.

(2.13) $$d^*_u(\alpha_u) \geq \mu_N(\Omega)/2 \; .$$

Indeed, by the definition of α_u, $d_u(t) > \mu(\Omega)/2$ for all $t < \alpha_u$ and consequently $d^*_u(t) > \mu(\Omega)/2$ for all $t < \alpha_u$. Since d^*_u is left continuous, we get (2.13). Q.E.D.

PROOF OF PROPOSITION 2.6. Take u and B as in proposition 2.6 and let α_u be such that (2.11) and (2.12) hold. We decompose B into a disjoint union

$$B = B^- \cup B^0 \cup B^+$$

where $B^- = B \cap [\min u, \alpha_u[$, $B^0 = B \cap \{\alpha_u\}$ and $B^+ =]\alpha_u, \max u]$. If we can show that for some constant $c = c(N,\Omega) > 0$, one has

$$\mu_N(u^{-1}(B^-)) \geq c \, \frac{\mu_1(B^-)^N}{\mathrm{Lip}(u)^N}$$

and

$$\mu_N(u^{-1}(B^+)) \geq c \, \frac{\mu_1(B^+)^N}{\mathrm{Lip}(u)^N} \; ,$$

then, by an argument similar to that used in the second part of the proof of proposition 2.1, we conclude that (2.8) holds (with the new constant cc_0^N).

We are thus reduced in this way to proving estimate (2.8) for functions u and Borelian sets B which satisfy in addition either

(2.14) $$\mu_N\{x \in \Omega \; ; \; u(x) \leq \inf B\} \geq \mu_N(\Omega)/2$$

or

(2.15) $$\mu_N\{x \in \Omega \; ; \; u(x) \geq \sup B\} \geq \mu_N(\Omega)/2.$$

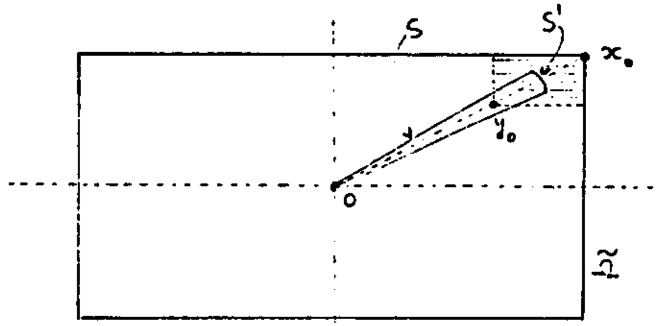
Let us first consider the case where (2.14) holds. Let $\tilde{u}$ be the Steiner rearrangement of u. Then max $\tilde{u}$ is attained at o while min $\tilde{u}$ is attained at each of the corners of the parallelipiped $\tilde{\Omega}$. Moreover, by (2.14),

(2.16) $$\mu_N\{x \in \tilde{\Omega} \; ; \; \tilde{u}(x) \leq \inf B\} \geq \mu_N(\Omega)/2.$$

Let x_0 be the corner of the parallelipiped $\tilde{\Omega}$ with positive coordinate and let $y_0 = x_0/2^{1/N}$. We claim that

(2.17) $$\tilde{u}(y_0) \leq \inf B.$$

Indeed, if (2.17) does not hold, then $\tilde{u}$ is > infB on a neighbourhood of the parallelipiped centered at o with corner y_0, which contradicts (2.16). We now deduce from (2.17) that $\tilde{u}$ is $\leq$ infB on $(y_0 + \mathbb{R}^N_+) \cap \tilde{\Omega}$, i.e. on the shaded parallelipiped in the following figure :



We can then construct a small sector S around the radius $[o,x_0]$, whose cap S' entirely lies in this shaded parallelipiped. The (N-1) measure of S' can be estimated from below by a positive number depending only on $\tilde{\Omega}$.

We are now in a position to adapt the argument of the proof of proposition 2.1. We claim that for any radius R of the above sector S,

$$\mu_1(B)=\int_{\tilde{u}^{-1}(B)\cap R} \left|\frac{\partial \tilde{u}}{\partial r}\right| \, dr \tag{2.18}$$

where r denotes the distance to the origin. This follows as in the proof of proposition 2.1, with the additional observation that since $\tilde{u}$ is $\leq$ infB on S', B entirely lies in the image of R by $\tilde{u}$. It follows from (2.18) that

$$\mu_1(\tilde{u}^{-1}(B)) \cap R) \geq \frac{\mu_1(B)}{\mathrm{Lip}(\tilde{u})} \tag{2.19}$$

for any radius R of S.

We now consider the set $\tilde{u}^{-1}(B)\cap S$ and compute its N-measure by the formula

$$\mu_N(\tilde{u}^{-1}(B)\cap S)=\int \left(\int_{\tilde{u}^{-1}(B)\cap R_\theta} r^{N-1} dr\right) d\theta,$$

where R_θ is the radius of S through the point θ of the unit sphere and where the outer integral is extended on that part of the unit sphere homothetic to the cap S'. Applying lemma 2.2 to the inner integral, we deduce from (2.19) that

$$\mu_N(\tilde{u}^{-1}(B)) \geq \frac{\mu_1(B)^N}{\mathrm{Lip}(\tilde{u})^N} \; c_3 \tag{2.20}$$

where c_3 is the N-measure of the sector of radius 1 through the cap S'. Of course c_3 is estimated from below by a positive number depending only on $\tilde{\Omega}$. Estimate (2.8) now follows by combining (2.20) with (2.9) and (2.10).

Estimate (2.8) is thus proved if u and B satisfy (2.14). Now if u and B satisfy (2.15), then -u and -B satisfy (2.14), so that (2.8) still holds in this case. Q.E.D.

REMARK 2.8. An estimate like (2.8) does not hold with a constant c depending only on N and the measure of the parallelipiped Ω. This can be seen by considering the domains

$$\Omega_k=\{(x,y) \in \mathbb{R}^2 \; ; \; |x| < k \text{ and } |y| < 1/k\} ,$$

the functions

$$u_k(x,y) = \begin{cases} -1 & \text{for} \quad x \leq -1 , \\ x & \text{for} \quad -1 \leq x \leq 1 , \\ +1 & \text{for} \quad x \geq 1 , \end{cases}$$

and the Borel set $]-1,1[$.

REMARK 2.9. An estimate like (2.8) does not hold in an arbitrary bounded open set Ω, even if u vanishes on a significant part of $\partial\Omega$. This can be seen by considering the domain.

$$\Omega=\{(x,y) \in \mathbb{R}^2 \; ; \; o < x < 1 \text{ and } -x^2 < y < x^2\},$$

the function $u(x,y)=1-x$ and the Borel sets $]1-1/k,1[$. One can also see here that both the Schwarz and the Steiner rearrangements of u are not Lipschitzian. Indeed $u^*(o,o)=1$ and $u^* \leq 1-t$ outside B(o,r) where $\pi r^2=2t^3/3$; letting $t\to o$ yields the conclusion for u^*. On the other hand, $\tilde{u}(o,o)=1$ and $\tilde{u}(o,y)=1-|y|^{1/2}$; letting $y\to o$ yields the conclusion for $\tilde{u}$.

3. PROOF OF THE THEOREMS.

PROOF OF THEOREM 1.2. It will be divided into several steps.

Step 1. We consider the functional ϕ defined by (1.6). It is easily verified that ϕ is weakly lower semi continuous on $H_0^1(\Omega)$ and we will show that ϕ is coercive on $H_0^1(\Omega)$, i.e. that $\phi(u) \to +\infty$ as $\|u\| \to \infty$. Here $\| \ \|$ denotes the $H_0^1(\Omega)$ norm. Assuming this for a moment, we deduce that the minimum of ϕ is achieved at some point u in $H_0^1(\Omega)$. Since ϕ is a C^1 functional on $H_0^1(\Omega)$ whose critical points are the weak solutions of (1.1), we obtain that u solves (1.1). Finally a standard bootstrap argument yields that any $H_0^1(\Omega)$ solution of (1.1) belongs to $W^{2,p}(\Omega)$.

Step 2. To prove that ϕ is coercive on $H_0^1(\Omega)$, we assume by contradiction the existence of a sequence u_n such that $\|u_n\| \to \infty$ and

$$(3.1) \qquad \phi(u_n) \leq c$$

for some constant c. Write

$$v_n = \frac{u_n}{\|u_n\|}.$$

Then, for a subsequence, $v_n \to v$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and a.e. in Ω.

We first show that $v \neq 0$. For that purpose observe that, by (3.1),

$$(3.2) \qquad \frac{1}{2}\int_\Omega |\nabla v_n|^2 - \int_\Omega \frac{F(x,u_n)}{\|u_n\|^2} - \int_\Omega \frac{hv_n}{\|u_n\|} \leq \frac{c}{\|u_n\|^2},$$

which implies, by (F_1),

$$(3.3) \quad \frac{1}{2}\int_\Omega |\nabla v_n|^2 - \left(\frac{\lambda_1}{2} + \epsilon\right)\int_\Omega v_n^2 - \int_\Omega \frac{b_\epsilon}{\|u_n\|^2} - \int_\Omega \frac{hv_n}{\|u_n\|} \leq \frac{c}{\|u_n\|^2}$$

If $v \equiv 0$, then the last four terms go to zero as $n \to \infty$. This leads to a contradiction since each v_n has norm 1 in $H_0^1(\Omega)$.

We claim that v is an eigenfunction corresponding to λ_1. Indeed, letting first $n \to \infty$ in (3.3) and then $\varepsilon \to 0$, we get

$$\int_\Omega |\nabla v|^2 \leq \lambda_1 \int_\Omega v^2,$$

which yields the claim.

We also deduce from (3.2), by using Poincaré's inequality, that

$$(3.4) \qquad \limsup \int_\Omega \left(\lambda_1 v_n^2 - \frac{2F(x,u_n)}{\|u_n\|^2}\right) \leq 0.$$

Write $\|u_n\| = t_n$. By the mean value theorem,

$$(3.5) \quad \int_\Omega \left(\frac{2F(x,t_n v_n)}{t_n^2} - \frac{2F(x,t_n v)}{t_n^2}\right) = \int_\Omega \frac{2}{t_n^2} f(x,t_n v + \theta(t_n v_n - t_n v))(t_n v_n - t_n v)$$

where $\theta = \theta(n,x) \in [0,1]$, and we observe, using (f_1), that the right-hand side of (3.5) goes to zero as $n \to \infty$. It thus follows from (3.4) that

$$(3.6) \qquad \limsup \int_\Omega \left(\lambda_1 v^2 - \frac{2F(x,t_n v)}{t_n^2}\right) \leq 0.$$

This will be used in step 4 to reach the final contradiction.

Step 3. We will show that for some $\delta > 0$ and all n sufficiently large,

$$(3.7) \quad \mu_N\{x \in \Omega \ ; \ \frac{2F(x,t_n v(x))}{t_n^2 v(x)^2} \leq \lambda_1 - \eta\} \geq \delta,$$

where η is the positive number entering assumption (F_2).

Since v is an eigenfunction corresponding to λ_1, we have either $v > 0$ on all Ω or $v < 0$ on all Ω. We consider the first case. The second case can be treated similarly, by using that part of assumption (F_2) relative to $-\infty$.

We apply proposition 2.1 to the open set Ω, the function u_n and the Borelian set

$$B_n = E(\Omega', \lambda_1 - \eta) \cap [o, t_n \max v],$$

where $E(\Omega', \lambda_1 - \eta)$ is the set entering assumption (F_2):

$$E(\Omega', \lambda_1 - \eta) = \bigcap_{x \in \Omega'} \{ s \in \mathbb{R}_o \ ; \ \frac{2F(x,s)}{s^2} \leq \lambda_1 - \eta \}.$$

This gives

(3.8) $$\mu_N\{x \in \Omega \ ; \ t_n v(x) \in B_n\} \geq c \frac{\mu_1(B_n)^N}{\mathrm{Lip}(t_n v)^N}$$

for some constant $c > o$. The left hand side of (3.8) is equal to

(3.9) $$\mu_N\{x \in \Omega' \ ; \ t_n v\ (x) \in B_n\}$$

and the set in (3.9) is clearly included in the set in (3.7). Moreover, assumption (F_2) tells us that for some $\delta_1 > o$ and all n sufficiently large,

$$\mu_1(B_n) \geq \delta_1 \max v t_n .$$

Finally

$$\mathrm{Lip}(t_n v) = t_n \mathrm{Lip}(v)$$

where $\mathrm{Lip}(v)$ is a constant. Collecting these informations, we deduce (3.7) from (3.8).

<u>Step 4</u>. We will now deduce a contradiction from (3.6) and (3.7). Let χ_n be the characteristic function of the set in (3.7). From (3.6) we have

$$\liminf \int_\Omega (\lambda_1 - \frac{2F(x,t_n v)}{t_n^2 v^2})\ v^2\ \chi_n$$
$$+ \liminf \int_\Omega (\lambda_1 - \frac{2F(x,t_n v)}{t_n^2 v^2})\ v^2\ (1-\chi_n) \leq o.$$

We apply the definition of χ_n to the first term and Fatou's lemm together with (F_1) to the second term to get

$$\eta \liminf \int_\Omega v^2 \chi_n \leq o.$$

This implies, for a subsequence,

$$\lim \int_\Omega v^2 \chi_n = o.$$

Thus $v^2 \chi_n \to o$ in $L^1(\Omega)$ and consequently in measure.

Recall that χ_n is the characteristic function of a varying set c measure $\geq \delta > o$. Choose $\varepsilon > o$ so small that

(3.10) $$\mu_N\{x \in \Omega \ ; \ v^2(x) < \varepsilon\} < \frac{\delta}{3}.$$

The existence of such an ε can be seen as follows: in the contrar for each k=1,2,...,

$$\mu_N\{x \in \Omega \ ; \ v^2(x) < \frac{1}{k}\} \geq \frac{\delta}{3} ;$$

calling A_k the set involved in this inequality and A the intersection c those A_k, we get $\mu_N(A) \geq \delta/3$ and v=o a.e. in A; this contradicts th unique continuation property of the eigenfunction v. We then deduc from (3.10) that

$$\mu_N\{x \in \Omega \ ; \ v^2(x) \geq \varepsilon\} \geq \mu_N(\Omega) - \delta/3$$

and consequently

$$\mu_N\{x\in\Omega\ ;\ \chi_n(x)v^2(x)\geq\varepsilon\}$$
$$=\mu_N(\{x\in\Omega\ ;\ v^2(x)\geq\varepsilon\}\cap\{x\in\Omega\ ;\ \chi_n(x)=1\})$$
$$\geq\frac{2\delta}{3}$$

This contradicts the convergence in measure of $\chi_n v^2$ towards zero. Q.E.D.

PROOF OF THEOREM 1.3. It is identical to that of theorem 1.2 except for step 3. The claim here is that for some $\eta > o$ and some $\delta > o$,

$$\mu_N\{x\in\Omega\ ;\ \frac{2F(x,t_n v(x))}{t_n^2 v(x)^2}\leq\lambda_1-\eta\}\geq\delta \tag{3.11}$$

for all n sufficiently large. As before we can consider only the case where v is >o on Ω.

Replacing if necessary the open set ω_+ of assumption $(F_3)_+$ by a smaller one, we can always suppose that ω_+ is a parallelipiped. Since the eigenfunction v is not constant on ω_+, we have, on ω_+,

$$o\leq \min v < \max v.$$

Let us write minv=a and maxv=b. We apply proposition 2.6 to the open set ω_+, the function $t_n v$ and the Borelian set

$$B_n=E(\omega'_+,\lambda_1-\eta)\cap[at_n,\ bt_n],$$

where η is associated to the number $\nu=a/b$ according to assumption $(F_3)^+$. This gives

$$\mu_N\{x\in\omega_+;t_n v(x)\in B_n\}\geq c\frac{\mu_1(B_n)^N}{\mathrm{Lip}(t_n v)^N} \tag{3.12}$$

for some positive constant $c > o$.

Estimate (3.11) can then be deduced from (3.12) as in the proof of theorem 1.2. Q.E.D.

REMARK 3.1. A natural question concerns the introduction in this problem of strong nonlinearities, as in [4]. This is possible under assumption (1.4), as was observed in [2], where this question is also studied with the operator $-\Delta$ replaced by the pseudo-laplacian $-\Delta_p$. The difficulty to include strong nonlinearities in our approach lies at the end of step 2, to derive (3.6) from (3.4). The same difficulty arises when one tries to relax the linear growth assumption for f into the more natural growth condition in $|s|^\sigma$.

REMARK 3.2. The method of this paper does not apply to the study of the Neumann problem

$$\begin{cases}-\Delta u=f(x,u)+h(x)\ \text{in}\ \Omega\ ,\\ \partial u/\partial n=o\ \text{on}\ \partial\Omega\end{cases}$$

near its first eigenvalue $\lambda_1=o$. This is due to the fact that the corresponding eigenfunctions are constant. A similar difficulty arises in the study of the periodic solutions of Liénard equation (cf.[6]).

REMARK 3.3. It is easily seen that theorems 1.2 and 1.3 still hold if assumptions (F_2) and $(F_3)_\pm$ are weakened in the following way. It suffices in (F_2) to assume that for some nonnegative function d(x) in $L^1(\Omega)$, the set

$$\bigcap_{x\in\Omega}\{s\in\mathbb{R}_0\ ;\ \frac{2F(x,s)}{s^2}\leq\lambda_1-\eta+\frac{d(x)}{s}\}$$

has a positive density at both $+\infty$ and $-\infty$. In $(F_3)_+$, one can replace the set $E(\omega'_+,\lambda_1-\eta)$ by the larger set

$$\bigcap_{x\in\omega'_+}\{s\in\mathbb{R}_0\ ;\ \frac{2F(x,s)}{s^2}\leq\lambda_1-\eta+\frac{d(x)}{s}\},$$

where the nonnegative function d(x) lies in $L^1(\omega_+)$ and possibly depends on ν. Of course a similar variant can be introduced for $(F_3)_-$.

REMARK 3.4. Assumption $(F_3)_+$ of theorem 1.3 can also be weakened in the following way, which involves the verification of (1.7) for only one choice of ν. Denote by v the positive normalized (i.e. $\|v\|=1$)

eigenfunction associated to λ_1. It suffices to require the existence of a full subset ω'_+ of an open parallelipiped $\omega_+ \subset \Omega$, a number ν with

$$\frac{\min_{\omega_+} v}{\max_{\omega_+} v} < \nu < 1$$

and a number $\eta > o$ such that (1.7) holds. Using lemma 4.5 of section 4 as well as the notations introduced there, we see that this will be satisfied if there exist a full subset ω'_+ of an open parallelipiped $\omega_+ \subset \Omega$ and a number $\eta > o$ such that

$$\frac{\underline{d}(E(\omega'_+,\lambda_1-\eta),o)}{\overline{d}(E(\omega'_+,\lambda_1-\eta),o)} > \frac{\min_{\omega_+} v}{\max_{\omega_+} v} .$$

A similar observation holds of course for $(F_3)_-$.

REMARK 3.5. Results similar to theorems 1.2 and 1.3 also hold for the problem

$$\begin{cases} Lu=f(x,u)+h(x) & \text{in } \Omega , \\ D^\alpha u=o & \text{on } \partial\Omega \quad \text{for } |\alpha|\leq m-1 \end{cases}$$

where L is a differential operator of the form

$$Lu=\sum_{|\alpha|,|\beta|\leq m} (-1)^{|\beta|} D^\beta(a_{\alpha\beta}(x)D^\alpha u)$$

with the following assumptions: $a_{\alpha\beta}=a_{\beta\alpha}$; $a_{\alpha\beta} \in C^{|\alpha|+|\beta|}(\overline{\Omega})$ for $|\alpha|+|\beta|>o$; $a_{\alpha\beta} \in L^\infty(\Omega)$ for $|\alpha|=|\beta|=o$; for some $\gamma > o$,

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^\alpha\xi^\beta \geq \gamma|\xi|^{2m}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$; the eigenfunctions of L associated to the first eigenvalue λ_1 have the unique continuation property, i.e. the only function $v \in H_o^m(\Omega)$ satifying $Lv=\lambda_1 v$ and vanishing on a set of positive measure is $v\equiv o$. Assumption $(F_3)_\pm$ must however be reinforced to take into account the fact that the eigenfunctions corresponding to

λ_1 may not be of constant sign on Ω. One requires that the open sets ω_+ and ω_- be equal.

4. EXAMPLES

We first give an example of a nonlinearity $f(x,s)=f(s)$ which satisfies the assumptions of theorem 1.2 together with (1.5).

EXAMPLE 4.1. Take a sequence

$$0 < a_1 < 2a_1 < a_2 < 2a_2 < \ldots < a_k < 2a_k < \ldots$$

such that $\lim a_k/a_{k+1}=0$ and consider the set

$$E = \bigcup_{k=1}^{\infty} [2a_k, a_{k+1}].$$

Letting

$$\zeta(r) = \frac{\mu_1(E \cap [0,r])}{\mu_1([0,r])},$$

we see that $\zeta(r)$ is increasing in each interval $[2a_k, a_{k+1}]$, decreasing in each interval $[a_k, 2a_k]$ and $\zeta(2a_k)=1/2\ \zeta(a_k)$; consequently

$$\liminf_{r\to\infty} \zeta(r) = \liminf_{k\to\infty} \zeta(2a_k)$$

$$= 1/2 \liminf_{k\to\infty} \zeta(a_k) \geq 1/2 \liminf_{k\to\infty} \frac{a_k-2a_{k-1}}{a_k} = 1/2$$

and

$$1 \geq \limsup_{r\to\infty} \zeta(r) = \limsup_{k\to\infty} \zeta(a_k)$$

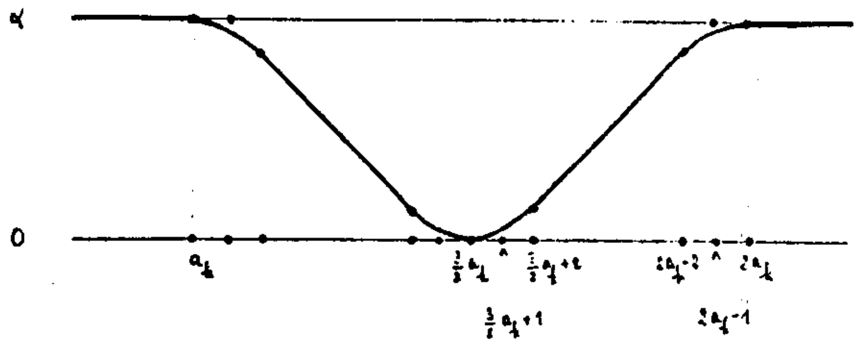
$$\geq \limsup_{k\to\infty} \frac{a_k-2a_{k-1}}{a_k} = 1.$$

Take $\alpha > 0$ and define a function ψ on $\mathbb{R}^+$ in the following way: put

$$\psi = \begin{cases} \alpha & \text{on } E, \\ 0 & \text{on } [0, 3/2\, a_1], \\ 0 & \text{at each point } 3/2\, a_k, \end{cases}$$

and connect these values in a smooth almost piecewise linear

oscillating manner, as described in the following figure :



This function is extended to all $\mathbb{R}$ by taking, for instance, $\psi(s)=+\psi(-s)$ for $s \in \mathbb{R}^-$. We then define the potential F by

$$F(s) = \lambda_1 \frac{s^2}{2} - \psi(s) \frac{s^2}{2}.$$

Clearly (F_1) and (1.5) hold, and, by the computation of $\liminf \zeta(r)$ above, the set

$$\{s\in\mathbb{R}_0 : \frac{2F(s)}{s^2} \leq \lambda_1-\alpha\} = E \cup (-E)$$

has positive density at both $+\infty$ and $-\infty$. It remains to verify (f_1). We have

$$\text{(4.1)} \qquad \frac{f(s)}{s} = \lambda_1-\psi(s)-\frac{s}{2}\psi'(s).$$

A simple calculation gives

$$\limsup_{s\to\infty} \frac{f(s)}{s} = \limsup_{s\to-\infty} \frac{f(s)}{s} = \lambda_1+\frac{3\alpha}{2},$$

$$\liminf_{s\to\infty} \frac{f(s)}{s} = \liminf_{s\to-\infty} \frac{f(s)}{s} = \lambda_1-3\alpha,$$

which yields (f_1).

We observe in this example that the quotient $f(s)/s$ does not lie asymptotically at the left side of λ_1 as $s \to \pm\infty$. Some sort of crossing of λ_1 occurs. However it is not of the Ambrosetti-Prodi type, which would mean

$$\limsup_{s\to-\infty} f(s)/s < \lambda_1 < \liminf_{s\to+\infty} f(s)/s.$$

It turns out that the crossing exhibited in the above example must always occur under our assumptions if (1.5) holds. This is a consequence of the following

PROPOSITION 4.2. <u>Let</u> $f:\mathbb{R}^+ \to \mathbb{R}$ <u>be continuous and denote by</u> F <u>the associated potential. Suppose that</u>

$$\limsup_{s\to\infty} \frac{2F(s)}{s^2} = \lambda_1$$

<u>and that, for some</u> $\eta > 0$, <u>the set</u>

$$\{s\in\mathbb{R}_0^+ \; ; \; \frac{2F(s)}{s^2} \le \lambda_1-\eta\}$$

<u>has positive density at infinity. Then</u>

$$\limsup_{s\to\infty} \frac{f(s)}{s} > \lambda_1.$$

PROOF. We write $2F(s)/s^2$ as $\lambda_1-\psi(s)$ so that

$$\liminf_{s\to\infty} \psi(s)=0 \tag{4.2}$$

and the set

$$\{s\in\mathbb{R}_0^+ \; ; \; \psi(s) \ge \eta\} \tag{4.3}$$

has positive density at infinity. By (4.1) we have to show that

$$\liminf_{s\to\infty} (\psi(s)+ \frac{s}{2}\, \psi'(s)) < 0 . \tag{4.4}$$

By (4.2) there exists a sequence $b_k \to \infty$ such that $\psi(b_k)=\varepsilon_k \to$ 0. Without loss of generality, one can always assume that ψ takes a value $\ge \eta$ at the left side of b_1 and that $\varepsilon_k < \eta$ for all k. Call a_k the largest $s < b_k$ such that $\psi(s)=\eta$. Thus $\psi(a_k)=\eta$ and $\psi < \eta$ on $]a_k,b_k]$. Define

$$\beta_k=\inf\{\psi(s)+s/2\ \psi'(s) \; ; \; s \in [a_k,b_k]\}.$$

Thus, on $[a_k,b_k]$,

$$\psi(s)+s/2\ \psi'(s) \ge \beta_k,$$

i.e.

$$(s^2\psi(s))' \ge 2s\beta_k,$$

which gives, after integration,

$$\varepsilon_k-\beta_k \ge \frac{a_k^2}{b_k^2}\,(\eta-\beta_k) . \tag{4.5}$$

We claim that for some constant $c > 0$ and all k sufficiently large,

$$\beta_k \le -c, \tag{4.6}$$

which will of course yield the conclusion (4.4).

To prove (4.6) we first observe that no subsequence of β converges to zero. Indeed, suppose by contradiction that there exist such a subsequence (still denoted by β_k). Then (4.5) implies that

$$\frac{a_k}{b_k} \to 0. \tag{4.7}$$

Since the set (4.3) does not intersect the union of the interval

$]a_k,b_k]$, lemma 4.3 below shows that (4.7) contradicts the assumption of positive density for (4.3). We now claim that no subsequence of β_k satisfies an inequality like

$$\beta_k \geq d > 0 \tag{4.8}$$

for some constant d. Indeed, suppose by contradiction that there exists such a subsequence (still denoted by β_k). Then either $\beta_k > \eta$ for an infinite number of values of k, or $\beta_k < \eta$ for an infinite number of values of k, or $\beta_k=\eta$ for an infinite number of values of k. We look separately at these values of k. In the first case (4.5) implies

$$\frac{a_k^2}{b_k^2} \geq \frac{\beta_k-\varepsilon_k}{\beta_k-\eta} > 1$$

since $\varepsilon_k < \eta$. This is contradictory since $a_k \leq b_k$. In the second case, (4.5) implies

$$\frac{a_k^2}{b_k^2} \leq \frac{\varepsilon_k-\beta_k}{\eta-\beta_k}$$

where the right hand side becomes < o since (4.8) holds. This is again contradictory. In the third case, (4.5) implies

$$\beta_k \leq \varepsilon_k$$

which, by (4.8), is also contradictory.

We have thus a sequence β_k such that no subsequence goes to zero and no subsequence satisfies (4.8). This implies that (4.6) holds. Q.E.D.

LEMMA 4.3. <u>Suppose that the measurable set</u> $G \subset \mathbb{R}^+$ <u>does not intersect a sequence of disjoint intervals</u> $]a_k,b_k[$ <u>where</u> a_k and $b_k \to \infty$. <u>If</u> G <u>has positive density at infinity, then there exists a constant</u> c <u>such that</u>

$$\frac{b_k}{a_k} \leq c \tag{4.9}$$

<u>for all</u> k.

PROOF. The quotient

$$\frac{\mu_1(G\cap[0,r])}{\mu_1([0,r])}$$

is multiplied by a_k/b_k when r crosses the hole $]a_k,b_k[$. It follows that if for a subsequence, $a_k/b_k \to 0$, then G does not have positive density at infinity. Q.E.D.

We now construct a nonlinearity f(x,s) which satisfies the assumptions of theorem 1.3 together with (1.5).

EXAMPLE 4.4. Take ψ as in example 4.1 and consider, for $0 < \eta \leq \alpha/2$, the set

$$E_\eta=\{s\in\mathbb{R}^+ \ ; \ \psi(s) \geq \eta\}.$$

We claim that for any $0 \leq \upsilon < 1$, there exists $\eta > 0$ such that

$$\liminf_{r\to\infty} \frac{\mu_1(E_\eta\cap[\upsilon r,r])}{\mu_1([\upsilon r,r])} > 0 \ . \tag{4.10}$$

For that purpose we first observe that the holes in E_η are intervals $]c_k,d_k[$ centered at $3/2\ a_k$, with length at most equal to

$$\frac{a_k\eta-4\eta+2\alpha}{\alpha} .$$

Letting

$$\zeta_\eta(r) = \frac{\mu_1(E_\eta\cap[0,r])}{\mu_1([0,r])}$$

and computing as in example 4.1, we obtain

$$\liminf_{r\to\infty} \zeta_\eta(r) = \liminf_{k\to\infty} \zeta_\eta(d_k)$$

$$= \liminf_{k\to\infty} \frac{c_k}{d_k} \zeta_\eta(c_k) \geq \liminf_{k\to\infty} \frac{c_k}{d_k} \zeta_\eta(a_k)$$

$$\geq \liminf \frac{c_k}{d_k} \frac{a_k - 2a_{k-1}}{a_k} \geq \frac{3\alpha-\eta}{3\alpha-\eta}$$

and

$$\limsup_{r\to\infty} \zeta_\eta(r) \geq \limsup_{r\to\infty} \zeta(r) = 1.$$

Consequently, given any υ with $0 \leq \upsilon < 1$, we can choose η such that

$$\frac{\liminf \zeta_\eta(r)}{\limsup \zeta_\eta(r)} > \upsilon$$

i.e., with the notations of lemma 4.5 below,

$$\frac{\underline{d}(E_\eta,0)}{\overline{d}(E_\eta,0)} > \upsilon .$$

The claim (4.10) is then a consequence of this lemma 4.5. We can now construct the desired example. Choose two balls B_+ and B_- in Ω and define

$$F(x,s) = \begin{cases} \lambda_1 \dfrac{s^2}{2} & \text{if } x \notin B_+ \cup B_- \\ & \text{or if } x \in B_+ \text{ and } s < 0 \\ & \text{or if } x \in B_- \text{ and } s \geq 0, \\ \lambda_1 \dfrac{s^2}{2} - \psi(s)\dfrac{s^2}{2} & \text{if } x \in B_+ \text{ and } s > 0 \\ & \text{or if } x \in B_- \text{ and } s < 0. \end{cases}$$

Clearly (F_1) and (1.5) hold, and the calculation in example 4.1 shows that (f_1) holds. Moreover (4.10) shows that $(F_3)_+$ and $(F_3)_-$ are satisfied on B_+ and B_- respectively.

LEMMA 4.5. _Let_ $G \subset \mathbb{R}^+$ _be measurable and denote, for_ $0 \leq \upsilon < 1$,

$$\underline{d}(G,\upsilon) = \liminf_{r\to\infty} \frac{\mu_1(G\cap[\upsilon r,r])}{\mu_1([\upsilon r,r])} ,$$

$$\overline{d}(G,\upsilon) = \limsup_{r\to\infty} \frac{\mu_1(G\cap[\upsilon r,r])}{\mu_1([\upsilon r,r])} .$$

Assume $\underline{d}(G,0) > 0$. _Then_ $\underline{d}(G,\upsilon) > 0$ _for all_ $0 \leq \upsilon < 1$ _such that_

$$\upsilon < \frac{\underline{d}(G,0)}{\overline{d}(G,0)} .$$

PROOF. Writing

$$\frac{\mu_1(G\cap[\upsilon r,r])}{(1-\upsilon)r} = \frac{\mu_1(G\cap[0,r])}{(1-\upsilon)r} - \frac{\mu_1(G\cap[0,\upsilon r])}{(1-\upsilon)r} ,$$

we obtain

$$\underline{d}(G,\upsilon) \geq \frac{1}{1-\upsilon}\underline{d}(G,0) - \frac{\upsilon}{1-\upsilon}\overline{d}(G,0) ,$$

which yields the conclusion. Q.E.D.

We now turn to a nonlinearity $f(x,s)=f(s)$ for which the glob result (Theorem 1.2) applies while the local one (Theorem 1.3) does no This illustrates the fact that the density requirement at infinity stronger in theorem 1.3 than in theorem 1.2.

EXAMPLE 4.6. Take a sequence

$$8 < b_1 < 4b_1 < b_2 < 4b_2 < \ldots$$

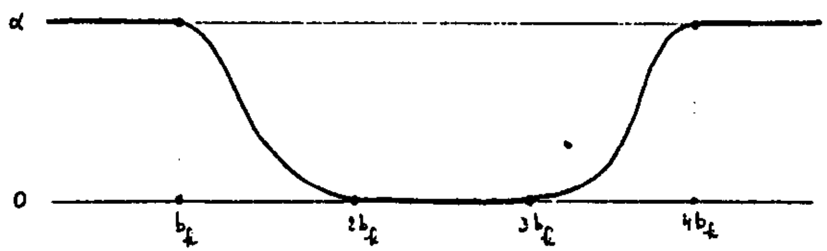
such that $\lim b_k/b_{k+1}=0$. This implies, as in example 4.1, that the set

$$G = \bigcup_{k=1}^{\infty} [4b_k, b_{k+1}]$$

has positive density at $+\infty$. Take $\alpha > 0$ and define a function φ on $\mathbb{R}^+$ in the following way: put

$$\varphi = \begin{cases} \alpha \text{ on } G, \\ 0 \text{ on } [0, 2b_1], \\ 0 \text{ on } \bigcup_{k=1}^{\infty} [2b_k, 3b_k], \end{cases}$$

and connect these values by a construction similar to that in example 4.1, as described in the following figure :



Then, for any $\eta > 0$, the set

$$G_\eta = \{s \in \mathbb{R}^+; \varphi(s) \geq \eta\}$$

does not intersect

$$\bigcup_{k=1}^{\infty} [2b_k, 3b_k],$$

and consequently

$$\underline{d}(G_\eta, 2/3) = 0.$$

A potential F(s) can then be constructed from this function φ, as in example 4.1, which satisfies (f_1), (F_1), (1.5), (F_2) but does not satisfy $(F_3)_\pm$.

Our next two examples concern nonlinearities whose potential oscillates below λ_1 but which fall outside our present results.

EXAMPLE 4.7. Consider

$$F(s) = \frac{3}{8}\lambda_1 s^2 + \frac{1}{8}\lambda_1 s^2 \sin s \,. \tag{4.11}$$

Clearly (F_1), (1.5) and (F_2) hold. However (f_1) is not verified since

$$f(s) = \frac{3}{4}\lambda_1 s + \frac{1}{4}\lambda_1 s \sin s + \frac{1}{8}\lambda_1 s^2 \cos s.$$

EXAMPLE 4.8. Consider

$$F(s) = \frac{3}{8}\lambda_1 s^2 + \frac{1}{8}\lambda_1 s^2 \sin\log(1+\log(1+|s|)). \tag{4.12}$$

Clearly (F_1) and (1.5) hold, as well as (f_1) since

$$\limsup_{s\to\infty} \frac{f(s)}{s} = \lambda_1 \,, \tag{4.13}$$

$$\liminf_{s\to\infty} \frac{f(s)}{s} = \frac{\lambda_1}{2} \,.$$

However (F_2) does not hold. This can be verified directly by using lemma 4.3. It also follows from proposition 4.2 and (4.13).

Observe that a little less damping in the oscillations in (4.12) leads to a nonlinearity which satisfies the assumptions of theorem 1.2:

EXAMPLE 4.9. Consider

$$F(s) = \frac{3}{8}\lambda_1 s^2 + \frac{1}{8}\lambda_1 s^2 \sin\log(1+|s|).$$

Clearly (F_1) and (1.5) hold, as well as (f_1) since

$$\lambda_1 < \limsup_{s\to\infty} \frac{f(s)}{s} < \frac{9}{8}\lambda_1 ,$$

$$\frac{3}{8}\lambda_1 < \liminf_{s\to\infty} \frac{f(s)}{s} < \frac{1}{2}\lambda_1 .$$

A direct computation show that (F_2) also holds.

To conclude this section, we show that a density condition at infinity for $f(s)/s$ implies a strict inequality at infinity for $2F(s)/s^2$. This was observed in a particular case in [1]. See also [2].

PROPOSITION 4.10. <u>Let</u> $f:\mathbb{R}^+ \to \mathbb{R}$ <u>be continuous with</u>

(4.14) $$\limsup_{s\to\infty} \frac{f(s)}{s} \le \lambda_1 .$$

<u>Suppose that for some</u> $\eta > 0$, <u>the set</u>

$$E=\{s \in \mathbb{R}_0 ; \frac{f(s)}{s} \le \lambda_1-\eta\}$$

<u>has positive density at</u> $+\infty$. <u>Then the associated potential</u> F <u>satisfies</u>

$$\limsup_{s\to+\infty} \frac{2F(s)}{s^2} < \lambda_1 .$$

PROOF. By (4.14) and the definition of E, we have

$$F(s) \le \int_{[0,s]\cap E}(\lambda_1 t-\eta t)+\int_{[0,s]\cap E^c}(\lambda_1 t+\varepsilon t+b_\varepsilon)$$

where the constant b_ε is associated to $\varepsilon > 0$ in the detailed expression of (4.14) and where E^c denotes the complement of E. Lemma 2.2. implies

$$\int_{[0,s]\cap E}\eta t \ge \int_0^{\mu_1(E\cap[0,s])}\eta t=\eta/2\ \mu_1(E\cap[0,s])^2,$$

where, by the positive density at infinity, the last term is $\ge \eta\delta s^2/2$ for some $\delta > 0$ and all s sufficiently large. Consequently, taking

$\varepsilon=\eta\delta/4$, we obtain

$$F(s) \le \lambda_1 \frac{s^2}{2} - \frac{\eta\delta}{4}\frac{s^2}{2}+sb_\varepsilon ,$$

which yields the conclusion. Q.E.D.

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