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THE EVOLUTION OF HARMONIC MAPS (PART II)

On the evolution of harmonic maps in higher dimensions

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On the evolution of harmonic maps in higher dimensions

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Abstract: We establish partial regularity results and existence of global regular solutions to the evolution problem for harmonic maps with small data. The key ingredient is a decay estimate analogous to the well-known monotonicity formula for energy minimizing harmonic maps.

1. Let M, N be (compact) Riemannian manifolds of dimensions m,n with metrics γ, g respectively. In local coordinates $x=(x^1,\dots,x^m)$, $u=(u^1,\dots,u^n)$ we denote $\gamma=(\gamma_{\alpha\beta})_{1\le\alpha,\beta\le m}$, $g=(g_{ij})_{1\le i,j\le n}$ and $(\gamma^{\alpha\beta})=(\gamma_{\alpha\beta})^{-1}$.

For a C^1-map $u: M\to N$ the energy of u is given by the intrinsic Dirichlet integral

$$E(u)=\int_M e(u)\,dM$$

with density

$$e(u;x)=\frac{1}{2}\gamma^{\alpha\beta}(x)\,g_{ij}(u)\,\frac{\partial}{\partial x^\alpha}u^i\,\frac{\partial}{\partial x^\beta}u^j$$

in local coordinates. A summation convention is used.

Since N is compact, N may be isometrically embedded into $\mathbb{R}^N$ for some N, and E becomes the standard Dirichlet integral of maps $u: M\to N\subset\mathbb{R}^N$.

u is harmonic iff E is stationary at u; in particular

$$\frac{d}{d\varepsilon}E(u+\varepsilon\phi)\Big|_{\varepsilon=0}=\int_U\left(-\Delta_M u+\Gamma_N(u)(\nabla u,\nabla u)_M\right)^i g_{ij}(u)\,\phi^j\,dx = 0 \qquad (1.1)$$

for any smooth variation ϕ with support in a coordinate neighborhood $U\subset\mathbb{R}^m$ and such that $(u+\varepsilon\phi)(U)$ is contained in a coordinate chart V in the target space, where

$$\Delta_M=\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}\,\gamma^{\alpha\beta}\frac{\partial}{\partial x^\beta}\,\cdot\right)$$

is the Laplace-Beltrami operator on M and the term

$$\left(\Gamma_N(u)(\nabla u,\nabla u)_M\right)^k=\gamma^{\alpha\beta}\,\Gamma^k_{ij}(u)\,\frac{\partial}{\partial x^\alpha}u^i\,\frac{\partial}{\partial x^\beta}u^j,\ 1\le k\le n$$

involves the Christoffel symbols of the metric g. I.e. u is harmonic iff u satisfies

$$-\Delta_M u+\Gamma_N(u)(\nabla u,\nabla u)_M=0. \qquad (1.2)$$

Regarding u as a map $u: M \to N \subset \mathbb{R}^N$, $E(u)$ as the ordinary Dirichlet integral, u is harmonic iff

$$\int_M - \Delta_M u \cdot \phi \, dM = 0$$

for all smooth $\phi: M \to \mathbb{R}^N$ tangent to N at u, i.e. such that $\phi(x) \in T_{u(x)}N$: the tangent space to N at $u(x), x \in M$. (Note that

$$\int_M \Gamma_N(u)(\nabla u, \nabla u)_M \cdot \phi \, dM = 0 \qquad (1.3)$$

for all such ϕ, i.e. $\Gamma_N(u)(\nabla u, \nabla u)_M$ is orthogonal to $T_u N$; cp. Schoen [8, § 1].)

Harmonic maps - in particular smooth E-minimizing maps - are distinguished representants of maps $M \to N$. In order to understand how much of the topological structure of a space N is captured by harmonic maps $M \to N$ it is natural to study the following

Problem 1: Given a (smooth) map $\chi_o: M \to N$, is there a harmonic map homotopic to χ_o?

In particular, we may ask for representations of the fundamental groups of N by harmonic maps:

Problem 2: Given a (smooth) map $\chi_o: S^m \to N$, is there a harmonic map homotopic to χ_o?

In dimensions $m = 2$ Sacks and Uhlenbeck [7] have given an (essentially) affirmative answer to problem 2. Moreover, the existence of harmonic 2-spheres turns out to be precisely the obstruction for solving problem 1 in general.

In dimensions $m > 2$ - apart from certain particular cases - essentially no significant progress has been made since the fundamental result by Eells and Sampson [2] in 1964:

Theorem 1.1: Suppose the sectional curvature of N: $\kappa_N \le 0$. Then for any (smooth) map $u_o: M \to N$ there is a (smooth) E-minimizing map $u: M \to N$ homotopic to u_o.

Their method is based on an analysis of the evolution problem

$$\partial_t u - \Delta_M u + \Gamma_N(u)(\nabla u, \nabla u)_M = 0 \;, \; u|_{t=0} = u_o \qquad (1.4)$$

which by (1.1) may be regarded as the L^2-gradient flow for E with respect to the metric $g(u)$. Eells and Sampson prove that under the above curvature restriction on the target (1.4) possesses a global regular solution $u(t)$, which as $t \to \infty$ converges to a harmonic map.

In [11] the latter result was generalized to arbitrary target manifolds in the case $m = 2$:

Theorem 1.2: Suppose $m = 2$. For any (smooth) map $u_o: M \to N$ there exists a (unique) global distribution solution to

(1.4) which is regular on $M \times [0,\infty[$ with exception of finitely many points $(x_k,t_k)_{1\le k\le K}$, $t_k \le \infty$. At a singularity $(\bar{x},\bar{t})$ a non-constant, smooth harmonic map $\bar{u}:\overline{\mathbb{R}^2} \cong S^2 \to N$ separates in the sense that for sequences

$$R_m \searrow 0 \;,\; x_m \to \bar{x} \;,\; t_m \nearrow \bar{t}$$

as $m \to \infty$

$$u_m(x) \equiv u(\exp_{x_m}(R_m x), t_m) \to \bar{u} \quad \text{in} \quad H^{1,2}_{loc}(\mathbb{R}^2;N) \;.$$

Moreover, $u(t)$ converges weakly in $H^{1,2}(M;N)$ to a smooth harmonic map $M \to N$ as $t \to \infty$ (strongly, if $t = \infty$ is regular).

Here $\exp_q : T_q M \to M$ denotes the exponential map,

$$H^{1,2}(M;N) = \{u \in H^{1,2}(M;\mathbb{R}^N)\,|\,u(M) \subset N \text{ a.e.}\}$$

and $H^{1,2}(M;\mathbb{R}^N)$ is the standard Sobolev space of square-integrable (L^2-) functions $u:M \to \mathbb{R}^N$ with distributional derivative $\nabla u \in L^2$. Remark that if $m = 2$ the space $H^{1,2}(M;N)$ coincides with the closure of the space $C^\infty(M;N)$ of smooth functions $u:M\to N$ in the $H^{1,2}$-norm.

For $m > 2$ this is no longer true. ([10, Example, p.267]; cp. however Proposition 7.2 below.)

The purpose of this note is to partially extend Theorem 1.2 to the case $m > 2$. In this case no existence and regularity results for (1.4) and arbitrary target manifolds are known unless certain a-priori restrictions relating the size

of the image $u(M \times \mathbb{R}_+)$ to a bound for the sectional curvature of N are satisfied, cp. e.g. [4]. However, unless M is a manifold with boundary ∂M and boundary conditions are posed on ∂M such conditions seem unnatural.

Imposing no a-priori restrictions on N or the range of u we prove partial regularity results (Theorem 6.1) and global existence and regularity results for smooth initial data with small energies (Theorem 7.1).

The basic ingredients are a monotonicity estimate Proposition 3.3 and the ε-regularity Theorem 5.1 which are reminiscent of the well-known monotonicity formula and ε-regularity theorem for minimizing harmonic maps in high dimensions, cp. Schoen-Uhlenbeck [9], Schoen [8].

For simplicity we restrict ourselves to the case $M = \mathbb{R}^m$. However, our results seem to carry over to compact manifolds M.

2. Notations

Let $z = (x,t)$ denote points in $\mathbb{R}^m \times \mathbb{R}$. For a distinguished point $z_o = (x_o,t_o)$, $R > 0$ let

$$B_R(x_o) = \{x \mid |x-x_o| < R\}$$

be an Euclidean ball centered at x_o, and let

$$P_R(z_o) = \{z = (x,t) \mid |x-x_o| < R \ , \ |t-t_o| < R^2\}$$

be a parabolic cylinder of radius R centered at z_o. Also let

$$S_R(t_o) = \{z = (x,t) \mid t = t_o-R^2\}$$

and

$$T_R(t_o) = \{z = (x,t) \mid t_o-4R^2<t<t_o-R^2\}$$

be horizontal sections, resp. horizontal layers in $\mathbb{R}^m \times \mathbb{R}$. Note that equation (1.4) is invariant under scaling

$$u \longmapsto u_R(x,t) = u(Rx, R^2 t)$$

and translation $x \longmapsto x-x_o$, $t \to t-t_o$. Using this invariance property we will often shift the center of attention

to the origin $z_o = 0$. In this case we simply write $P_R(0) = P_R$, etc.

Weighted estimates will involve the fundamental solution

$$G_{z_o}(z) = \frac{1}{(4\pi(t_o-t))^{m/2}} \exp\left(-\frac{(x-x_o)^2}{4(t_o-t)}\right) \ , \ t < t_o$$

to the (backward) heat equation with singularity at z_o. (Again $G_o(z) = G(z)$, for simplicity.)

δ denotes the parabolic distance function

$$\delta\big((x,t),(y,s)\big) = \max\{|x-y| \ , \ \sqrt{|s-t|}\} \ .$$

The letters c, C denote generic constants.

A map $u : \mathbb{R}^m \times [t_o,t_1] \to \mathbb{R}^N$ is regular iff u and ∇u are uniformly bounded and $\partial_t u, \nabla^2 u \in L^p_{loc}$ for all $p < \infty$.

<u>Remark</u> 2.1: With this definition, by [5;Theorem IV.9.1,p.341 f.] any regular solution u to (1.4) on an interval $[0,t_o]$ may be extended to a regular solution of an equation $(\partial_t-\Delta)u \in L^\infty_{loc}$ on $\mathbb{R}_+$ by letting u solve $(\partial_t-\Delta)u = 0$ for $t > t_o$. Lemma 3.1, Lemma 3.2, resp. Proposition 3.3 and 4.1 below will also apply to the extended function u.

Moreover, for a regular solution u of (1.4), also $\partial_t u$, $\nabla^2 u$, etc. will be uniformly bounded, if the initial data u_o are smooth.

3. Energy estimates and monotonicity formula

Let $u: \mathbb{R}^m \times [0,T] \to N$ be a regular solution to (1.4) with $E(u(t)) < \infty$ for $t \in [0,T]$. The following estimate is well-known:

Lemma 3.1:

$$\sup_{0 \le t \le T} E(u(t)) + \int_0^T \int_{\mathbb{R}^m} |\partial_t u|^2 dx dt \le E(u_o) .$$

Proof: Simply multiply (1.4) by $\partial_t u$ and integrate by parts. By (1.3) and since $E(u(t)) < \infty$ for all t the non-linear term and boundary integrals vanish..

qed

We also need a weighted decay estimate analogous to Lemma 3.1. This is our key result.

Lemma 3.2: Let $u: \mathbb{R}^m \times [0,T] \to N$ be a regular solution to (1.4) with $|\nabla u(x,t)| \le c < \infty$ uniformly. Then for any point $z_o = (x_o, t_o) \in \mathbb{R}^m \times]0,T[$ the function

$$\Phi(R;u) = \frac{1}{2} R^2 \int_{S_R(t_o)} |\nabla u|^2 G_{z_o} \, dx$$

is non-decreasing for $0 < R \le R_o = \sqrt{t_o}$.

Proof: By translation we may achieve that $z_o = 0$. We establish that

$$\frac{d}{dR} \Phi(R;u) \Big|_{R=R_1} \ge 0 .$$

By scale invariance

$$\Phi(R;u) = \Phi(1;u_R) ,$$

where $u_R(x,t) = u(Rx, R^2 t)$; also it suffices to consider $R_1 = 1$.

By the exponential decay of G and regularity of u we may differentiate under the integral sign:

$$\frac{d}{dR} \Phi(R;u) \Big|_{R=1} = \frac{d}{dR} \Phi(1;u_R) \Big|_{R=1}$$

$$= \int_{S_1} \nabla u \nabla \left(\frac{d}{dR} u_R \Big|_{R=1} \right) G \, dx$$

$$= \int_{S_1} \nabla u \nabla (x \cdot \nabla u + 2t\partial_t u) \, G \, dx$$

$$= - \int_{S_1} \Delta u (x \cdot \nabla u + 2t\partial_t u) \, G \, dx$$

$$- \int_{S_1} \nabla u (x \cdot \nabla u + 2t\partial_t u) \nabla G \, dx .$$

The vector $x\cdot\nabla u+2t\partial_t u$ is tangent to N at u; hence by (1.3-4) and using the explicit form of G:

$$\frac{d}{dR}\Phi(R;u)\Big|_{R=1} =$$

$$= -\int_{S_1} 2t|\partial_t u|^2 G\,dx - \int_{S_1} \frac{1}{2t}|x\cdot\nabla u|^2 G\,dx$$

$$-2\int_{S_1} \partial_t u(x\cdot\nabla u)\, G\,dx \geq 0 .$$

In order to obtain the last estimate we have used Young's inequality

$$2(\partial_t u)(x\cdot\nabla u) \leq 2|t|\,|\partial_t u|^2 + \frac{1}{2|t|}|x\cdot\nabla u|^2 .$$

Also note that $t = -1$ on S_1.

<u>qed</u>

In particular, Lemma 3.2 implies the following monotonicity formula for solutions to (1.4):

<u>Proposition</u> 3.3: Suppose $u:\mathbb{R}^m \times [0,t_o=4R_o^2]\to N$ is a regular solution to (1.4) with $|\nabla u(x,t)| \leq c < \infty$ uniformly. Then for any point $z_o = (x_o,t_o)$ the function

$$\Psi(R;u) := \int_{T_R(z_o)} |\nabla u|^2 G_{z_o}\,dxdt$$

is non-decreasing for $0 < R < R_o$.

<u>Proof</u>: Shift $z_o = 0$ and compute for $0 < R < R_1 < R_o$ (with $\frac{r'}{r} = \frac{R_1}{R} =: \lambda$):

$$\Psi(R;u) = \int_{-4R^2}^{-R^2}\int_{\mathbb{R}^m} |\nabla u|^2 G\,dxdt = 4\int_R^{2R} r^{-1}\Phi(r;u)\,dr$$

$$= 4\int_{R_1}^{2R_1} \frac{\Phi(r'/\lambda;u)}{\Phi(r';u)} r'^{-1}\,\Phi(r';u)\,dr' \leq \Psi(R_1;u)$$

by Lemma 3.2.

<u>qed</u>

4. A Bochner-type estimate

Suppose $u:Q \to N$ is a regular solution of (1.4) in an open space-time region $Q \subset \mathbb{R}^m \times \mathbb{R}$. Taking the gradient of both sides of (1.4) and multiplying by ∇u we obtain

$$\partial_t \nabla u \cdot \nabla u - \Delta \nabla u \cdot \nabla u =$$

$$= (\partial_t - \Delta)\left(\frac{|\nabla u|^2}{2}\right) + |\nabla^2 u|^2$$

$$= -\nabla(\Gamma_N(u)(\nabla u, \nabla u)) \cdot \nabla u \le \varepsilon\, |\nabla^2 u|^2 + C(\varepsilon)\, |\nabla u|^4 .$$

Choosing $\varepsilon = 1$ yields the following differential inequality for the energy density $e(u) = \frac{1}{2} |\nabla u|^2$ of u :

<u>Proposition</u> 4.1: Let $u:Q \to N$ be a regular solution to (1.4) in Q with energy density $e(u)$. Then there holds

$$(\partial_t - \Delta)\, e(u) \le c\, e(u)^2$$

with a constant c depending only on N and m.

<u>Remark</u> 4.2: By the maximum principle for the heat equation, Proposition 4.1 implies an a-priori estimate for $|\nabla u|$ on

a small time interval, for any regular solution u of (1.4) with regular initial data u_0. This guarantees the existence of solutions to (1.4), locally. If $E(u_0) < \infty$, by Lemma 3.2 also $E(u(t)) \le c < \infty$ uniformly, locally near $t = 0$, and also the energy inequality Lemma 3.1 will hold.

5. The ε-regularity theorem

Our monotonicity formula Proposition 3.3 allows to use ideas of Schoen-Uhlenbeck [9] and Schoen [8] to prove the following:

Theorem 5.1: There exists a constant $\varepsilon_o > 0$ depending only on N and m such that for any regular solution $u: \mathbb{R}^m \times [-4R_o^2, 0] \to N$ of (1.4) with $E(u(t)) \le E_o < \infty$, uniformly in t, the following is true:

If for some $R \in]0, R_o[$ there holds

$$\Psi(R;u) := \int_{T_R} |\nabla u|^2 G \, dxdt < \varepsilon_o ,$$

then

$$\sup_{P_{\rho R}} |\nabla u|^2 \le c\,(\rho R)^{-2}$$

with constants $\rho > 0$ depending on N, m, E_o, and $\inf\{R, 1\}$, and c depending on N and m, only.

Proof: We closely follow Schoen's proof [8; Theorem 2.2] for the analogous result in the stationary case.

Let $r_1 = \delta R$, $\delta \in]0, \frac{1}{2}]$ to be determined in the sequel. For $r, \sigma \in]0, r_1[$, $r + \sigma < r_1$, and $z_o = (x_o, t_o) \in P_r$ our monotonicity formula (for the extended function u, cp. Remark 2.1) implies

$$\sigma^{-n} \int_{P_\sigma(z_o)} |\nabla u|^2 dxdt \le c \int_{P_\sigma(z_o)} |\nabla u|^2 G_{(x_o, t_o + 2\sigma^2)} dxdt$$

$$\le c \int_{T_\sigma(t_o + 2\sigma^2)} |\nabla u|^2 \, G_{(x_o, t_o + 2\sigma^2)} dxdt \tag{5.1}$$

$$\le c \int_{T_R} |\nabla u|^2 \, G_{(x_o, t_o + 2\sigma^2)} dxdt .$$

But on T_R, given $\varepsilon > 0$, if $\delta > 0$ is small enough:

$$G_{(x_o, t_o + 2\sigma^2)}(x,t) \le \frac{c}{(4\pi |t|)^{m/2}} \exp\left(- \frac{|x - x_o|^2}{4(t_o + 2\sigma^2 - t)} \right)$$

$$\le C \exp\left(\frac{|x|^2}{4|t|} - \frac{|x - x_o|^2}{4|t_o + 2\sigma^2 - t|} \right) G(x,t)$$

$$\le C \exp\left(c\, \delta^2 \frac{|x|^2}{R^2} \right) G(x,t) \le \begin{cases} C\, G(x,t) , & \text{if } |x| \le \frac{R}{\delta} \\ C\, R^{-m} \exp(-c\delta^{-2}) , & \text{if } |x| \ge \frac{R}{\delta} \end{cases}$$

$$\le C\, G(x,t) + C\, R^{-2} \exp\left((2-m) \log R - c\delta^{-2} \right)$$

$$\le C\, G(x,t) + \varepsilon R^{-2} . \tag{5.2}$$

Remark that $\delta \sim |\ell n R|^{-1/2}$ for small R and may be chosen independent of R, if $R \ge 1$. Hence

$$\sigma^{-n} \int_{P_\sigma(z_o)} |\nabla u|^2 dxdt \le c\,\Psi(R) + c\,\varepsilon\, E_o \le c\,(\varepsilon_o + \varepsilon\, E_o) \;. \tag{5.3}$$

There exists $\sigma_o \in [0, r_1[$ such that

$$(r_1-\sigma_o)^2 \sup_{\overline{P}_{\sigma_o}} e(u) = \max_{0\le\sigma\le r_1} (r_1-\sigma)^2 \sup_{\overline{P}_\sigma} e(u) \;;$$

moreover, there exists $(x_o,t_o) \in \overline{P}_{\sigma_o}$ such that

$$\sup_{\overline{P}_{\sigma_o}} e(u) = e(u)(x_o,t_o) = e_o \;.$$

Set $\rho_o = \frac{1}{2}(r_1-\sigma_o)$. By choice of $\sigma_o, (x_o,t_o)$

$$\sup_{P_{\rho_o}(x_o,t_o)} e(u) \le \sup_{P_{\sigma_o+\rho_o}} e(u) \le 4\, e_o \;.$$

Introduce

$$r_o = \sqrt{e_o} \cdot \rho_o$$

and define a smooth map $v\colon P_{r_o} \to N$ by letting

$$v(x,t) = u\left(\frac{x-x_o}{\sqrt{e_o}} \;,\; \frac{t-t_o}{e_o}\right)$$

v solves (1.4) in P_{r_o} ; moreover, v satisfies

$$e(v)(0,0) = 1 \;,$$

$$\sup_{P_{r_o}} e(v) \le 4 \;.$$

By our Bochner-type estimate Proposition 4.1 therefore $e(v)$ satisfies

$$(\partial_t - \Delta)\, e(v) \le c_1\, e(v)$$

with a constant c_1 depending only on m and N . Thus, if instead of $e(v)$ we consider the function $f(x,t) = \exp(-c_1 t)\, e(v)$ in P_{r_o} and if $r_o \ge 1$, Moser's Harnack inequality [6; Theorem 1, p. 102] implies the estimate

$$1 = e(v)(0,0) \le c \int_{P_1} e(v)\, dxdt$$

But, scaling back, by (5.3) and since $\frac{1}{\sqrt{e_o}} + \sigma_o \le \rho_o + \sigma_o < r_1$

$$\int_{P_1} e(v)\, dxdt = \left(\sqrt{e_o}\right)^n \int_{P_{\frac{1}{\sqrt{e_o}}}(x_o,t_o)} e(u)\, dxdt \le c(\varepsilon_o + \varepsilon\, E_o)$$

and we obtain a contradiction for small $\varepsilon_o, \varepsilon > 0$. Hence we may assume $r_o \le 1$. But then the Harnack-inequality gives

$$1 = e(v)(0,0) \le C\, r_o^{-n-2} \int_{P_{r_o}} e(v)\,dxdt$$

$$= C\, r_o^{-2} \rho_o^{-n} \int_{P_{\rho_o}(x_o,t_o)} e(u)\,dxdt \ ,$$

i.e. by (5.1-2) and since $\rho_o + \sigma_o = \frac{1}{2}(r_1+\sigma_o) < r_1$:

$$\rho_o^2 e_o = r_o^2 \le C\,\Psi(R) + C\varepsilon\, E_o \le C \ .$$

Finally, by choice of σ_o this implies

$$\max_{0\le\sigma\le r_1} (r_1-\sigma)^2 \sup_{P_\sigma} e(u) \le 4\rho_o^2 e_o \le 4\,C \ .$$

Hence, we may choose $\sigma = \frac{1}{2} r_1 = \frac{\delta}{2} R$ and devide by σ^2 to complete the proof.

qed

Remark 5.2: Instead of (5.2) we may estimate for $K > 0$, $R > 0$, uniformly on T_R :

$$G_{(x_o,t_o+2\sigma^2)}(x,t) \le \frac{C}{R^m} \le c(K)G(x,t), \text{ if } |x| \le KR$$

resp.

$$G_{(x_o,t_o+2\sigma^2)}(x,t) \le c\cdot\exp\left(\frac{|x|^2}{4|t-R^2|} - \frac{|x-x_o|^2}{4|t-t_o-2\sigma^2|}\right) G_{(0,R^2)}(x,t)$$

$$\text{(5.4)} \qquad \le c\,\exp\left(-\,c^{-1}K^2\right) G_{(0,R^2)}(x,t), \text{ if } |x| \ge KR$$

provided $(x_o,t_o) \in P_\sigma$, $\sigma < R/2$. Hence we obtain that for any $\varepsilon > 0$ there holds

$$G_{(x_o,t_o+2\sigma^2)} \le C(\varepsilon)\, G + \varepsilon\, G_{(0,R^2)} \ ,$$

uniformly on T_R , uniformly in $R > 0$. Thus, instead of (5.3) we obtain

$$\sigma^{-n} \int_{P_\sigma(z_o)} |\nabla u|^2 dxdt \le C(\varepsilon)\Psi(R) + c\varepsilon \int_{T_R} |\nabla u|^2 G_{(0,R^2)}\,dxdt \ .$$

If now either $E(u(t)) \le E_o < \infty$ or $|\nabla u| \le C < \infty$ we may apply Proposition 3.3 to the term on the far right and deduce that

$$\sigma^{-n} \int_{P_\sigma(z_o)} |\nabla u|^2 dxdt \le C(\varepsilon)\varepsilon_o +$$

$$+ c\,\varepsilon \int_{T_{R_o}} |\nabla u|^2 G_{(0,R^2)}\, dxdt$$

$$\le C(\varepsilon)\varepsilon_o + c\varepsilon R_o^{2-m} E_o .$$

With this modification and leaving the remainder of the proof of Theorem 5.1 unchanged we obtain the following variants of this result:

Theorem 5.3: For any $R_o > 0$, E_o there exists a constant $\varepsilon_o > 0$ depending on R_o, E_o, $\mathcal{N}$, and m such that for any regular solution $u : \mathbb{R}^m \times [-4R_o^{\,2},0] \to \mathcal{N}$ of (1.4) with $E(u(t)) \le E_o < \infty$ the following is true:

If for some $R \in]0,R_o[$ there holds

$$\Psi(R;u) = \int_{T_R} |\nabla u|^2 G\, dxdt < \varepsilon_o ,$$

then

$$\sup_{P_{R/2}} |\nabla u|^2 \le C R^{-2}$$

with a constant c depending on $\mathcal{N}$ and m only.

Theorem 5.4: For any $C_o > 0$ there exists a constant $\varepsilon_o > 0$ depending on C_o, $\mathcal{N}$, and m such that for any regular solution $u: \mathbb{R}^m \times [-4R_o^2,0] \to \mathcal{N}$ of (1.4) with $|\nabla u| \le c < \infty$ uniformly the following is true:

If for some $R \in]0,R_o[$ there holds

$$\psi(R;u) = \int_{T_R} |\nabla u|^2 G\, dxdt < \varepsilon_o$$

while

$$\int_{T_R} |\nabla u|^2 G_{(0,R^2)}\, dxdt \le C_o ,$$

then

$$\sup_{P_{R/2}} |\nabla u|^2 \le C R^{-2}$$

with a constant c depending only on $\mathcal{N}$ and m .

6. Partial regularity

Using the a-priori estimate obtained previously we can prove partial regularity of weak solutions u to (1.4) with finite energy and which can be weakly approximated by smooth global solutions to (1.4):

Theorem 6.1: Suppose $u: \mathbb{R}^m \times \mathbb{R}_+ \to \mathcal{N}$ is limit of a sequence $\{u_k\}$ of regular solutions u_k to (1.4) with uniformly finite energy

$$E(u_k(t)) \le E_o < \infty \,, \; \forall\, k \in \mathbb{N} \,, \; t > 0$$

in the sense that $E(u(t)) \le E_o$ a.e. and

$$\nabla u_k \to \nabla u \quad \text{weakly in} \quad L^2(Q)$$

for any compact $Q \subset \mathbb{R}^m \times \mathbb{R}_+$.

Then u solves (1.4) in the classical sense and is regular on a dense open set $Q_o \subset \mathbb{R}^m \times \mathbb{R}_+$ whose complement Σ has locally finite m-dimensional Hausdorff-measure (with respect to the parabolic metric δ). Moreover, there exists $t_o > 0$ (depending on $\mathcal{N}$, m, and E_o) such that $\Sigma \cap (\mathbb{R}^m \times [t_o, \infty[) = \emptyset$. Finally, $u(t) \to u_\infty \equiv p \in \mathcal{N}$ in C^1_{loc} as $t \to \infty$, where $u_\infty \equiv p$ is a constant map.

Proof: This proof is modelled on [8, proof of Corollary 2.3].

Define

$$\Sigma = \bigcap_{R>0} \left\{ z_o \in \mathbb{R}^m \times \mathbb{R}_+ \,\middle|\, \liminf_{k\to\infty} \int_{T_R(z_o)} |\nabla u_k|^2 G_{z_o} \, dxdt \ge \varepsilon_o \right\} ,$$

where $\varepsilon_o > 0$ is the constant determined in Theorem 5.1. Σ is closed. Indeed, if $\{z_\ell\}$ is a sequence of points in Σ converging to $z_o \in \mathbb{R}^m \times \mathbb{R}_+$, for any $R > 0$, $\ell \in \mathbb{N}$ we have

$$\liminf_{k\to\infty} \int_{T_R(z_\ell)} |\nabla u_k|^2 \, G_{z_\ell} \, dxdt \ge \varepsilon_o \, .$$

Since $G_{z_\ell} \to G_{z_o}$ uniformly away from $z_o = (x_o, t_o)$ and since $E(u_k) \le E_o$ uniformly, this implies that for any $\delta > 0$

$$\liminf_{k\to\infty} \int_{t_o-\delta-4R^2}^{t_o+\delta-R^2} \int_{\mathbb{R}^m} |\nabla u_k|^2 \, G_{z_o} \, dxdt \ge \varepsilon_o \, .$$

By Proposition 3.3 and since $R, \delta > 0$ were arbitrary this implies that

$$\liminf_{k\to\infty} \int_{T_R(z_o)} |\nabla u_k|^2 G_{z_o}\, dxdt \geq \varepsilon_o$$

for all $R > 0$, whence $z_o \in \Sigma$ as claimed.

Σ has locally finite m-dimensional Hausdorff-measure with respect to the metric δ, given by

$$m\text{-meas}(\Sigma) = \lim_{R>0} \inf(c(m) \sum_i R_i^m) .$$

The infimum here is taken with respect to all covers J of Σ by cylinders $P_{R_i}(z_i)$ of radius $R_i \leq R$.
It will suffice to show that

$$m\text{-meas}\ (\Sigma \cap Q) \leq c(Q,E_o)$$

for all compact regions $Q \subset \mathbb{R}^m \times \mathbb{R}_+$. Let $R > 0$ be given and let $J = \{P_{R_i}(z_i)\}$ be a cover of $\Sigma \cap Q$ with $R_i \leq R$. We may assume $z_i \in \Sigma$: By Vitali's covering lemma (cp. Caffarelli-Kohn-Nirenberg [1 ; Lemma 6.1, p. 806] for a parabolic version) there exists a subfamily $J' = \{P_i = P_{R_i}(z_i')\}$ of J such that $P_i \cap P_j = \emptyset$ if $i \neq j$ and such that the collection $\{P_{5R_i}(z_i')\}$ covers $\Sigma \cap Q$.

Note that for sufficiently small $R \leq R(d,E_o)$ and arbitrary $z_o = (x_o,t_o)$, $k \in \mathbb{N}$, $\varepsilon > 0$, by (5.4) there is a constant $C(\varepsilon)$ such that:

$$\int_{T_R(z_o)} |\nabla u_k|^2 G_{z_o}\, dxdt \leq c\, R^{-m} \int_{P_{C(\varepsilon)R}(z_o)} |\nabla u_k|^2\, dxdt$$

$$+ \varepsilon \int_{T_R(z_o)} |\nabla u_k|^2\, G_{(z_o+(0,R^2))}\, dxdt .$$

Applying Lemma 3.2 the last term may be dominated for sufficiently small $R > 0$, $\varepsilon > 0$:

$$\varepsilon \int_{T_R(z_o)} |\nabla u_k|^2\, G_{(z_o+(0,R^2))}\, dxdt \leq$$

$$\leq \varepsilon\, c\left(t_o+R^2\right) \int_{\mathbb{R}^m} |\nabla u_k|^2\, G_{(z_o+(0,R^2))}\, dx \Big|_{t=0}$$

$$\leq \varepsilon\, c\,(Q)\, E_o \leq \frac{1}{2}\, \varepsilon_o .$$

Thus for $z_o \in \Sigma \cap Q$, $0 < R < R(d,E_o)$ we can choose a cylinder $P_{R_o}(z_o)$ of radius $R_o < R$ such that for sufficiently large k

$$\int_{P_{R_o}(z_o)} |\nabla u_k|^2 dxdt \geq c(Q,E_o)^{-1} R_o^m \varepsilon_o . \qquad (6.2)$$

Since Σ is closed we may cover $\Sigma \cap Q$ by finitely many such cylinders $P_{R_i}(z_i)$ from which we extract a dispoint finite sub-family $J' = \{P_i = P_{R_i}(z_i')\}$ as above. We choose $k \in \mathbb{N}$ such that (6.2) is satisfied on each cylinder P_i.
By summation over i

$$\sum_i R_i^m \leq c(Q,E_o)\, \varepsilon_o^{-1} \sum_i \int_{P_i} |\nabla u_k|^2 dxdt =$$

$$= c(Q,E_o)\, \varepsilon_o^{-1} \int_{\bigcup_i P_i} |\nabla u_k|^2 dxdt \leq c(Q,E_o) < \infty .$$

Moreover, the collection $\{P_{5R_i}(z_i')\}$ covers $\Sigma \cap Q$ with $\sup_i R_i < R$. Hence

$$\text{m-meas}(\Sigma \cap Q) \leq \lim_{R \to 0} \left\{ \inf_{J} \left\{ c(m) \sum_i R_i^m \right\} \right\} \leq c(Q,E_o) ,$$

as was to be shown.
Next, for $z_o \notin \Sigma$ there exists $R > 0$ such that

$$\int_{T_R(z_o)} |\nabla u_k|^2 G_{z_o}\, dxdt \leq \varepsilon_o$$

for infinitely many $k \in \mathbb{N}$. By Theorem 5.1 then also $|\nabla u_k| \leq C$ uniformly in a uniform neighborhood of z_o, and a-priori bounds for higher derivatives may be derived from (1.4). It follows that a subsequence $u_k \to u$ in $C^2_{loc}(\mathbb{R}^m \times \mathbb{R}_+ \setminus \Sigma; N)$ and u is a regular solution of (1.4) off Σ.

Finally, using Proposition 3.3, for large $t_o, 4R^2 \le t_o$, we may estimate

$$\int_{T_R(z_o)} |\nabla u_k|^2 G_{z_o} \, dxdt \le$$

$$\le \int_0^{t_o/4} \int_{\mathbb{R}^m} |\nabla u_k|^2 G_{z_o} \, dxdt \le C \, t_o^{\frac{2-m}{2}} E_o < \varepsilon_o$$

uniformly in k, and we obtain full regularity for $t_o > C(E_o/\varepsilon_o)^{\frac{2}{m-2}}$. Moreover, choosing R as large as possible and applying Theorem 5.1 we infer the uniform decay

$$|\nabla u(x,t)|^2 \le C/t$$

for large t, and $u(t) \to u_\infty \equiv \text{const}(t \to \infty)$.

qed

7. Small initial data

In particular, Theorem 6.1 can be turned into a global existence and regularity result for smooth initial data with small energy:

Theorem 7.1: There exists a constant $\varepsilon_1 > 0$ depending on C_1, N and m such that for initial data $u_o \in H^{1,2}_{loc}(\mathbb{R}^m; N)$ with $\nabla u_o \in L^\infty$ and

$$||\nabla u_o||_\infty \le C_1 \, , \; E(u_o) < \varepsilon_1$$

there exists a unique smooth solution u of (1.4) which as $t \to \infty$ converges to a constant map $u_\infty \equiv p \in N$.

The proof is a consequence of Theorem 6.1 and the following approximation result for functions $u_o \in H^{1,2}_{loc}(\mathbb{R}^m; N)$ with finite energy and satisfying (7.1) below. (This result is analogous to an approximation result of Schoen-Uhlenbeck [10, Proposition, p. 267] in the case $m = 2$.)

Proposition 7.2: There exists $\varepsilon_2 > 0$ such that any map $u \in H^{1,2}_{loc}(\mathbb{R}^m; N)$ satisfying the condition

$$\sup_{R < R_o} R^{2-m} \int_{B_R(x_o)} |\nabla u|^2 dx \le \varepsilon_2 \, , \tag{7.1}$$

uniformly for all $x_o \in \mathbb{R}^m$ and for some $R_o > 0$, can be approximated in $H^{1,2}_{loc}(\mathbb{R}^m; N)$ by smooth maps $u_k \in C^\infty(\mathbb{R}^m; N)$.

Moreover, if u has finite energy, resp. $|\nabla u| \in L^\infty$, we may choose u_k with finite energy and $E(u_k) \le c\, E(u)$ with a constant c depending only on N, resp. $\|\nabla u_k\|_{L^\infty} \le c \|\nabla u\|_{L^\infty}$.

Proof: There is $\delta_o > 0$ such that any point $q \in \mathbb{R}^N$ at distance $< \delta_o$ from N has a unique nearest neighbor $\pi(q) \in N$. Moreover, this projection π from the δ_o-neighborhood $U_{\delta_o}(N)$ of N onto N is smooth.

For $R < R_o$ let $\phi = \phi_R$ be a mollifier:

$$\phi_R \in C_o^\infty(B_R), \quad 0 \le \phi_R \le CR^{-m}, \quad \int_{B_R} \phi_R \, dw = 1 .$$

For u as above, $R < R_o$ let

$$u_R(\bar{x}) = (u * \phi_R)(\bar{x}) = \int_{B_R(\bar{x})} u(x)\, \phi_R(\bar{x}-x)\, dx .$$

It is well-known that $u_R \in C^\infty$ and $u_R \to u$ in $H^{1,2}_{loc}(\mathbb{R}^m, \mathbb{R}^N)$ as $R \to 0$. Hence if we show that $u_R : \mathbb{R}^m \to U_{\delta_o}(N)$ for sufficiently small R, the family $v_R = \pi \circ u_R$, $0 < R < R_o$, will lie in $C^\infty(\mathbb{R}^m; N)$, and will converge to u in $H^{1,2}_{loc}(\mathbb{R}^m, N)$, as required.

As in [10], for any $\bar{x} \in \mathbb{R}^m$ we estimate

$$\operatorname{dist}(u_R(\bar{x}), N)^2 \le CR^{-m} \int_{B_R(\bar{x})} |u_R(\bar{x}) - u(x)|^2 dx$$

$$\le CR^{2-m} \int_{B_R(\bar{x})} |\nabla u|^2 dx \le C\varepsilon_2 , \text{ if } R < R_o ,$$

which will be $< \delta_o^2$ if $\varepsilon_2 > 0$ is small enough.

Finally, by smoothness of π and Fubini's theorem

$$\int_{\mathbb{R}^m} |\nabla v_R|^2 dx \le C \int_{\mathbb{R}^m} |\nabla u_R|^2 dx$$

$$= C \int_{\mathbb{R}^m} \Big| \int_{\mathbb{R}^m} \nabla u(y)\, \phi_R(x-y)\, dy \Big|^2 dx$$

$$\le C \int_{\mathbb{R}^m} \Big(\int_{\mathbb{R}^m} \phi_R(x-y)\, dy \Big) \Big(\int_{\mathbb{R}^m} |\nabla u|^2(y)\, \phi_R(x-y)\, dy \Big) dx$$

$$= C \int_{\mathbb{R}^m} |\nabla u|^2 dx$$

where $C = \|\nabla \pi\|_\infty = C(N)$. The estimate for $\|\nabla u_k\|_{L^\infty}$ is obtained in a similar way.

qed

Proof of Theorem 7.1: If $\varepsilon_1 > 0$ is sufficiently small, by Proposition 7.2 there is a sequence $u_{ko} \in C^\infty(\mathbb{R}^m; N)$ of smooth functions approximating u_o in $H^{1,2}_{loc}$ and with $E(u_{ko}) \le C\, E(u_o) = E_o$. Remark that by convolution also

$$||\nabla u_{ko}||_\infty \le ||\nabla u_o||_\infty .$$

We will show that for $\varepsilon_1 > 0$ sufficiently small

$$\sup_{x_o, R>0} R^2 \int_{\mathbb{R}^m} |\nabla u_{ko}|^2 G_{(x_o,R^2)}\, dx < \varepsilon_o$$

which by Theorem 5.1 and Lemma 3.2 will imply the existence of smooth global solutions u_k to (1.4) with initial data u_{ko}.

But using the explicit formula for G, for $0 < R < e^{-m}$

$$\int_{\mathbb{R}^m} |\nabla u_{k_o}|^2 G_{(x_o,R^2)}\, dx \le$$

$$\le C\, R^{-m} \int_{B_{2R|\ell n R|}(x_o)} |\nabla u_{ko}|^2 dx + C\, R^{-m+|\ell n\ R|} E_o$$

$$\le C\, |\ell n\ R|^m\, ||\nabla u_o||^2_\infty + C\, E_o$$

and this is $< R^{-2}\varepsilon_o$ if $R < R_1 = R_1(||\nabla u_o||_\infty, E_o)$;

while for $R > R_1$ we can achieve

$$\int_{\mathbb{R}^m} |\nabla u_{ko}|^2 G_{(x_o,R^2)}\, dx \le C\, R^{-m}\, E_o < R^{-2}\varepsilon_o$$

if $E_o < C\, R_1^{m-2}\varepsilon_o$.

Hence Theorem 5.1, Lemma 3.2 and our monotonicity formula Proposition 3.3 guarantee uniform global a-priori bounds $|\nabla u_k(x,t)|^2 \le C/t$. Since by Remark 4.2, cp. also [3], (1.4) for smooth initial data u_{ko} admits smooth solutions locally, we thus obtain global smooth solutions u_k to (1.4) with data u_{ko}. Morevoer, $\{u_k\}$ is uniformly bounded in C^1, hence relatively compact in C^o_{loc} with uniform limit u solving (1.4) with initial data u_o. Since u is continous u is also regular. (This follows from standard results in regularity theory for parabolic systems, cp. [5].)

qed

Remark 7.3: Inspection of the proof shows that $\nabla u_o \in L^{m+\mu}_{loc}$ and uniform local boundedness

$$\sup_{x_o} \int_{B_1(x_o)} |\nabla u_o|^{m+\mu} dx \le C$$

for some $\mu > 0$ would suffice instead of $\nabla u_o \in L^\infty$.

8. Tangent maps

The appearance of singularities can be related to non-constant harmonic mappings of $(m-1)$-dimensional spheres, as in the case of locally minimizing weakly harmonic maps, cp. Schoen-Uhlenbeck [9 , Theorem III, p. 310]:

Theorem 8.1: Suppose $u : \mathbb{R}^m \times [0,t_o[\to N$ with uniformly finite energy $E(u(t)) < E_o < \infty$ is a locally regular solution to (1.4) which develops a singularity as $t \nearrow t_o$. Then there exist sequences $R_k \to 0$, $\bar{R}_\ell \to \infty$, $x_k \in \mathbb{R}^m$, $t_k \nearrow t_o$ such that

$$u_k(x,t) \equiv u(x_k + R_k\bar{R}_\ell x, t_k + R_k^2\bar{R}_\ell^2 t) \to u_\infty \text{ in } C^1_{loc}(\mathbb{R}^m \times]-\infty,0];)$$

it first $k \to \infty$ and then $\ell \to \infty$, where either

$$u_\infty(x,t) \equiv v_\infty(x/|x|) \ . \qquad (8.1)$$

is induced by a non-constant harmonic map $v_\infty : S^{m-1} \to N$, or

$$u_\infty(x,t) \equiv v_\infty\left(x/\sqrt{|t|}\right), \qquad (8.2)$$

where u_∞ is a non-constant solution to (1.4) in the half-space $\{t<0\}$ and homogenous on curves $t = cx^2$.

Proof: Suppose there exists R_o , $4R_o^2 < t_o$ such that for all $z_o = (x_o,t_o)$ there holds

$$\int_{T_{R_o}(z_o)} |\nabla u|^2 G_{z_o} \, dxdt < \varepsilon_o \ .$$

Then by Theorem 5.1 ∇u remains uniformly bounded as $t \nearrow t_o$, contradicting the hypothesis.
Thus, given a sequence of radii R_k, $R_k \to 0$ $(k \to \infty)$, there exist points $z_k = (x_k,t_k)$, $t_k < t_o$ such that

$$\int_{T_{R_k}(z_k)} |\nabla u|^2 G_{z_k} \, dxdt = \sup_{\substack{\bar{z}=(\bar{x},\bar{t}) \\ \bar{t}\le t_k, \bar{R}\le R_k \\ \bar{t}-4\bar{R}^2 \ge 0}} \int_{T_{\bar{R}}(\bar{z})} |\nabla u|^2 G_{\bar{z}} \, dxdt = \varepsilon_o \ .$$

Moreover, since $|\nabla u| \le C$ uniformly for $t \le \bar{t} < t_o$, it follows that $t_k \nearrow t_o$.

Rescale, letting

$$u_k(x,t) = u(x_k + R_k x, t_k + R_k^2 t) \ .$$

Then $u_k : \mathbb{R}^m \times]-\frac{t_k}{R_k^2}, 0[\to N$ solves (1.4) and satisfies

$$\sup_{\substack{\bar{z}=(\bar{x},\bar{t}) \\ \bar{t}\le 0, \bar{R}\le 1 \\ \bar{t}\ge 4\bar{R}^2 - t_k/R_k^2}} \int_{T_{\bar{R}}(\bar{z})} |\nabla u_k|^2 G_{\bar{z}} \, dxdt = \int_{T_1} |\nabla u|^2 G dxdt = \varepsilon_o \ .$$

By Theorem 5.4 the family (u_k) is uniformly bounded in C^1_{loc}. Passing to a subsequence we may assume that $u_k \to \bar{u}$ uniformly locally (and hence in C^1, cp. the proof of Theorem 6.1), where $\bar{u}: \mathbb{R}^m \times]-\infty,0] \to N$ is a non-constant, regular solution of (1.4) such that

$$\sup_{\substack{\bar{z}=(\bar{x},\bar{t})\\ \bar{t}\le 0,\bar{R}\le 1}} \int_{T_{\bar{R}}(\bar{z})} |\nabla\bar{u}|^2 G_{\bar{z}}\,dxdt = \int_{T_1} |\nabla\bar{u}|^2 G\,dxdt = \varepsilon_o .$$

Moreover, by Proposition 3.3 for any $\bar{z} = (\bar{x},\bar{t})$, $\bar{t} \le 0$, any $\bar{R} > 0$

$$\int_{T_{\bar{R}}(\bar{z})} |\nabla\bar{u}|^2 G_{\bar{z}}\,dxdt = \lim_{k\to\infty} \int_{T_{\bar{R}}(\bar{z})} |\nabla u_k|^2 G_{\bar{z}}\,dxdt$$

$$= \lim_{k\to\infty} \int_{T_{\bar{R}R_k}(x_k+R_k\bar{x},\,t_k+R_k^2\bar{t})} |\nabla u|^2 G_{(x_k+R_k\bar{x},\,t_k+R_k^2\bar{t})}\,dxdt$$

$$\le \lim_{k\to\infty} \sup_{x_o\in\mathbb{R}^m} \int_{T_1(x_o,t_o)} |\nabla u|^2 G_{(x_o,t_o)}\,dxdt \le C E_o$$

uniformly in $\bar{R}$, $\bar{z}$.

It follows that, letting

$$\Psi(R,\bar{u}) = \int_{T_R} |\nabla\bar{u}|^2 G\,dxdt$$

as above, we have by Proposition 3.3

$$\int_0^\infty \frac{d}{dR}\Psi(R;\bar{u})\,dR = \int_0^\infty \left|\frac{d}{dR}\Psi(R,\bar{u})\right| dR < \infty ;$$

and there exists a sequence $\bar{R}_k \ge 0$ such that

$$\frac{d}{dR}\Psi(\bar{R}_k;\bar{u}) \to 0 \quad (k\to\infty) .$$

Let

$$\bar{u}_k = \bar{u}_{\bar{R}_k} \equiv \bar{u}(\bar{R}_k x, \bar{R}_k^2 t) ;$$

then as in the proof of Lemma 3.2

$$\frac{d}{dR}\bar{\Psi}(\bar{R}_k,\bar{u}) = \frac{d}{dR}\bar{\Psi}(1,\bar{u}_k)$$

$$= 2\int_{T_1} \nabla\bar{u}_k\, \nabla(x\cdot\nabla\bar{u}_k + 2t\cdot\partial_t\bar{u}_k)\,G\,dxdt$$

$$= -2\int_{T_1} \partial_t\bar{u}_k(x\cdot\nabla\bar{u}_k + 2t\cdot\partial_t\bar{u}_k)\,G\,dxdt$$

$$-2\int_{T_1} \frac{x\cdot\nabla\bar{u}_k}{2t}\left(x\cdot\nabla\bar{u}_k + 2t\partial_t\bar{u}_k\right) G\,dxdt$$

$$= \int_{T_1} \frac{1}{|t|}\,(x\cdot\nabla\bar{u}_k + 2t\partial_t\bar{u}_k)^2\, G\,dxdt .$$

It follows that either

$$\partial_t \bar{u}_k \ , \ x \cdot \nabla \bar{u}_k \to 0 \quad \text{in} \quad L^2_{loc}$$

in which case (using Theorem 5.4 again)

$$\bar{u}_k \to \bar{u}\ (x,t) \equiv \bar{v}_\infty \left({}^{x}\!/_{|x|} \right)$$

converges to a map $\bar{u}_\infty$ induced by a non-constant harmonic map $\bar{v}_\infty : S^{m-1} \to N$;

or

$$\bar{u}_k \to \bar{u}_\infty$$

where $\bar{u}_\infty$ is a non-constant solution to (1.4) on $\mathbb{R}^m \times]-\infty, 0[$ with

$$\partial_t \bar{u}_\infty = \frac{x \cdot \nabla \bar{u}_\infty}{2|t|} \ ,$$

i.e.

$$\bar{u}_\infty(x,t) = \bar{v}_\infty \left({}^{x}\!/_{\sqrt{|t|}} \right) .$$

qed

Note that by Theorem 6.1 if a solution u of (1.4) behaves irregularly as $t \to \bar{t} \leq \infty$, necessarily a singularity must be encountered in finite time.

A natural question is whether homogenous solutions of the kind (8.2) may appear.

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