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SYMPLECTIC TOPOLOGY PART I

Symplectic Rigidity, Holomorphy and Hamiltonian Dynamics : A Survey

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SYMPLECTIC RIGIDITY, HOLOMORPHY AND HAMILTONIAN DYNAMICS: A SURVEY

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CONTENTS

I. INTRODUCTION

Since the end of the seventies a dramatic progress has been achieved in Hamiltonian dynamics and symplectic geometry. In this survey we shall look at several topics which play a key role in this development. We are however far from being complete and we refer the reader to [B], [Gr3] and [Ra–Am–Ek–Ze] for other aspects. The topics are presented here in their logical, rather than their chronological order. Before we go into details however we give a short "historical" sketch.

The breakthrough in the variational theory of finding periodic solutions of Hamiltonian systems (if we forget the more special geodesic problem) is due to Rabinowitz and Weinstein in 78, [Ra1], [We1]. In his paper Rabinowitz showed that the well known classical variational principle can be effectively used to show the existence of periodic solutions for Hamiltonian systems. This came as a big surprise since this principle was considered as very degenerate and as useless for existence questions (see for example J. Moser's remark in [Mo2]). Motivated by Rabinowitz's result there was a flood of papers studying different types of Hamiltonians, see [Ra3], [Ze1] and generally [Ra–Am–Ek–Ze] for a survey. Moreover it lead to the study of new classes of abstract functionals, see [Be–Ra2], [H1]. On the other hand the dual variational principle was introduced in [Cl1] and led to the celebrated Ekland–Lasry result [E–L], Ekelands Morse–theory for convex Hamiltonian systems [E1], the solution of the minimal period problem in [E–H1] for convex Hamiltonian after preliminary results by Ambrosetti and Mancini [A–M] and Clarke–Ekeland [Cl–E] and recently in [E–H3] to global and local symplectic invariants of convex

Hamiltonian energy surfaces and their periodic trajectories. A very recent breakthrough is due to C. Viterbo [Vi] who proves the so called Weinstein conjecture in the $\mathbb{R}^{2n}$–case. This conjecture is concerned with the existence of periodic solutions for a Hamiltonian system on a prescribed energy surface. Motivated by [Vi] in [H–Ze] a new phenomenon was detected – The Almost–Existence–Mechanism. In [H–Vi] and [F–H–V] more cases of the Weinstein conjecture have been solved. The method of proof in [F–H–V] is particularly interesting since one uses a new ingredient: first order elliptic systems. This brings us to symplectic geometry and Gromov's key contribution to this subject.

Symplectic geometry had developed separately from Hamiltonian dynamics. Though a global symplectic geometry had been conjectured in 65 by Arnold [Ar1], [Ar2], the results stayed local for a long time, see [We1], [Mo1]. In fact embedding results due to Gromov, see [Gr2,3] indicated a high flexibility of the symplectic notion. In 82 Conley and Zehnder [Co–Ze1] succeeded in proving one of the Arnold conjectures using methods from the variational theory of Hamiltonian dynamics and the homotopy index of C. Conley [Co]. Using this "hook–up" M. Chaperon proved a Lagrangian intersection result for the standard torus, [Ch], and in [H2] the most general case for Lagrangian intersection theory in cotangent bundles of compact manifolds was established, see [L–S] for a simpler proof. [Co–Ze1] also inspired A. Floer and J. Sikorav in proving some of the Arnold conjectures in certain spaces [Fl1], [Si]. A very important contributions is then made in 85 by M. Gromov [Gr1]. He introduces his theory of almost holomorphic curves. A. Floer quickly realized that Gromov's

(nonvariational) ideas can be connected with the "classical" variational approach used in [Co–Ze1], [H2], [Fl1], [Si]. This leads to Floer's Morse theory for Lagrangian intersections [F2–F5] and then to the Lusternik Schnirelman theory, simultaneously developed in [F7] and [H3]. A phenomenon well–known in the theory of harmonic maps and detected in this set up by Sacks and Uhbenbeck [S–U] and for holomorphic maps in [Gr1], namely the bubbling off of harmonic or holomorphic spheres plays a key technical role and causes difficulties if certain topological conditions are not met. Recently Floer obtain some very interesting results in this direction [Fl6].

In another direction, in [E–H5], the local rigidity of symplectic maps has been studied, giving a new phenomenon. Rigidity in symplectic geometry had only be accepted and expected as a global phenomenon, see remarks in [B] and [Gr3] concerning the celebrated Eliashberg Gromov result which says in a variant that the symplectic diffeomorphism group of a compact symplectic manifold is C^0–closed in the diffeomorphism group. Very interestingly the results in [E–H5] show a close relationship between symplectic embedding problems and Hamiltonian dynamics on energy surfaces.

In this survey we shall start with the local rigidity property of symplectic geometry, connect it with Hamiltonian dynamics on an energy surface and study the Weinstein conjecture. Finally, we look at the Arnold conjecture and survey the recent results and explain some of the key technical ingredients.

II. LOCAL RIGIDITY IN SYMPLECTIC GEOMETRY

We denote by (V,ω) the standard symplectic vectorspace $\mathbb{R}^{2n}$ equipped with the symplectic form $\omega = \langle J\cdot,\cdot\rangle$, where

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \in \mathcal{L}(V)$$

and $\langle\cdot,\cdot\rangle$ denotes the standard inner product. We call a linear map $\phi: V \to V$ symplectic if it preserves ω, i.e.

$$\phi^*\omega = \omega, \ (\phi^*\omega)(\xi,\eta) = \omega(\phi(\xi), \ \phi(\eta))$$

We denote by $Sp(V)$ the group of symplectic linear isomorphisms. Given $\phi \in Sp(V)$ we have $\phi^*\omega^n = \omega^n$ so that $\det(\phi) = 1$. Given a smooth map $h: U \to V$, defined on an open subset U of V we write $h^*\omega$ for the 2-form on U defined by

$$(h^*\omega)(x)(\xi,\eta) = \omega(h'(x)\xi, h'(x)\eta)$$

where $h'(x)$ is the derivative at $x \in U$. A map h as described above is called symplectic iff $h^*\omega = \omega$ on U. Assume now U_1 and U_2 are open subsets of V. We study the question, when there exists a symplectic embedding $h: U_1 \to U_2$? By the preceeding discussion h will be volume preserving so that the existence of such an embedding implies $\mathrm{vol}(U_2) \geq \mathrm{vol}(U_1)$. A volume condition in V is a "2n-dimensional" condition in contrast to the 2-dimensional condition of being symplectic. Can we say more than $\mathrm{vol}(U_2) \geq \mathrm{vol}(U_1)$ (if $n \geq 2$)? For $r \in [0,+\infty]$ denote by $B(r)$ the open Euclidean ball in V of radius r, where we have $B(0) = \emptyset$ and $B(\infty) = V$. By $\Sigma(r)$ we denote the open symplectic cylinder of radius r, i.e.

$$\Sigma(r) = \{x \in V \mid x_1^2 + x_{n+1}^2 < r^2\}.$$

We ask now the simple looking question when can $B(r)$ be symplectically embedded into $\Sigma(r')$. Obviously an embedding (even linear) is possible if $r' = r$. Can we perhaps squeeze $B(r)$ into some $\Sigma(r')$ with $r' < r$. It turns out that this is a very deep question. In [Gr1] proved the following result (among many other things).

THEOREM 1 If there exists a symplectic embedding $h: B(r) \to \Sigma(r')$ then $r' \geq r$.

This result is proved using nonlinear first order elliptic systems (Cauchy–Riemann type operators). Motivated by Gromov's result the notion of a symplectic capacity function has been introduced in [E–H5]. The idea is to put the embedding problem $h: U_1 \to U_2$ on an axiomatic foundation. This approach actually leads to an interesting hook up with the fixed energy problem in Hamiltonian dynamics and in particular this connection can be used to give an alternative proof of Theorem 1 using periodic solution of Hamiltonian systems on a prescribed energy surface, see [E–H5] for more details.

1. SYMPLECTIC CAPACITY FUNCTIONS

We denote by $\wp$ the power set of V.

Definition 1 A map $\alpha: \wp \to [0,+\infty]$ satisfying the following three axioms is called a symplectic capacity function

(A1) (normalisation) $\alpha(\Sigma(1)) = \alpha(B(1)) = \pi$

(A2) (monotoncity) $S \subset T \Rightarrow \alpha(S) \leq \alpha(T)$

(A3) (conformality) If $\psi: V \to V$ is a diffeomorphism satisfying $\psi^* \omega = c\,\omega$ for some $c > 0$ then, $\alpha(\psi(S)) = c\,\alpha(S)$ for $S \in \varphi$.

It is not difficult to show that the existence of a symplectic capacity function α is equivalent to Gromov's result. Let us define, with D being the symplectic diffeomorphism group of V,

(1) $\dot{\alpha}(S) = \inf\{\pi r^2 \in [0,+\infty] \mid \text{there exists } \psi \in D \text{ with } \psi(S) \subset \Sigma(r)\}$

$\alpha(S) = \sup\{\pi r^2 \in [0,+\infty] \mid \text{there exists } \Psi \in D \text{ with } \psi(B(r)) \subset S\}$.

An easy theorem is

THEOREM 2 $\hat{\alpha}$ and $\check{\alpha}$ are symplectic capacity functions satisfying $\check{\alpha} \leq \dot{\alpha}$. Moreover given any symplectic capacity function α we have

$$\check{\alpha} \leq \alpha \leq \hat{\alpha}.$$

So $\check{\alpha}$ is the minimal and $\dot{\alpha}$ the maximal symplectic capacity function.

We call a subset S of V a linear ellipsoid if there exists a quadratic form q, with $q(x) \geq 0$, $x \in V$, such that

$$S = \{x \in V \mid q(x) < 1\} =: S_q$$

The collection of all linear ellipsoids will be denoted by φ_{LE}. An ellipsoid is a set S such that

$$S = \psi(T)$$

for some $\psi \in D$ and $T \in \varphi_{LE}$. φ_E denotes the collection of all ellipsoids. Note that in "short hand"

$$\varphi_E = D \circ \varphi_{LE}$$

For $o < r_1 \leq r_2 \ldots \leq r_1$ define with $r = (r_1, \ldots r_n)$ $S(r) \in \varphi$ by

$$S(r) = \{x \in V \mid \sum_{i=1}^{n} \frac{1}{r_i^2}(x_i^2 + x_{i+n}^2) < 1\}.$$

It is a standard result in linear symplectic geometry, see [Ar2], that given any $S \in \varphi_{LE}$ which is bounded there exists r and $\psi \in Sp(V)$ such that

$$\psi(S) = S(r)$$

Using this fact one easily verifies that

$$\dot{\alpha}|\varphi_E = \check{\alpha}|\varphi_E$$

So there is only one symplectic capacity function on φ_E. This leads to a good open problem.

PROBLEM 1 What is the maximal class φ_M of φ such that $\hat{\alpha}|\varphi_M = \check{\alpha}|\varphi_M$? A result which we will obtain later suggest that if S is symplectomorphic to an open convex set, then $\hat{\alpha}(S) = \check{\alpha}(S)$.

Given a symplectic capacity function α we can study homeomorphisms in V preserving α. We start in the linear category

DEFINITION 2 By $A(V)$ we denote the subgroup of $GL(V)$ consisting of all maps ϕ such that

(2) $\alpha(\phi(S)) = \alpha(S)$

for every bounded $S \in \varphi_{LE}$.

Note that the definition does not depend on the choice of α since on φ_{LE} $\check{\alpha} = \dot{\alpha} = \alpha$.

The key linear result is

THEOREM 3 $A(V) = Sp(V) \cup \Gamma \circ Sp(V)$, where Γ is a linear isomorphism $(V,\omega) \to (V,-\omega)$.

For the proof which is basically straight forward but somewhat tedious we refer the reader to [E–H5].

A useful result which we need later on is

PROPOSITION 1 If ϕ: $V \to V$ is linear and $\alpha(\phi(S)) > 0$ for every nonempty bounded $S \in \varphi_{LE}$, then $\phi \in GL(V)$.

Next we move to the nonlinear category.

2. A LOCAL RIGIDITY THEOREM AND THE ELIASHBERG–GROMOV–THEOREM

Having a symplectic capacity function and the key linear result Theorem 3 we start with a technical result.

THEOREM 4 Assume (h_k): $B(1) \to V$ is a sequence of continuous maps converging uniformly to a continuous map h: $B(1) \to V$ which

is differentiable at 0 with derivative $h'(0)$. Suppose

$$\alpha(h_k(S)) = \alpha(S) \text{ , } k \in \mathbb{N}$$

for all $S \in \varphi_{LE}$, $S \subset B(1)$. Then $h'(0) \in A = A(V)$.

An easy corollary of Theorem 4 is the result we are aiming at

THEOREM 5 (Local symplectic rigidity) Assume (h_k): $B(1) \to V$ is a sequence of symplectic embeddings converging uniformly to a continuous map h: $B(1) \to V$ which is differentiable at 0 with derivative $h'(0)$. Then $h'(0) \in Sp(V)$.

Assuming Theorem 4 for the moment we know since the h_k are α-preserving that $h'(0) \in A$. If n is odd then Γ is orientation reversing. Since $h'(0)$ has to be orientation preserving we must have $h'(0) \in Sp(V)$. If n is even consider the symplectic vectorspace $(V \oplus \mathbb{R}^2, \omega_V \oplus \omega_{\mathbb{R}^2})$ which is symplectic isomorphic to $(\mathbb{R}^{2n+2}, \omega_{\mathbb{R}^{2n+2}})$, say by a map γ. Define $\tilde{h}_k$ by

$$\tilde{h}_k = \gamma \circ (h_k \times I_{\mathbb{R}^2}) \circ \gamma^{-1}$$

Then $\tilde{h}_k$ is a symplectic embedding and $\tilde{h}_k$ converges uniformly to $\tilde{h}$ defined by

$$\tilde{h} = \gamma \circ (h \times I_{h^{2^2}}) \circ \gamma^{-1}$$

Now $n + 1$ is odd and $\tilde{h}'(0) \in Sp(\mathbb{R}^{2n+2})$ by the previous argument. Since

$$\tilde{h}'(0) = \gamma \circ (h'(0) \times I_{\mathbb{R}^2}) \circ \gamma^{-1}$$

we see that $h'(0) \times I_{\mathbb{R}^2}$ is symplectic which is only possible if $h'(0) \in Sp(V)$.

Next we sketch a proof of Theorem 4.

Given $S \in \varphi_{LE}$, $S \subset B(1)$, and $\epsilon > 0$ we find $T \in \varphi_E$ such that

$$cl(h(S)) \subset T \ , \ \dot{\alpha}(T) \leq \dot{\alpha}(h(S)) + \epsilon$$

Since for k large enough we have $h_k(S) \subset T$ we see that

$$\begin{aligned} \dot{\alpha}(S) &= \alpha(S) \\ &= \alpha(h_k(S)) \\ &\leq \dot{\alpha}(h_k(S)) \\ &\leq \dot{\alpha}\,(T) \\ &\leq \dot{\alpha}\,(h(S)) + \epsilon. \end{aligned}$$

$\epsilon > 0$ being arbitrary implies

(1) $\dot{\alpha}(S) \leq \dot{\alpha}(h(S))$

For $t \in (0,1)$ using (1) we compute

(2) $$\begin{aligned} \dot{\alpha}(S) &= \dot{\alpha}(\tfrac{1}{t}(tS)) \\ &= \frac{1}{t^2}\,\dot{\alpha}(tS) \\ &\leq \frac{1}{t^2}\,\dot{\alpha}(h(tS)) \\ &= \dot{\alpha}(\tfrac{1}{t}h(tS)) \end{aligned}$$

If now $t \downarrow 0$ the maps $\frac{1}{t}h(t\cdot)$ converge uniformly to $h'(0)$. Using

the same arguments as above we see that

(3) $$\limsup_{t \downarrow 0} \ \dot{\alpha}(\tfrac{1}{t}h(tS)) \leq \dot{\alpha}(h'(0)S)$$

combining (2) and (3) gives

(4) $\dot{\alpha}(S) \leq \dot{\alpha}(h'(0)S)$

for every $S \in \varphi_{LE}$ with $S \subset B(1)$. By Proposition 1 this implies that $h'(0) \in GL(V)$. Next we have to show the reversed inequality in (4). Since h is differentiable at 0 we find a nondecreasing map $\epsilon\colon (0,1) \to (0,+\infty)$ such that $\epsilon(s) \to 0$ as $s \to 0$ and assuming $h(0) = 0$

$$|h(x) - h'(0)x| \leq \epsilon\,(|x|)\ |x|.$$

Given $\delta \in (0,1)$ we find $k(\delta)$ such that for $k \geq k(\delta)$

$$|h_k(x) - h'(0)x| \leq \epsilon(|x|)\ |x| + \delta.$$

Pick $\gamma > 0$. For $\tau \in (0,1)$ small enough we have $(1+\gamma)\tau < 1$ which we assume in the following. Take an arbitray $S \in \varphi_{LE}$ with $S \subset B(1)$. We shall show that

(5) $h_k((1+\gamma)\tau S) \supset h'(0)(\tau S)$

for k large enough and τ small enough. (5) will be a simple consequence of a Bouwer degree argument which we leave to the reader. Hence from (5) we infer

$$\begin{aligned} \dot{\alpha}(h'(0)(\tau S)) &= \alpha(h'(0)(\tau S)) \\ &\leq \alpha(h_k((1+\gamma)\tau S)) \end{aligned}$$

$$= \alpha((1+\gamma)\tau S)$$
$$\leq \dot{\alpha}\ ((1+\gamma)\tau S)$$
$$\leq (1+\gamma)^2\tau^2\ \dot{\alpha}(S)$$

Hence we obtain for $S \in \varphi_{LE}$, $S \subset B(1)$

(6) $\dot{\alpha}(h'(0)S) \leq \dot{\alpha}(S)$

Now combining (4) and (6) and using that $h'(0)$ is linear we must have

(7) $\dot{\alpha}(S) = \dot{\alpha}(h'(0)S)$ $\quad S \in \varphi_{LE}$ bounded.

This shows that $h'(0) \in A$.

Finally we give as a corollary the celebrated Eliashberg–Gromov result. The result had been announced in 1981 by Eliashberg, but had never been published. According to Gromov the result had been proved by Eliashberg using a very complicated combinatorial argument based on Poincare's proof of the last geometric theorem. In [Gr2] Gromov sketches a (difficult) proof based on the Nash–Moser implicit function theorem and Theorem 1. Because of its proof the result had been seen as a global phenomenon see [B] and [Gr3]. However the proof used here shows that it is indeed a very local result.

THEOREM 6 (Eliashberg–Gromov) Let (M,ω) be a symplectic manifold. Then the symplectic diffeomorphism group $D_\omega(M)$ is for the C^0–compact open topology closed in the diffeomorphism group Diff(M).

PROOF: Using Darbousk charts we can reduce the result to

Theorem 5.

3. RELATIONSHIP TO THE FIXED ENERGY PROBLEM

In this part we shall show that there is a relation to the fixed energy problem in Hamiltonian dynamics. This relation is not well understood today and a better understanding could be mutally beneficial for both fields.

Symplectic capacity functions give us an obstruction against embedding one set symplectically into another set. The obstruction is a number $\alpha(S)$ and naively one would guess the obstruction to be visible as a class of 2–dimensional subsets in S. Since, while embedding S into T the problems should arise at the boundary of S, the geometric obstruction set should have a trace on the boundary. The trace of a nice 2–dimensional set should be 1–dimensional and in nice cases it should be a loop on ∂S.

Assuming ∂S to be smooth, there is a class of nice loops on ∂S, namely periodic Hamiltonian trajectories. To be more precise we need some notation.

Given a connected compact smooth hypersurface Λ in V the normal bundle is trivial as a consequence of Alexander duality, [Sp]. Moreover $V\backslash\Lambda$ has a bounded and an unbounded component. We shall denote the bounded component by B_Λ. There exists a canonical one–dimensional distribution $\mathcal{L}_\Lambda \to \Lambda$, $\mathcal{L}_\Lambda \subset T\Lambda$, defined by

$$\mathcal{L}_\Lambda = \{(x,\xi) \in T\Lambda \mid \xi \perp_\omega T_x\Lambda\}$$

$\mathcal{L}_\Lambda \to \Lambda$ is called the characteristic distribution.

DEFINITION 3 A closed characteristic or periodic Hamilitonian trajectory on Λ is a submanifold P of Λ diffeomorphic to S^1 such that $TP = \mathcal{L}_\Lambda | P$. $\mathcal{P}(\Lambda)$ denotes the collection of all closed characteristic on Λ.

We need further the definition of contact type, [We3], and restricted contact type, [E–H4].

DEFINITION 4 A compact connected smooth hypersurface $\Lambda \subset V$ is said to be of contact type if there exists a 1–form λ on Λ such that $\lambda(x,\xi) \neq 0$ for $(x,\xi) \in \mathcal{L}_\Lambda$ and $\xi \neq 0$ so that $d\lambda = \omega/\Lambda$. If λ can be extended to V such that $d\lambda = \omega$ we say Λ is of restricted contact type. The collection of all contact type hypersurfaces in V will be denoted by ℓ, the subcollection of restricted contact type by ℓ_r. One has the following result

THEOREM 7 There exists a symplectic capacity function α such that for $\Lambda \in \ell_r$ there exists a positive integer k and a closed characteristic $P \in \mathcal{P}(\Lambda)$ with

(1) $\alpha(B_\Lambda) = k | \int \lambda | P |$

Moreover for every $S \in \varphi$ we have

$$\alpha(S) = \inf \{\alpha(U) | \ U \text{ open } \quad U \supset S\}$$

and for every nonempty open U

$$\alpha(U) = \sup \{\alpha(B_\Lambda) \ | \ B_\Lambda \subset U \ , \ \Lambda \in \ell_r\}.$$

Some ideas concerning the proof of Theorem 7 will be given in III. 4.

Here are some open problems.

PROBLEM 2 Can we pick $P \in \mathcal{P}(\Lambda)$ such that

$$\alpha(B_\Lambda) = | \int \lambda | P |$$

i.e can we take $k = 1$. If B_Λ is symplectomorphic to a convex domain this is actually possible.

PROBLEM 3 Assume $\Lambda \in \ell_r$ is homeomorphic to S^{2n-1}. Is it true that there exists $\psi \in D$ such that

$$\psi(B_\Lambda) \subset \dot{\Sigma}(\alpha(B_\Lambda)) \ ?$$

If yes, is it true that $\psi(\Lambda) \cap \partial\dot{\Sigma}(\alpha(B_\Lambda))$ contains a closed characterstics on $\psi(\Lambda)$?

III. THE FIXED ENERGY PROBLEM IN HAMILTONIAN DYNAMICS AND THE WEINSTEIN CONJECTURE.

As already mentioned in the introduction the break through in questions of existence of periodic solutions on a prescribed energy surface is due to Rabinowitz. Motivated by some further result by Rabinowitz, Weinstein formulated a conjecture which carries his name. A very recent break through in proving the Weinstein conjecture is due to C. Viterbo, [Vi]. Here we shall start with an almost existence phenomenon which had been detected by E. Zehnder and the author and which was motivated by Viterbo's Theorem, [H–Ze]. Moreover we shall survey strong extensions of these results to the cotangent bundle case [Ho–Vi] and the $P \times \mathbb{C}^{\ell}$–case [F–H–V], where P is a compact symplectic manifold. Then we indicate how one can refine the method to tackle some new kind of fixed point theorem, [E–H4]. Finally we give an idea how a symplectic capacity function based as the fixed energy problem can be constructed.

1. AN ALMOST EXISTENCE RESULT FOR PERIODIC SOLUTIONS

We use the notation as introduced in II.3.

DEFINITION 5 Given a connected compact smooth hypersurface Λ in V we call a diffeomorphism $\psi: (-1,1) \times \Lambda \to U$ onto an open bounded neighborhood U of Λ in V a parametrized family of compact hypersurfaces modeled on Λ provided $\psi(0,x) = x$ for every $x \in \Lambda$. We also put $\Lambda_\epsilon = \psi(\{\epsilon\} \times \Lambda)$ and we shall write (Λ_ϵ) or $(\Lambda_\epsilon)_{\epsilon\in(-1,1)}$ instead of ψ.

Denote by λ a 1–form on V such that $d\lambda = \omega$ and write

$a(P)$ for $|\int \lambda|P|$ where $P \in \mathcal{P}(\Lambda)$. We have the following surprising almost existence result for periodic solutions, see [H–Ze].

THEOREM 8 (Almost existence) Given $\mathcal{T} = (\Lambda_\epsilon)$ where $\Lambda = \Lambda_0$ is a compact connected hypersurface in V there exists a constant $d = d(\mathcal{T}) > 0$ such that for given $\delta \in (0,1)$ there exists an $|\epsilon| < \delta$ so that the following is true

$$\mathcal{P}(\Lambda_\epsilon) \neq \emptyset \text{ and there exists } P_\epsilon \in \mathcal{P}(\Lambda_\epsilon) \text{ such that } a(P_\epsilon) \leq d.$$

Theorem 8 has at least two interesting consequences. Assume first Λ is of contact type. As it was shown in [We3] there exists a vectorfield η defined on an open neighborhood of Λ in V which is transverse to Λ so that $L_\eta\omega = \omega$. Using the flow associated to η we can construct a $\psi: (-1,1) \times \Lambda \to U$ such that

$$(1) \quad T\psi_\epsilon(\mathcal{L}_\Lambda) = \mathcal{L}_{\Lambda_\epsilon}$$

where $\psi_\epsilon: \Lambda \to \Lambda_\epsilon : \psi_\epsilon(x) := \psi(\epsilon,x)$. Cleary (1) implies

$$(2) \quad \psi_\epsilon(\mathcal{P}(\Lambda)) = \mathcal{P}(\Lambda_\epsilon)$$

Since by Theorem 8 there is a $|\epsilon_0| < 1$ with $\mathcal{P}(\Lambda_{\epsilon_0}) \neq \emptyset$ we see that $\mathcal{P}(\Lambda) \neq \emptyset$. This proves

THEOREM 9 (Viterbo) If Λ is of contact type then $\mathcal{P}(\Lambda) \neq \emptyset$.

Theorem 9 is a generalization of the Weinstein conjecture.

DEFINITION 6 We say a compact smooth hypersurface Λ admits an a priori estimate if there is a family (Λ_ϵ) as in definition 5, $\Lambda_0 = \Lambda$, and a constant $c > 0$ such that

$$(3) \quad \ell(P) \le \frac{1}{c}\, a(P)$$

for every $P \in \bigcup_{\epsilon \in (-1,1)} \mathcal{P}(\Lambda_\epsilon)$. It is of course not assumed that $\bigcup \mathcal{P}(\Lambda_\epsilon) \neq \emptyset$. As a consequence of the almost existence result we have

THEOREM 10 ([H-Ze]) If Λ admits an a priori estimate then $\mathcal{P}(\Lambda) \neq \emptyset$.

This follows immediately from Theorem 8 over the Ascoli-Arzela-Theorem. Theorem 10 had been (vaguely) conjectured by P. Rabinowitz: a priori estimates imply existence. Defintion 6 makes precise what a priori estimate means.

We sketch now a proof of the almost existence result. The idea is to construct a Hamiltonian system $H\colon V \to \mathbb{R}$ such that

$$H^{-1}((0,b)) = \bigcup_{|\epsilon|<\delta} \Lambda_\epsilon$$

for a suitable large number b and

$$H^{-1}(c) = \Lambda_{\epsilon(c)} \qquad \text{for } c \in (0,b) \quad ,$$

so that the following holds: If $x\colon [0,1]/\{0,1\} \to V$ is a smooth 1-periodic loop satisfying

$$\dot{x} = X_H(x) \; ,$$

where X_H is the Hamiltonian vectorfield associated to H via $dH = \omega(X_H,\cdot)$ and if

$$\phi_H(x) = \frac{1}{2}\int_0^1 \langle -J\dot{x}, x\rangle - \int_0^1 H(x) \;\; > 0$$

then $x([0,1]) \subset \Lambda_\epsilon$ for some $|\epsilon| < \delta$. Clearly this implies Theorem 8 up to the a priori bound for the action $a(P)$. So the crucial point is the construction of the Hamiltonian. Fix $\delta \in (0,1)$ and denote by B the bounded component of $V \backslash \psi([-\delta,\delta] \times \Lambda)$ and by A the unbounded component. We may assume $0 \in B$. We define

$$(4) \quad \gamma = \operatorname{diam}(U)$$

and fix numbers r,b such that

$$(5) \quad \gamma < r < 2\gamma$$
$$\frac{3}{2}\pi^2 < b < 2\pi^2$$

Note that γ,r,b do not depend on the choice of δ. Next we pick smooth functions $j\colon (-1,1) \to \mathbb{R}$, $g\colon (0,\infty) \to \mathbb{R}$ satisfying

$$(6) \quad j|(-1,-\delta] = 0,\; j|[\delta, 1) = b,\; j'(s) > 0 \text{ for } -\delta < s < S$$
$$g(s) = b \text{ for } s \le r,\; g(s) = \frac{3}{2}\pi s^2 \;\; s \text{ large},$$
$$g(s) \ge \frac{3}{2}\pi s^2 \text{ for } s > r,\; 0 < g'(s) \le 3\pi s \text{ for } s > r.$$

Next we define a very special Hamiltonian $H \in C^\infty(V,\mathbb{R})$ by

$$(7) \quad H(x) = \begin{cases} 0 & \text{if } x \in B \\ j(\epsilon) & \text{if } x \in \Lambda_\epsilon,\; -\delta \le \epsilon \le \delta \\ b & \text{if } x \in A^\epsilon,\; |x| \le r \\ g(|x|) & \text{if } |x| > r \end{cases}$$

We note that

$$(8)\quad -b + \tfrac{3}{2}\pi\, |x|^2 \le H(x) \le b + \tfrac{3}{2}\pi\, |x|^2$$
for $x \in V$

If $x: [0,1]/\{0,1\} \to V$ is a smooth loop we define $\phi(x)$ by

$$(9)\quad \phi(x) = \tfrac{1}{2} \int_0^1 \langle -J\dot{x}, x\rangle\, dt - \int_0^1 H(x(t))dt$$

We have the following crucial observation

LEMMA 1 Assume x is a 1-periodic solution of the Hamiltonian system $\dot{x} = X_H(x)$ satisfying $\phi(x) > 0$. Then $P := x([0,1])$ is a closed characteristic in some $\mathfrak{P}(\Lambda_\epsilon)$ with $|\epsilon| < \delta$.

The proof is simple. If x is constant we have $\phi(x) \le 0$. If x is nonconstant and $|x(t)| \ge r$ for some t we have $|x(t)| = |x(0)|$. Then x satisfies

$$-J\dot{x} = \frac{g'(|x|)}{|x|}\, x$$

With $v = |x(0)|$ we compute

$$\begin{aligned}\phi(x) &= \tfrac{1}{2} g'(v)v - g(v)\\ &\le \tfrac{3}{2}\pi\, v^2 - g(v)\\ &\le \tfrac{3}{2}\pi\, v^2 - \tfrac{3}{2}\pi\, v^2\\ &= 0\end{aligned}$$

Hence we can only have $x([0,1]) \subset \Lambda_\epsilon$ for some $\epsilon \in (-\delta,\delta)$, which immediately implies our assertion.

Now 1-periodic solutions of $\dot{x} = X_H(x)$ can be found as critical points of ϕ on a suitable Hilbertspace of loops. In view of Lemma

1 it is enough to find a critical point x of ϕ satisfying $0 < \phi(x) \le b$. In fact if $P = x([0,1])$ we find since $P \in \mathfrak{P}(\Lambda_\epsilon)$ for some $|\epsilon| < \delta$

$$\begin{aligned}a(P) &\le \tfrac{1}{2} \int_0^1 \langle -J\dot{x}, x\rangle dt\\ &= \phi(x) + \int_0^1 H(x)dt\\ &\le b + b\\ &= 2b\end{aligned}$$

Define $d(\mathfrak{J}) := 16\pi\, \text{diam}\, (U)^2$ $(> 2b)$.

In order to find critical points for ϕ we take a variational set up as in [Be-Ra2]. Denote by $E = H^{\frac{1}{2}}(S',V)$ the Hilbertspace of all functions $x \in L^2(o,1\,;V)$ satisfying

$$x(t) = \sum_{k\in\mathbb{Z}} \exp(2\pi k t J)x_k \quad , \quad x_k \in V$$
$$\sum_{k\in\mathbb{Z}} |k|\, |x_k|^2 < \infty$$

As an inner product we take

$$(x,y) = (2\pi \sum |k| \langle x_k,y_k\rangle) + \langle x_o,y_o\rangle$$

and define $\|x\| = (x,x)^{\frac{1}{2}}$. We have an orthogonal decomposition $E = E^- \oplus E^0 \oplus E^+$, where E^-, E^0, E^+ correspond to the subspaces $k < 0$, $k = 0$, and $k > 0$. We denote by P^-, P^0 and P^+ the corresponding orthogonal projections. the functional ϕ previously defined for smooth loops extends to a smooth functional on E and is given by

$$(10)\quad \phi(x) = \tfrac{1}{2}((-P^- + P^+)x,x) - \int_0^1 H(x(t))dt$$

Using the compact embedding $E \subset L^p$, $1 \le p < \infty$, we see that the gradient $\phi'\colon E \to E$ of ϕ is a compact map, which has linear growth since H is quadratic outside a big ball. We have

$$(11) \quad \phi(0) = 0,\ \phi'(0) = 0 \quad \text{and} \quad \phi''(0) = -P^- + P^+$$

An easy consequence of (11) is

LEMMA 2 There exist $\alpha \in (0,1)$ and $\beta > 0$ such that

$$\phi|\Gamma \ge \beta \quad \text{where} \quad \Gamma = \{x \in E^+ \mid \|x\| = \alpha\}.$$

Define $e(t) = \frac{1}{\sqrt{2\pi}} \exp(2\pi t J)(1,0,\dots,0)$ and put $\dot{E} = E^- \oplus E^0 \oplus \mathbb{R}e$. We define a bounded subset Σ of E by

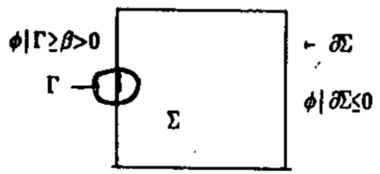
$$\Sigma = \{x = x^- + x^0 + se \mid \|x^- + x^0\| \le \tau \ ,\ s \in [0,\tau]\}.$$

where $\tau > 1$ will be specified later.

LEMMA 3 Denote by $\partial\Sigma$ the boundary of Σ in $\dot{E}$. If $\tau > 1$ is sufficiently large we have $\phi|\partial\Sigma \le 0$.

This is an immediate consequence of the properties of H.

What we have established so far is the existence of what is called a linking, see [Be–Ra2], [H1]. Here we have the following picture



Now we solve the ordinary differential equation $x' = -\phi'(x)$ in

order to obtain a global flow $\mathbb{R} \times E \to E$: $(t,x) \to x \cdot t$. Clearly

$$\frac{d}{dt}\phi(x\cdot t) = -\|\phi'(x\cdot t)\|^2 \le 0$$

Since $t \to \phi(x\cdot t)$ is a nonincreasing map we see that

$$(\partial\Sigma) \cdot t \cap \Gamma = \emptyset \quad \text{for} \quad t \ge 0$$

Our aim is to show that

$$\Sigma \cdot t \cap \Gamma \ne \emptyset \quad \text{for} \quad t \ge 0$$

So we need

LEMMA 4 $\Sigma \cdot t \cap \Gamma \ne \emptyset$ for $t \ge 0$.

The statement is equivalent to the solvability of

$$(12) \quad \underset{t \ge 0\ ,\ x \in \Sigma}{(P^- + P^0)(x\cdot t) = 0}, \quad \|x\cdot t\| = \alpha$$

Applying the variation of constant formula to $x' = -\phi'(x)$ we can write

$$x \cdot t = e^t x^- + x^0 + e^{-t} x^+ + B(t,x)$$

where the map $B\colon \mathbb{R} \times E \to E$ is smooth and maps bounded sets into precompact sets. Hence (12) is equivalent to

$$(13) \quad \begin{aligned} 0 &= x^- + x^0 + (e^{-t}P^- + P^0)\, B(t, x^- + x^0 + se) \\ 0 &= se - (s + \alpha - \|(x^- + x^0 + se) \cdot t\|)e \end{aligned}$$

Clearly (13) is of the form

(14) $0 = z + T_t(z) \quad z \in \Sigma$

We know already that $0 \neq z + T_t(z)$ for $z \in \partial\Sigma$ and $t \geq 0$. Since T is compact we can apply Leray-Schauder-Degree-Theory to find with $\dot{\Sigma}$ = interior of Σ in E

$$\begin{aligned} & \deg(I + T_1, \dot{\Sigma}, 0) \\ &= \deg(I - \alpha e, \dot{\Sigma}, 0) \\ &= \deg(I, \dot{\Sigma}, \alpha e) \\ &= 1 \end{aligned}$$

since $\alpha e \in \dot{\Sigma}$

LEMMA 5 $c = \lim\limits_{t\to\infty} \sup\limits_{x\in\Sigma} \phi(x \cdot t)$ exists.

Moreover $+\infty > c \geq \beta > 0$ and c is a critical level of ϕ.

PROOF The asymptotic behavior of H implies immediately that ϕ satisfies the Palais-Smale condition, see [Am-Ze], [H-Ze]. Since $\phi(\Sigma)$ is bounded in $\mathbb{R}$ we have $c < +\infty$. Since the map $t \to \sup\limits_{x\in\Sigma} \phi(x \cdot t)$ is nonincreasing c exists.

From Lemma 4 we infer that $\Sigma \cdot t \cap \Gamma \neq \emptyset$ for $t \geq 0$ which implies

$$\sup \phi(\Sigma \cdot t) \geq \inf \phi(\Gamma) \geq \beta > 0$$

Hence $0 < c < +\infty$. Assuming that c is not a critical level we find using the PS-condition that

$$\|\phi'(x)\| \geq \epsilon \quad \text{if} \quad \phi(x) \in [c-\epsilon, c+\epsilon]$$

for some $\epsilon > 0$. So there exists $T > 0$ such that $\phi^{c+\epsilon} \cdot T \subset \phi^{c-\epsilon}$, where $\phi^r = \phi^{-1}((-\infty, r])$. If $t > 0$ is large enough we have by the definition of c

$$\Sigma \cdot t \subset \phi^{c+\epsilon}$$

Hence

$$\begin{aligned} \Sigma \cdot (t+T) &= (\Sigma \cdot t) \cdot T \\ &\subset \phi^{c+\epsilon} \cdot T \\ &\subset \phi^{c-\epsilon} \end{aligned}$$

showing that $c \leq c - \epsilon$ which is a contradiction.

2. EXTENSIONS TO MORE GENERAL SPACES AND REMARKS

Let us first formulate the general Weinstein conjecture. If we replace (V,ω) by a symplectic manifold (M,ω) and Λ is a compact smooth hypersurface in M we can define again contact type.

CONJECTURE (Weinstein) If Λ is a smooth compact hypersurface of contact type in (M,ω) then $\mathcal{P}(\Lambda) \neq \emptyset$.

Actually Weinstein also assumed that $H^1(\Lambda; \mathbb{R}) = 0$. This hypothesis seems however not to be crucial.

Moreover we can ask when does an almost existence result hold, which is a more general concept then the Weinstein conjecture.

First we consider the contangent bundle case. Let $M = T^*N$ be the cotangent bundle of a compact connected smooth manifold of dimension at least two. Given a connected compact hypersurface $\Lambda \subset M$ we say Λ encloses the zero section if Λ does not intersect the zero section $N \subset T^*N$ and $M \setminus \Lambda$ has exactly two components, so that the bounded component contains N. M is equipped with the standard symplectic form $d\lambda$, where λ = "pdq". Again we can define a parametrized family. We have

THEOREM 11 ([H–Vi]) Assume N is a compact connected smooth manifold of dimensions at least two. Let $\mathfrak{F} = (\Lambda_\epsilon)$ be a parmetrized family modelled on a compact connected hypersurface Λ enclosing the zero section. Then there is a constant $d(\mathfrak{F}) > 0$ such that for every $\delta \in (0,1)$ there is a $|\epsilon| < \delta$ with $\mathcal{P}(\Lambda_\epsilon) \neq \emptyset$ containing P_ϵ and

$$0 < | \int \lambda | P_\epsilon | \leq d$$

In particular the Weinstein conjecture holds and the existence mechanism based on a priori estimates.

Another result is the following

THEOREM 12 ([F–H–V]) We have an almost existence result for families (Λ_ϵ) modelled on a compact connected smooth hypersurface $\Lambda \subset P \times \mathbb{C}^\ell (\ell \geq 1)$, with trivial normal bundle. Here $P \times \mathbb{C}^\ell$ is equipped with the symplectic structure $\omega_P \oplus \omega_{\mathbb{C}^\ell}$ and (P, ω_P) is a compact symplectic manifold so that ω_P vanishes on $\pi_2(P)$. $\mathbb{C}^\ell \simeq \mathbb{R}^{2\ell}$ is equipped with the standard structure.

The proof of Theorem 11 is very technical and based on methods developed in [H2] in the proof of one of the Arnold conjectures.

Theorem 12 is proved with the aid of first order elliptic systems. This is conceptual much simpler though the technical difficulties are still nontrivial. A proof of Theorem 11 using the machinery of Theorem 12 should be possible.

There is obviously one open problem:

PROBLEM 4 Does there exist a smooth compact hypersurface Λ in V with $\mathcal{P}(\Lambda) = \emptyset$?

3. GENERALIZED FIXED POINT PROBLEMS

An interesting question in the global perturbation theory of Hamiltonian systems is the following. Consider an autonomous Hamiltonian system

$$\text{(US)} \quad -J\dot{x} = H'(x)$$

having $\Lambda = H^{-1}(1)$ as a compact regular energy surface. We call (US) the unperturbed system. Assume now (US) is perturbed between time zero and time one by a nonautonomous Hamiltonian system giving the perturbed equation

$$\text{(PS)} \quad -J\dot{x} = H'(x) + \dot{\epsilon}_t'(x)$$

where $\dot{\epsilon}$ has compact support and $\text{supp}(\dot{\epsilon}) \subset [0,1] \times V$. We are interested in solutions of (PS) which agree before time zero and after time one with the unperturbed movement up to a phase shift. More precisely do there exist solutions x of (US) and y of (PS) and a number $\delta \in \mathbb{R}$ such that

$$H(x(t)) = 1 \quad \text{for all} \quad t \in \mathbb{R}$$

$$y(t) = x(t) \quad \text{for all} \quad t \le 0$$
$$y(t) = x(t+\delta) \quad \text{for} \quad t \ge 1$$

If $\mathcal{L}_\Lambda \to \Lambda$, $\Lambda = H^{-1}(1)$, is the characteristic line bundle we denote by $L_\Lambda(x)$ for $x \in \Lambda$ the leaf through x. If ψ is the time-one-map of (PS) (we assume it exists) then the problem described is equivalent to the following generalized fixed point problem

$$\psi(x) \in L_\Lambda(x) \ , \quad x \in \Lambda$$

Here we give two results which can be proved by a tricky extension of the method first described. The details can be found in [E–H4]. See also [Mo1] for local results in this direction.

We denote by G the group $\mathbb{Z}_2$ acting in the usual way by $x \to -x$ on V.

THEOREM 12 Assume Λ is a compact connected smooth hypersurface which is G–invariant and of restricted contact type. Let ψ be a G–equivariant symplectic diffeomorphism in V. Then there exists $x \in \Lambda$ with $\psi(x) \in L_\Lambda(x)$.

What happens if we drop the group action. As the following Theorem shows the situation is subtle and far from being understood.

THEOREM 13 Assume Λ is a compact connected smooth hypersurface of restricted contact type in V enclosing 0. Let $t \to \psi_t$ be an isotopy of the identity in D_ω fixing 0 and let $H: [0,1] \times V \to \mathbb{R}$ be the Hamiltonian generating $t \to \psi_t$ and

normalized by $H(t,0) = 0$ for all $t \in [0,1]$. We assume

$$H(t,x) - \tfrac{1}{2}\langle H''(t,0)\, x,x\rangle \ge 0$$

for every $t \in [0,1]$ and x enclosed by Λ

and

$$\tfrac{1}{2} \int_0^1 \langle -J\dot{x},x\rangle dt - \int_0^1 H(t,x(t))dt \le 0$$

for every fixed point x_0 of ψ_1 which is enclosed by Λ where $x(t) = \psi_t(x_0)$. Then there exists $x \in \Lambda$ satisfying $\psi_1(x) \in L_\Lambda(x)$.

If one relaxes the condition somewhat one can have nonexistence as an example in [E–H4] shows. Theorem 13 implies generalization of results due to F. Clarke [Cl1] and J. Moser [Mo1]. It is not known how good Theorem 13 is, i.e. how sharp are the hypotheses?

4. A RELATED SYMPLECTIC CAPACITY FUNCTION

We sketch now a proof of the existence of a particular symplectic capacity function as given in II.3 Theorem 7. Prompted by the proof of Lemma 4 in III. 1 we introduce a subgroup β of the homeomorphism group of $E = H^{\frac{1}{2}}(S^1; V)$. Namely

(1) $h \in \beta \iff h: E \to E$ is a homeomorphism such that
$$h(x) = e^{\gamma^+(x)}x^+ + x^0 + e^{\gamma^-(x)}x^- + K(x)$$

where $\gamma^+, \gamma^-: E \to \mathbb{R}$ are continuous and map bounded sets into bounded sets. Moreover K is continous and maps bounded sets into precompact sets. Further

(2) $K(x) = 0,\ \gamma^+(x) = \gamma^-(x) = 0$ if $\|x\|$ large
or $\int_0^1 \langle -J\dot{x},x\rangle dt \leq 0$

That β is a subgroup of homeo (E) is not difficult to verify.

Given $\Lambda \in \ell_r$, i.e. Λ is of restricted contact type, we denote by $\mathcal{F}(\Lambda)$ the set of all $H: V \to [0,+\infty)$ such that

(3) $H(x) = 0$ for all x in an open neighborhood of $B_\Lambda \cup \Lambda$.

and

(4) $H(x) = k|x|^2$ for $|x|$ large for some $k \in (0,+\infty)$

Finally we define for $\Lambda \in \ell_r$

(5) $$\tau(\Lambda) = \inf_{H\in\mathcal{F}(\Lambda)} \sup_{h\in\beta} \inf_{x\in S^+} \phi_H(h(x))$$

where S^+ is the unitsphere in E^+ and ϕ_H is defined by

$$\phi_H(x) = \frac{1}{2}\,((-P^- + P^+)x,x) - \int_0^1 H(x(t))dt$$

Then we put

$$\alpha(B_\Lambda) = \tau(\Lambda) \quad \text{for } \Lambda \in \ell$$

and for $U \neq \emptyset$ open

$$\alpha(U) = \sup\{\alpha(B_\Lambda) \mid B_\Lambda \subset U,\ \Lambda \in \ell_r\}$$

and finally for $S \in \wp$

$$\alpha(S) = \inf\{\alpha(U) \mid U \supset S,\ U \text{ open}\}$$

One verifies then (which is quite technical) that α is a symplectic capacity function having the properties stated in Theorem 7.

IV. THE ARNOLD CONJECTURES

In the introduction we have already described some of the history of the Arnold conjectures. In the first section we describe the problems and introduce the necessary notation. We also like to suggest the reading of [Ar1].

1. SYMPLECTIC FIXED POINT THEORY AND LAGRANGIAN INTERSECTION THEORY

In the following we assume that (M,ω) is a compact symplectic manifold. We are interested in studying fixed points for symplectic diffeomorphisms on M. One cannot expect in general that every symplectic map has a fixed point. Just look at the map on $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ induced by $x \to x + (\frac{1}{2},0,0,\dots,0)$. However maps generated by a time dependent Hamiltonian vectorfield are good candidates. We say $\psi \in \mathcal{D}_\omega(M)$ is an exact symplectic diffeomorphism iff there exists a Hamiltonian H so that Ψ is the time-1-map for the flow associated to the Hamiltonian vectorfield

$\dot{x} = X_{H_t}(x),\ dH_t = \omega(X_{H_t},\cdot)$.

Arnold conjectured in [Ar1] certain lower bounds for the number of fixed points of an exaxt symplectic map. A more general concept than fixed point theory is Lagrangian intersection theory. A submanifold L of M is called Lagrangian if $2\dim L = M$ and $\omega/L = 0$. Denote by $M \times M$ the symplectic manifold with symplectic form $\omega \oplus (-\omega)$. One easily verifies that the graph of a symplectic map ψ, say graph (ψ) is a compact Lagrangian submanifold, also $\mathrm{diag}(M\times M)$, the diagonal is Lagrangian. Clearly

we have a natural bijection.

$$\mathrm{Fix}(\psi) \xrightarrow{\sim} \mathrm{graph}(\psi) \cap \mathrm{diag}(M\times M)$$

So the study of the fixed point problem is equivalent to the intersection problem. There are of course intersection problems which cannot be formulated as a fixed point problem.

The result whose proof we are going to sketch is the following, see [F7], [H3].

THEOREM 14 Let (M,ω) be a compact symplectic manifold. Assume L is a compact Lagrangian submanifold such that $\omega/\pi_2(M,L) = 0$. Assume L' is an exact deformation of L. Then $\#(L\cap L') \geq c(L)$, where $c(L)$ is the cohomological $\mathbb{Z}_2$-category which is defined below.

Here L' is an exact deformation of L iff $L' = \psi(L)$ for an exact symplectic diffeomorphism on M.

DEFINITION 7 The cohomological $\mathbb{Z}_2$-category is the smallest number k so that there exists an open covering U_1, U_k of L so that the inclusion maps $\epsilon_i: U_i \to (L,\{x_0\})$ for some fixed $x_0 \in L$ induce the zero maps $\check{\epsilon}_i: \check{H}(L,\{x_0\})_i\ \mathbb{Z}_2) \to \check{H}(U_i)\ \mathbb{Z}_2)$ in Cech cohomology with coefficients in $\mathbb{Z}_2$.

A trivial consequence, as already sketched, is the following Theorem.

THEOREM 15 Let (M,ω) be a compact symplectic manifold such that $\omega|\pi_2(M) = 0$. Then an exact symplectic diffeomorphism on M

has at least $c(M)$ fixed points.

It is possible to replace $c(M)$ by the cohomological category where we maximize the number k in the definition 7 over all commutative rings R which we take as coefficients. Such a generalization is not possible for Theorem 14.

We sketch now a proof of Theorem 14. We can construct an almost complex structure J on M, i.e. $J(x): T_xM \to T_xM$. $J(x)^2 = -Id$ such that $g = \omega \circ (J\times I)$ is a Riemannian metric. We call such an almost complex structure positive (better ω-positive). Denote by Z the strip $\{s + it \mid s \in \mathbb{R}, t \in [0,1]\}$ in $\mathbb{C}$. We shall study the first order elliptic system on Z given by

$$(1)\quad \begin{aligned} &u_s + J(u)u_t = 0, \quad u: Z \to M \\ &u(\mathbb{R}) \subset L, \quad u(i+\mathbb{R}) \subset L' \\ &\int_Z (|u_s|^2 + |u_t|^2)dsdt < \infty \end{aligned}$$

A solution of (1) is automatically smooth by standard elliptic regularity theory. Denote by $\Omega = \Omega_J(L,L')$ the set of solutions of (1). We equip Ω with the compact-open C^∞-topology induced from $C^\infty(Z,M)$ and which is of course metrizable, say d is a metric. We have a continuous map $\Omega \xrightarrow{\pi} L: u \to u(o)$. The key step in proving Theorem 14 is the following

THEOREM 16 Under the hypothesis of Theorem 14 (Ω,d) is a compact metric space. Moreover $\check{\pi}: \check{H}(L) \to \check{H}(\Omega)$ is injective.

Next observe that we have a continous flow on Ω defined by

$$\mathbb{R} \times \Omega \to \Omega: (\tau,u) \to u\cdot\tau \quad , (u,t)(s+it) = u(s-\tau+it)$$

Moreover it is not difficult to show that there exists a strict Liapunov function $\alpha: \Omega \to \mathbb{R}$, i.e. $\tau \to \alpha(u\cdot\tau)$ is strictly decreasing unless u is a fixed point for the flow. Clearly u can only be fixed point for the flow if $u(s+it) = \text{const} \in L \cap L'$ since $\int |u_s|^2 + |u_t|^2 < \infty$. In fact one has

$$\alpha(u\cdot\tau) - \alpha(u) = -\frac{1}{2}\int_{[-\tau,o]+i[0,1]} (|u_s|^2 + |u_t|^2)dsdt$$

Summing up we have a gradient like flow on a compact metric space. Now using methods as in [Co-Ze2] one sees easily that # Rest points of "$\cdot$" $\geq c(L)$. Since Rest points of "$\cdot$" $= L \cap L'$ the proof of Theorem 14 is complete.

2. GROMOV'S THEORY OF ALMOST HOLOMORPHIC CURVES

As we have seen the differential equation $u_s + J(u)\,u_t = 0$ plays an important role in the study of Lagrangian intersection problems. Note that this differential operator is basically a nonlinear Cauchy-Riemann operator. We shall therefore write

$$\bar{\partial}u := u_s + J(u)\,u_t$$

It was Gromov who pointed out in his seminal paper [Gr1] the importance of almost holomorphic curves in symplectic geometry. Gromov's method was non variational. Floer however was able to combine Gromov's ideas with the variational calculus, more precisely a homological Conley index.

There are two important analytical facts concerning certain maps u: $B(\epsilon) \to M$, where $B(\epsilon)$ is the ϵ-ball around 0 in $\mathbb{C}$. Using a theorem due to Gromov and Rohlin, see [Gr2] we may assume that (M,g) is isometrically embedded in some large $(\mathbb{R}^N,<\cdot,\cdot>)$.

With other words we may assume that we are given a compact submanifold M of some $\mathbb{R}^N$. Moreover M is equipped with an almost complex structure J so that $\omega = <+J\cdot,\cdot>$ is a symplectic form on M.

Denote by $H^j(G,\mathbb{R}^N)$; $j = 0,1,2,$ the standard Sobolev spaces. By $H^j(G,M)$ we denote the subset consisting of these maps in $H^j(G,\mathbb{R}^N)$ which have their image in M. If $j \geq 2$ $H^j(G,\mathbb{R}^N)$ can be actually seen as a submanifold of the Hilbertspace $H^j(G,\mathbb{R}^N)$ provided G is bounded. We give now two important facts.

FACT 1 Let (u_k) be a sequence of maps in $H^j(B(\epsilon), M)$, $j \geq 2$, such that $\|\bar\partial u_k\|_{j-1,B(\epsilon)} \to 0$ as $k \to \infty$. Assume $\|u_k\|_{j,B(\epsilon)} \leq c$ for some constant $c > 0$. There exists a $\delta \in (0,\epsilon)$ such that $(u_k \mid B(\delta))$ is precompact in $H^j(B(\delta), M)$.

Fact 1 is a consequence of basic linear elliptic estimates for $\bar\partial$.

The difficulty to apply Fact 1 is that one needs the a priori bound on $\|u_k\|_{j,B(\epsilon)}$. Here fact 2 is useful.

FACT 2 Let (u_k) be a sequence of maps in $H^j(B(\epsilon), M)$ such that $\|\bar\partial u_k\|_{j-1,B(\epsilon)} \to 0$ as $k \to \infty$. Assume $\|u_k\|_{1,B(\epsilon)} \leq c$ for some $c > 0$ and all k. If $\omega|\pi_2(M) = 0$ there exists $\delta \in (0,\epsilon)$ such that $(u_k|B(\delta))$ is precompact in $H^j(B(\delta), M)$.

Similarly one has the above estimates if $B(\epsilon)$ denotes the ball around 0 in $\mathbb{C}^+ = \{z \in \mathbb{C} \mid im(z) \geq 0\}$ and u_k maps the

(boundary of $B(\epsilon)) \cap \partial\mathbb{C}^+$ into a compact Lagrangian submanifold, see [H3].

In fact 2 the assumption $\omega|\pi_2(M) = 0$ comes in as follows. If the a priori estimate does not hold we are able to use the conformal invariance of the operator $\bar\partial$ to construct a map $u: \mathbb{C} \to M$ $u \neq$ constant, $\int |u_s|^2 + |u_t|^2 < \infty$. One can apply a removable singularly theorem [P] to obtain an almost holomorphic map $\bar{u}: S^2 \to M$. Since $\bar{u}$ is non constant we have

$$\int_{S^2} \bar{u}^* \omega = \frac{1}{2} \int_{\mathbb{C}} |u_s|^2 + |u_t|^2 > 0$$

On the other hand $\omega|\pi_2(M) = 0$ so that $\int_{S^2} \bar{u}^* \omega = 0$ giving a contradiction. So we obtain a contradiction since bubbling off of a holomorphic sphere does not occur if $\omega|\pi_2(M) = 0$.

Note that Fact 1 and Fact 2 with the remark thereafter imply immediately the first part of Theorem 16 if one can show the existence of a constant $c_1 > 0$ such that every $u \in \Omega$ satifies

$$\frac{1}{2} \int_Z (|u_s|^2 + |u_t|^2) \leq C_1$$

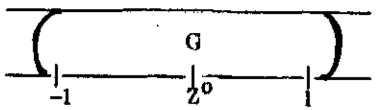
or equivalently

$$\int_Z u^* \omega \leq C_1$$

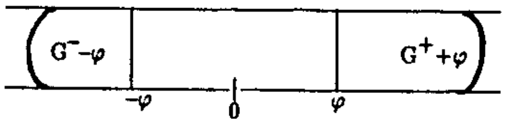
This is however purely topological and follows from the calculus of differential forms, see [H3], using that L' is an exact deformation of L.

3. LUSTERNIK-SCHNIRELMAN-THEORY AND ALMOST HOLOMORPHIC DISKS

In order to be able to obtain a Lusternik-Schnirelman-Theory we have to show that the map Υ induces an injection in Čech-Cohomology. In order to do so we use the continuity property of Čech cohomology together with an approximation result derived in [H3]. We choose a compact domain with smooth boundary ∂G in $\mathbb{C}$ such that $G \subset Z$ and $\partial G \cap \partial Z \supset [-1,1] + i\{0,1\}$. We also assume that G is convex. So G looks like



Now we stretch G to obtain domains G_φ for $\varphi > 0$ as follows. We cut G into two halfs G^-, G^+ along the axis $i\mathbb{R}$ and shift G^- to the left and G^+ to the right so that we can insert a piece $(-\varphi,\varphi) + i[0,1]$.



Now using that L' is an exact deformation of L, say there is an exact isotopy ψ_t of the identity in $D_\omega(M)$ such that $\psi_1(L) = L'$ we define a map $\partial G_\varphi \to$ Lagrangian submanifolds of M by $x \to \psi_{\beta(x)}(L)$ having the following property. For simplicity write L_x

instead of $\psi_{\beta(x)}(L)$

$$L_x = \begin{cases} L & x \in [-\varphi-1,\ \varphi+1] \\ L' & x \in [-\varphi-1,\ \varphi+1] + i \\ \text{interpolation between } L \text{ and } L' \text{ for the remaining } x. & \end{cases}$$

If we enlarge φ we can take the same interpolation between L and L' and just keep $L_x = L$ or $L_x = L'$ on a corresponding longer interval. We study now the elliptic boundary value problem

$$(P_\varphi) \quad \begin{aligned} &\bar\partial u = 0 \quad \text{on } G_\varphi \\ &u(x) \in L_x \quad \text{for } x \in \partial G_\varphi \end{aligned}$$

It is not difficult to see that we have a uniform H^1-bound for topological reasons independent of $\varphi > 0$ (just use some calculus with differential forms.) Using the estimates in section 2 one gets uniform C^∞-bounds on every compact subset K of Z for the restriction of the solution of (P_φ) (φ large enough) to K. In view of the Ascoli-Arzela-Theorem this is of course a strong compactness statement. Let $\gamma: \mathbb{R} \to [0,1]$ be a smooth map such that $\gamma(s) = 1$ for $s \leq 1$ and $\gamma'(s) < 0$ for $1 < s < 2$ and $\gamma(s) = 0$ for $s \geq 2$. For u a solution of (P_φ) we define a new map $r_\varphi(u): Z \to M$ by

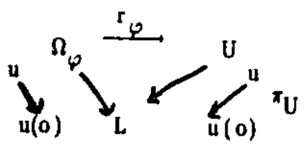
$$r_\varphi(u)(s + it) = u(\gamma(\tfrac{2|s|}{\varphi})\, s + it)$$

provided φ is large enough. It is an easy exercise that r_φ defines a continuous map

$$\{u \in C^\infty(G_\varphi,M) \mid u(x) \in L_x \ x \in \partial G_\varphi\} \to \{u \in C^\infty(Z,M) \mid u(\mathbb{R}) \subset L, \ u(\mathbb{R}+i) \subset L'\}$$

Denote by Ω_φ the solution set of (P_φ). What we said about a priori estimates concerning (P_φ) immediately implies the following approximation result.

THEOREM 17 Under the hypothesis of Theorem 14 we find for a given open neighborhood U of $\Omega(L,L')$ in $\{u \in C^\infty(Z,M) \mid u(\mathbb{R}) \subset L, u(\mathbb{R} + i) \subset L'\}$ a number $\varphi_0 > 0$ such that for every $\varphi \geq \varphi_0$ we have $r_\varphi(\Omega_\varphi) \subset U$. In particular we have the commutative diagram



Assume now we can show for every $\varphi > 0$ that the map π_φ defined by

$$\pi_\varphi\colon \Omega_\varphi \to L: \quad u \to u(o)$$

induces an injective map in Cech cohomology. Then we see since

$$\check{r}_\varphi \check{\pi}_U = \check{\pi}_\varphi$$

that $\check{\pi}_U$ is injective as well. This implies since $\Omega(L,L')$ is a compact subset of a metric space by the continuity property of Čech cohomology that $\check{\pi}\colon \check{H}(L) \to \check{H}(\Omega(L,L'))$ is injective. In view of this remark we have to study (P_φ) and show that $\pi_\varphi\colon \Omega_\varphi \to L$ induces an injection in cohomology. The advantage of (P_φ) in contrast to the elliptic problem defining $\Omega(L,L')$ is that (P_φ) is a problem on a bounded domain. We freeze now φ and consider the problem

$$(P) \quad \bar{\partial} u = 0 \text{ on } G_\varphi$$
$$u(x) \in L_x \text{ for } x \in \partial G_\varphi$$

We embed the problem (P) into a one parameter family of problems as follows: Recall that $L_x = \psi_{\beta(x)}(L)$ for a suitable map $\beta\colon \partial G_\varphi \to [0,1]$. Define for $\tau \in [0,1]$ a new family $x \to L_x^\tau$ by

$$L_x^\tau = \psi_{(\tau\beta(x))}(L)$$

Then $L_x^1 = L_x$ and $L_x^0 = L$ for every $x \in \partial G_\varphi$.

We consider now the family of nonlinear elliptic boundary value problems.

$$(P^\tau) \quad \bar{\partial} u = o \text{ on } G_\varphi$$
$$u(x) \in L_x^\tau \text{ for } x \in \partial G_\varphi$$

For $\tau = 0$ we in fact know all the solutions, namely all constant maps with image in L. Moreover our a priori estimates imply that we can consider (P^τ) as a problem for a family of proper Fredholm operators. Define a Hilbertmanifold Λ^j for $j \geq 2$ by

$$\Lambda^j = \{u \in H^j(G_\varphi,M) \mid u(\partial G_\varphi) \subset L\}$$

Moreover define for $\tau \in [0,1]$

$$\Lambda_\tau^j = \{u \in H^j(G_\varphi,M) \mid u(x) \in L_x^\tau \text{ for } x \in \partial G_\varphi\}.$$

We define a Hilbertspace bundle $E^{j-1} \to H^j(G_\varphi, M)$ by

$$E_u^{j-1} = H^{j-1}(u^* TM)$$

where u^*TM is the pullback of the tangent bundle via $u: G_\varphi \to M$. By Kuipers theorem, [Ku], there exists a trivialization

$$E^{j-1} \xrightarrow{\Theta} H^j(G_\varphi, M) \times H$$

where H is an (abstract) separable Hilbertspace. We can construct a smooth family of diffeomorphism of M, say $\phi_{\tau,x}: M \to M$, $x \in G_\varphi$ such that $\Gamma_\tau: \Lambda^j \to H^j(G_\varphi, M)$ defined by

$$\Gamma_\tau(u)(x) = \phi_{\tau,x}(u(x))$$

induces a diffeomorphism $\Lambda^j \xrightarrow{\sim} \Lambda^j_\tau$ still denoted by Γ_τ. We consider now the 1-parameter family of smooth maps $f_\tau: \Lambda^j \to H$ defined by

$$f_\tau(u) = pr_2 \circ \Theta \circ \bar\partial \circ \Gamma_\tau(u)$$

where pr_2: $H^j(G_p, M) \times H \to H$ is the projection onto the second factor. It turns out that $f_\tau: \Lambda^j \to H$ is proper with respect to an open neighborhood of 0 in H. Hence degree theory is available. Moreover f_τ is a Fredholm operator of index n. We define a family of Fredholm operators $\tilde f_\tau: \Lambda^j \to H \times L$ by

$$\tilde f_\tau(u) = (f_\tau(u) , \pi(u))$$

We see that $\tilde f_0^{-1}(0, \ell_0)$ for some fixed $\ell_0 \in L$ consists precisely of the constant solution $u_0(x) = \ell_0$. Moreover

$T\tilde f_\tau(u_0): T_{u_0}\Lambda^j \to H \times T_{\ell_0}L$ is an isomorphism. Hence denoting the $\mathbb{Z}_2$-degree by $d_{\mathbb{Z}_2}$ we infer

$$d_{\mathbb{Z}_2}(\tilde f_0, (0,\ell_0)) = 1.$$

By homotopy invariance of the degree we obtain

$$d_{\mathbb{Z}_2}(\tilde f_1, (0,\ell_0)) = 1.$$

Now the desired result concerning the behavior of π on cohomological level follows from the following abstract result, [H3].

THEOREM 18 Let $f: V \to H$ be a smooth Fredholm map of index n defined on a separable Hilbertmanifold V with image in a separable Hilbertspace H. Assume f is proper with respect to a zero neighborhood in H. Assume there exists a smooth map $\pi: V \to L$ into a compact manifold of dimension n such that

$$d_{\mathbb{Z}_2}(\tilde f, (0,\ell_0)) = 1$$

where $\tilde f(x) = (f(x), \pi(x))$ and $\ell_0 \in L$ is fixed. Then

$$\pi^*: \check H(L) \to \check H(f^{-1}(0))$$

is injective.

For a simple proof see [H3].

4. MORSE-THEORY AND FLOER HOMOLOGY

In 3 we gave a relatively simple proof of the Lusternik-Schnirelman theory for Lagrangian intersections. The Morse-theory developed prior to the LS-theory by A. Floer is very difficult. Actually the idea is quite simple but the technical realization is very hard. Here we shall sketch the simple idea. The idea was motivated by an influential paper of Witten [Wi]. The reader should read also [F5]. Consider a time dependent family of positive almost complex structures, say $t \to J_t$. We have an associated family of Riemannian metrics g_t on M.

We assume L and L' are compact Lagrangian submanifolds so that L' is an exact deformation of L and the intersections in $L \cap L'$ are transversal. We define for $J = (J_t)_{t\in[0,1]}$

$$\Omega_J = \{u \in C^\infty(Z,M) \mid u_s + J_t(u)u_t = 0 \ , \int |u_s|^2 + |u_t|^2 < \infty \\ u(\mathbb{R}) \subset L \ , u(\mathbb{R}+i) \subset L'\}$$

First one derives the same estimate we derived for the LS-theory. Then one shows that for a generic family J Ω_J can be written as the union of finite manifolds $M(x,y)$ where $(x,y) \in (L\cap L')^2$

$$\Omega_J = \cup M(x,y)$$

where $M(x,y)$ consists of those $u \in \Omega_J$ such that $u \to x,y$ as $s \to \pm\infty$. Moreover there exists a unique map μ: $L \cap L' \to K$ defined up to an additive constant such that

$$\dim M(x,y) = \mu(x) - \mu(y)$$

for generic J. Moreover we have an obvious $\mathbb{R}$-action on $M(x,y)$. Denote by $\dot{M}(x,y) = M(x,y)/\mathbb{R}$ the reduced space. Clearly

$$\dim \dot{M}(x,y) = \mu(x) - \mu(y) - 1$$

It turns now out that under the assumption $\pi_2(M,L) = 0$, the set $\dot{M}(x,y)$ is finite if $\mu(y) - \mu(x) = 1$ (for generic J).

Denote by C the free $\mathbb{Z}_2$-vectorspace generated by the points in $L \cap L'$. C is a graded vectorspace where the grading is given by the map μ, i.e.

$$C^P = \text{span}\ \{x \mid \mu(x) = p\}.$$

So $C = \oplus C^P$. We define a coboundary operator δ: $C \to C$ of degree 1 by

$$\delta x = \sum_{\mu(y)=\mu(x)+1} < y,\delta x > y$$

where $<y,\delta x>$ is the number of elements in $\dot{M}(y,x)$ mod 2. (recall that $\dot{M}(y,x)$ is finite if $\mu(y) = \mu(x) + 1$. A crucial observation is then $\delta^2 = 0$. So we can define the cohomology of the complex (C,δ)

$$I^*(L,L') = \text{kern}(\delta)/\text{im}(\delta)$$

$I^*(L,L')$ is called the Floer Cohomology of the intersection problem $(M,(L,L'))$.

Assume now L_τ is an exact homotopy of $L = L_0$. In [F2] it is shown that

(1) $I^*(L,L')$ does not depend on the choice of J as long as J is generic.

(2) $I^*(L,L_{\tau_1}) \simeq I^*(L,L_\tau)$ provided L and L_{τ_1}, L_τ intersect transversally.

(3) $I^*(L,L_\tau) \simeq H^*(L,\mathbb{Z}_2)$.

Clearly if we have only transversal intersections in $L \cap L'$ and we associate to an intersection point x the monomial $t^{\mu(x)+c}$ for some fixed integer c (which we assume to be zero), we obtain the equality

$$\sum_{x\in L\cap L'} t^{\mu(x)} = P(t) + (1+t)Q(t)$$

where $P(t)$ is the $\mathbb{Z}_2$–Poincare polynomial of L. So we have the following

THEOREM 19 If (M,ω) is a compact symplectic manifold, and L a compact Lagrangian submanifold satisfying $\omega|\pi_2(M,L)$ and L' an exact deformation of L intersecting L' transversally then there exists a unique map $\mu\colon L \cap L' \to \mathbb{Z}$ such that

$$\sum_{x\in L\cap L'} t^{\mu(x)} = P(t) + (1+t)Q(t)$$

where P is the Poincare' Polynomial of L associated to H with $\mathbb{Z}_2$–coefficients and $Q(t)$ is a polynomial with nonnegative coefficients in $\mathbb{Z}$.

If $\pi_2(M,L) \neq 0$ one can give easily counter examples for the Lagrangian intersections problem. However one expects some results for the fixed point problem if $\pi_2(M) \neq 0$. The first such result was given in [Fo] by B. Fortune. Recently in [F6] Floer proves some further results. The difficulty is that one does not have in general compactness and the phenomenon of Bubbling off of holomorphic spheres can occur and has to be taken into account.

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