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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

SMR 281/21

COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS
(11 January - 5 February 1988)

NONLINEAR SUBDIFFERENTIAL ANALYSIS

PART I

Evolution Equation for the Eigenvalue Problem for the Laplace Operator with Respect to an Obstacle

A.MARINO
Dipartimento di Matematica
Universita degli Studi di Pisa
PISA
ITALY

EVOLUTION EQUATION FOR THE EIGENVALUE PROBLEM FOR THE LAPLACE OPERATOR WITH RESPECT TO AN OBSTACLE.

Čobanov G. - Marino A. - Scolozzi D.

INTRODUCTION

Many problems concerning evolution equations or inequations of parabolic type, as is well known, can very well be fitted in the important theory of monotone operators, eventually perturbed with suitable terms (see H.Brezis, "opérateurs maximaux monotones", NOTES DE MATHEMATICA (50), NORTH HOLLAND 1973).

This previous theory constitutes an important scheme of reference which often permits to interprete in a very significant manner some of the important aspects of the particular problem in consideration.

In the "variational" case, the fundamental requirement of the scheme of the previous theory are given by a Hilbert space H, a function f defined in H which, for example, is of the type:

$$f = \mathcal{E} + \Gamma$$

where $\mathcal{E}$ is a lower semi-continuous convex function and Γ is a function of class $C^{1,1}$, and by the equation:

$$(*) \qquad -U'(t) \in \partial\mathcal{E}(U(t)) + \operatorname{grad}\Gamma(U(t))$$

where $\partial\mathcal{E}(u)$ is the subdifferential at u, with respect to the scalar product in H, of the convex function $\mathcal{E}$. U is the unknown function of the real variable t.

We can say that (*) is the evolution equation associated to the function $\mathcal{E}+\Gamma$ and that U is a curve of maximal slope for $\mathcal{E}+\Gamma$.

It is well known that, making use of the nice and fairly natural definition of subdifferential, the evolution equation associated to $\mathcal{E}+\Gamma$ on a closed convex obstacle K can also be formulated in this manner; that is, the curve of maximal slope for the restriction of $\mathcal{E}+\Gamma$ to K. In this case, it is enough to substitute the function $\mathcal{E}$ by the function $\mathcal{E}+I_K$, where I_K is defined by $I_K(u) = 0$ if $u \in K$, $I_K(u) = +\infty$ if $u \in H\setminus K$.

$\mathcal{E} + I_K$ is again convex and lower semi-continuous.
In other words the definition of subdifferential allows us to include the closed convex constraint K (which, in general, is not regular) in the convex and semi-continuous function $\mathcal{E}$, which thus takes also the value $+\infty$.

There are several problems in which it is necessary to consider certain constraints V which are not convex. We are thus led to consider the following function defined on H:

$$f = \mathcal{E} + \Gamma + I_V$$

where $\mathcal{E}$ is convex and lower semi-continuous, Γ is of class $C^{1,1}$ and $I_V(u) = 0$ if $u \in V$, $I_V(u) = +\infty$ if $u \notin H\, V$.
In this case, we substitute the equation (*) by:

(**) $$-U'(t) \in \partial^- f(U(t))$$

where $\partial^- f(u)$ represents a natural extension (see (1.1)) of the notion of subdifferential mentioned earlier.
It is evident that f can not be expressed, in general, as a sum of a convex function and a regular function because $\{u \mid f(u) < +\infty\}$ is not a convex set if V is not convex.
In this case also, the constraint is included in the function f and this permits us to consider, as for the (*), non regular constraints provided that f belongs to suitable classes. A problem of this type has been considered in [8], to study geodesics with respect an obstacle.

In order to illustrate the problem considered in this paper, we consider a bounded open set Ω in $\mathbb{R}^n$, a suitable function $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, two functions ϕ_1 and ϕ_2 on Ω with $\phi_1 \le 0 \le \phi_2$ on $\partial\Omega$ and a number $\rho > 0$.

Setting

$$K = \{u \in L^2 \mid \phi_1 \le u \le \phi_2\},\quad S_\rho = \{u \in L^2 \mid \int_\Omega u^2(x)dx = \rho^2\}$$

we consider

$$f_o(u) = 1/2 \int_\Omega |Du|^2 dx + \int_\Omega \left[\int_0^u g(x,t)dt\right]dx \qquad \text{for } u \in H^1_o \cap K$$

$$f_o(u) = +\infty \quad \text{for } u \in L^2 \setminus H^1_o \cap K \qquad (Du = (D_{x_1}u, D_{x_2}u, \ldots, D_{x_n}u)$$

and f defined by

$$f(u) = f_o(u) \quad \text{if} \quad u \in S_\rho,\quad f(u) = +\infty \quad \text{if} \quad u \in L^2 \setminus S_\rho .$$

Clearly the functional f_o is of the type considered in (*), under suitable hypothesis on g.
But the functional f is not of the same type as the set $\{u \mid f(x) < +\infty\}$ is not convex.
We shall be interested in studying, for this functional, the problem:

$$-U'(t) \in \partial^- f(U(t)) .$$

We emphasize the fact that we are interested in considering the subdifferential $\partial^- f$ of this functional with respect to the scalar product of $L^2(\Omega)$.
For example in the classical case of the convex functionals:

$$\mathcal{E}(u) = 1/2 \int_\Omega |Du|^2 dx \quad \text{if} \quad u \in H^1_o(\Omega)$$

the evolution equation for $\mathcal{E}$ with respect to L^2 of type (*), coincides with the heat equation

$$U'(t) = \Delta U(t) \quad , \quad U(t) = 0 \quad \text{on } \partial\Omega ,$$

because, with respect to the scalar product L^2, we have $\partial\mathcal{E}(u) \neq \emptyset$ if and only if $\Delta u \in L^2(\Omega)$ and, if $\partial\mathcal{E}(u) \neq \emptyset$, we have $\partial\mathcal{E}(u) = \{-\Delta u\}$ where $\partial\mathcal{E}(u)$ denotes the subdifferential of the convex function $\mathcal{E}$ in the sense of (*).

In order to study our problem we shall introduce a class of functions, namely, the class C(p,q) (see (1.6)) which is more particular than the class of ϕ-convex functions introduced in [6], and which seems to us is particularly adapted to our goal.
We shall study the behaviour of functions of such a class under suitable constraints (see the Theorem (1.13) on stationary points with constraints): this gives raise to some problem of "non tangency" between the domain of the assigned function (that is, the set of the u on which it takes finite values) and the constraint (in the example mentioned the function is f_o and the constraint is S_ρ).
For the abstract equation (**) we shall use a theorem on evolution, valid for ϕ-convex functions (see (2.4)).
The results of this paper will be used also to obtain a multiplicity result (which we shall explain in [3]) for the solutions of the equation:

$$(***) \qquad 0 \in \partial^- f(u)$$

under some suitable symmetry assumptions.
We can say that the solutions u of (***) are the eigenfunctions of $-\Delta + g(x,\cdot)$ with respect to the obstacles ϕ_1 and ϕ_2.

§1. SUBDIFFERENTIAL UNDER CONSTRAINTS.

In this section we recall the notion of subdifferential of a function, which was already introduced, for example in [5], and we shall consider the class C(p,q) of functions, not necessarily continuous, but adapted to the problem with constraint considered in Section 4. It is concerned with a subclass of ϕ-convex functions treated in [6], which naturally have properties stronger that the general ϕ-convex functions. However, for some other problems this class is not sufficient enough (see for example [8][10][11][12]). We shall study some properties of the class of functions C(p,q) which are useful for the problem under consideration and, in particular, we shall show that a result on "constrained stationary points" holds.

Throughout this section we suppose that H is a Hilbert space with scalar product $\langle\ ,\ \rangle$ and norm $\|\ \|$, that W is an open subset of H and that f is a function from W into $\mathbb{R}\cup\{+\infty\}$; we use the notation $D(f) = \{u \in W \mid f(u) < +\infty\}$.

(1.1) DEFINITION. Let u be in D(f). We call subdifferential of f at u the set $\partial^- f(u)$ of all α in H, such that

$$\liminf_{v\to u} \frac{f(v) - f(u) - \langle\alpha, v-u\rangle}{\|v-u\|} \geq 0 .$$

f is said to be subdifferentiable at u if $\partial^- f(u) \neq \emptyset$; in this case we call (lower) sub-gradient of f at u the element with minimal norm $\mathrm{grad}^- f(u)$ of $\partial^- f(u)$ ($\partial^- f(u)$ is convex and closed as is easily seen).

If f is convex or regular this definition coincides with the usual one.

(1.2) DEFINITION. If E is a subset of H we call indicator function of E the function I_E defined as:

$$I_E(u) = 0 \quad \text{if} \quad u \in E \quad , \quad I_E(u) = +\infty \quad \text{if} \quad u \notin E \ .$$

If $u \in E$, then we call $\partial^- I_E(u)$ the cone of normals to E at u. (Evidently $0 \in \partial^- I_E(u)$ for all $u \in E$). If $u \in E$, then we call cone of tangents the set

$$T_u(E) = \{ v \in H \mid \langle v, \nu \rangle \leq 0 \quad \forall \ \nu \in \partial^- I_E(u) \} \ .$$

(Note that $\partial^- I_E(u)$ and $T_u(E)$ are dual cones; cf. for instance [16], p.237).

(1.3) REMARKS. a) If f and g are two functions from W into $R \cup \{+\infty\}$ and if $u \in D(f+g)$, then

$$\partial^- f(u) + \partial^- g(u) \subseteq \partial^-(f+g)(u) \ .$$

Simple examples show, in general, that equality is not true. It is easy to verify that if f is differentiable at u, then

$$\partial^- g(u) \neq \emptyset \quad \text{if and only if} \quad \partial^-(f+g)(u) \neq \emptyset$$

and in this case

$$\partial^-(f+g)(u) = \text{grad } f(u) + \partial^- g(u)$$

holds. Furthermore

$$\text{grad}^-(f+g)(u) = \text{grad } f(u) + P(-\text{grad } f(u))$$

where P is the projection of H onto the convex set $\partial^- g(u)$.

b) if E is a subset of H, $u \in D(f) \cap E$ and $\partial^-(f+I_E)(u) \neq \emptyset$ then $-\text{grad}^-(f+I_E)(u) \in T_u(E)$.

If furthermore f is differentiable at u, then

$$\text{grad}^-(f+I_E)(u) = -Q(-\text{grad } f(u)) \ ,$$

where Q is the projection of H onto the convex cone $T_u(E)$. //

It is expedient to recall here a theorem proved in [6] .

(1.4) THEOREM. Let the function $h : H \to R \cup \{+\infty\}$ with $D(h) \neq \emptyset$ be lower semi-continuous and bounded from below. If $u \in H$, then for every $\lambda > 0$ and every $\sigma > 0$ there exists $u' \in H$ with $\| u - u' \| < \sigma$, such that the function

$$v \in H \to h(v) + \frac{1}{2\lambda} \| v - u' \|^2$$

attains a unique minimum in H.

(1.5) COROLLARY. If the function f is lower semi-continuous, then the set $\{u \in D(f) \mid \partial^- f(u) \neq \emptyset\}$ is dense in $D(f)$.

An easy proof can be found in [6], but for the sake of completeness we recall it here.

PROOF (of corollary (1.5)): First we consider the case $W = H$ and f bounded from below. According to the preceding theorem, for fixed u in $D(f)$ there exists a sequence $(u'_k)_k$ converging to u, such that the function

$$v \in H \to f(v) + \frac{1}{2\lambda}\| v - u'_k \|^2 \qquad (\lambda = \frac{1}{k})$$

has a minimum. If v_k is the point which achieves this minimum, then according to a) in (1.3) $\partial^- f(v_k) \neq \emptyset$ since the norm of H is differentiable. Obviously

$$f(v_k) + \frac{k}{2}\| v_k - u'_k \|^2 \le f(u) + \frac{k}{2}\| u - u'_k \|^2 \quad .$$

On the other hand, one can suppose that $\| u - u'_k \| \le \frac{1}{k}$ (by taking $\sigma = \frac{1}{k}$). This implies that $\sup_k (k \| v_k - u'_k \|^2) < +\infty$. Hence $(v_k)_k$ converges to u.

The general case can be reduced to the previous in an obvious manner.//

We shall now introduce the class $C(p,q)$.

(1.6) DEFINITION. Let p and q be two real continuous functions defined

on $D(f)$.

f is said to be of class $C(p,q)$ if:

for every u in $D(f)$ with $\partial^- f(u) \neq \emptyset$ and every α in $\partial^- f(u)$ we have

$$f(v) \ge f(u) + \langle \alpha, v-u \rangle - [p(u)\|\alpha\| + q(u)]\| v - u \|^2 , \quad \forall v \in W$$

We note that the subdifferentiability of f at u is not required explicitly.

(1.7) REMARKS. a) It is clear that if $f_o : H \to R \cup \{+\infty\}$ is convex and if $g: W \to R$ is of class $C^{1,1}$, then $f_o + g$ is of class $C(0,q)$ with some suitable constant q.

b) It is easy to verify that the indicator function of a submanifold M in H of class C^2 (or $C^{1,1}_{loc}$) is of class $C(p,0)$ and

$$\partial^- I_M(u) = \{v \in H \mid \langle v,w \rangle = 0 \quad \forall w \in T_u(M)\} .$$

c) If f is of class $C(p,q)$ and is lower semi-continuous, then the following property of $\partial^- f$ holds:

if $(u_h)_h$ is a sequence in $D(f)$ converging to an u in $D(f)$, and for every h, $\alpha_h \in \partial^- f(u_h)$ and $(\alpha_h)_h$ converges weakly to α, then $\lim_h f(u_h) = f(u)$ and $\alpha \in \partial^- f(u)$.

This property is an obvious consequence of definition (1.6).

d) It is evident that for $f \in C(p,q)$, if u and v are in $D(f)$ and if $\alpha \in \partial^- f(u)$ and $\beta \in \partial^- f(u)$, then

$$\langle \beta-\alpha, v-u\rangle \geq -[p(u)\|\alpha\| + p(v)\|\beta\| + q(u) + q(v)]\|u - v\|^2 .$$

e) If $f \in C(p,q)$ and is lower semicontinuous, then there exist a and b in R such that

$$f(v) \geq -a - b\|v\|^2 \qquad \forall v \in W$$

Indeed, if $D(f) \neq \emptyset$, then according to (1.5) there exists u in D(f), such that $\partial^- f(u) \neq \emptyset$. The conclusion now follows from the definition (1.6).//

In (1.8) and (1.9) we set forth a simple but important property of functions of class C(p,q). A somewhat weaker property for a more general class of functions is proved in [6] .

For the sequel we set $B(u_o,\rho) = \{u \in H \mid \|u - u_o\| < \rho\}$.

(1.8) THEOREM. Let f be a lower semi-continuous function of class C(p,q). Suppose that $\sup p = \bar{p} < +\infty$ and $\sup q = \bar{q} < +\infty$. For u in H let $\delta(u) = d(u,D(f))$ and let

$$U = \{u \in W \mid \delta(u) < d(u,H-W), 2\bar{p}\delta(u) < 1\} .$$

Then there exists a function $\lambda_o : U \to R$, with $\lambda_o(u) > 0$ for every u of U, such that for u in U and λ in $]0,\lambda_o(u)[$, the function

$$v \to f(v) + \frac{1}{2\lambda}\|v - u\|^2$$

attains a unique minimum at a point $r_\lambda(u)$ (in D(f)). Moreover

$\lim_{\lambda\to 0} \|r_\lambda(u) - u\| = \delta(u)$. Finally, for an u and v in U and λ with $0 < \lambda \leq \lambda_o(u) \wedge \lambda_o(v)$, the inequality

$$\|r_\lambda(u) - r_\lambda(v)\|[1-\bar{p}(\|u-r_\lambda(u)\|+\|v-r_\lambda(v)\|) - 2\lambda\bar{q}] \leq \|u - v\|$$

holds.

PROOF. I) Given u in U, let ρ be a real number with $\delta(u) < \rho < d(u,H-W)$ and $2\bar{p}\rho < 1$. Since $B(u,\rho) \cap D(f) \neq \emptyset$ and since the inequality e) in (1.7) holds, one can verify easily that there exists a $\lambda_o > 0$ such that for every λ in $]0,\lambda_o]$ it follows that

$$\inf\{f(u) + \frac{1}{2\lambda}\|v - u\|^2 \mid v \in B(u,\rho)\} <$$

$$< \inf\{f(v) + \frac{1}{2\lambda}\|v - u\|^2 \mid v \in W-B(u,\rho)\} .$$

It is possible then to suppose that λ_o is so small that $2\bar{p}\rho + 2\bar{q}\lambda_o < 1$.

II) According to theorem (1.4) there exists a sequence $(u_k)_k$ converging to u, such that every function

$$v \to f(v) + \frac{1}{2\lambda}\|v - u_k\|^2$$

attains a minimum at some point v_k in $B(u,\rho)$ (one can redefine f with value $+\infty$ in $H - \overline{B(u,\rho)}$). Since the norm of H is differentiable, according to a) in (1.3) we have

$$\alpha_k = -\frac{1}{\lambda}(v_k - u_k) \in \partial^- f(v_k) .$$

Using d) of (1.7) we obtain:

$$\|v_k - v_h\|[1-\lambda(p(v_k)\|\alpha_k\| + p(v_h)\|\alpha_h\| + q(v_k) + q(v_h))] \le \|u_k - u_h\| .$$

Now $\lim_h \sup (\lambda p(v_k)\|\alpha_h\| = \lim_h \sup (p(v_h)\| v_h - u_h\|) \le \bar{p}\rho$ and $q(v_h)\lambda \le \lambda\bar{q}$ for every h. Since $2\bar{p}\rho + 2\bar{q}\lambda_o < 1$, $(v_h)_h$ is a Cauchy sequence and obviously its limit point v belongs to $B(u,\rho)$ and is a point of minimum.

III) Using again d) in (1.7) one can prove the uniqueness of this point and the inequality in the statement. Since ρ can be chosen arbitrarily close to $\delta(u)$, we have also proved that $\lim_{\lambda\to 0} \|r_\lambda(u) - u\| = \delta(u)$.//

These facts permits as to prove also the following statement:

(1.9) <u>THEOREM</u>. If f is lower semicontinuous and of class $C(p,q)$, then for every u_o in $D(f)$ there exists $\rho_o > 0$ and $\lambda_o > 0$, such that for every u in $B(u_o,\rho_o)$ and every λ in $]0,\lambda_o]$, the function

$$v \to f(v) + \frac{1}{2\lambda}\|v - u\|^2$$

admits one and only one point $r_\lambda(u)$ (in $D(f)$), at which a minimum is attained. Moreover the function r_λ: $B(u_o,\rho_o) \to D(f)$ is lipschitz continuous.

We recall an important property which we shall make use of also in Sections 3 and 4.

(1.10) <u>PROPOSITION</u>. Let f be lower semi-continuous.

a) If f is in $C(0,0)$, then it is convex on the convex parts of W (cf. also [6]).

b) If f is of class $C(0,q)$, then $D(f)$ is locally convex, i.e. for every u in $D(f)$ there exists a $\rho > 0$ such that $B(u,\rho) \cap D(f)$ is convex.

If furthermore $\sup q = \bar{q} < +\infty$, then for any $\bar{u}$ in H the function $v \to f(v) + \bar{q}\|v - \bar{u}\|^2$ is in $C(0,0)$ and hence is convex on the convex parts of W.

<u>PROOF</u>. To prove a) we consider u and v in $D(f)$ such that $u_t = u + t(v-u) \in W$ for every t in $[0,1]$.
Clearly it is enough to prove that $u_t \in D(f)$ and that

$$f(u_t) \le tf(v) + (1-t)f(u)$$

for all $t \in [0,1]$, such that $d(u_t,D(f)) < d(u_t,H-W)$. If now u_t has this property, then according to Theorem (1.8) (with $\bar{p} = \bar{q} = 0$), ρ fixed such that $d(u_t,D(f)) < \rho < d(u_t,H-W)$ and λ small enough, the function

$$v \to f(v) + \frac{1}{2\lambda}\|v - u_t\|^2$$

has a minimum in $B(u_t,\rho)$. Denoting by v_λ the point where this minimum is achieved we have:

$$\alpha_\lambda = -\frac{1}{\lambda}(v_\lambda - u_t) \in \partial^- f(v_\lambda)$$

and moreover $f(v) \ge f(v_\lambda) + \langle\alpha_\lambda, v-v_\lambda\rangle$ and $f(u) \ge f(v_\lambda) + \langle\alpha_\lambda, u-v_\lambda\rangle$.
Hence

$$tf(v) + (1-t)f(u) \ge f(v_\lambda) + \langle\alpha_\lambda, u_t-v_\lambda\rangle = f(v_\lambda) + \frac{1}{\lambda}\|u_t - v_\lambda\|^2 .$$

Since $\inf_\lambda f(v_\lambda) > -\infty$ we have $\limsup_{\lambda\to 0} \frac{\|u_t - v_\lambda\|^2}{\lambda} < +\infty$, i.e. $\lim_{\lambda\to\infty} v_\lambda = u_t$ and the assertion follows from the semicontinuity of f.
As to b), we have to note that the first part follows from the second and a), and that the second part is obvious.//

(If $\sup q = +\infty$, then the assertion of b) can be false for example if: $f(x) = \frac{1}{x^2}$).

In Sections 3 and 4 we need to consider functions of the type just discussed but under a regular constraint defined by a hypersurface.

For the sequel we need to study when a convex set K and a hypersurface M are not "tangent". We prefer however to introduce here the notion of tangency (and nontangency) between two arbitrary sets, since it enables us to obtain more general results than those contained in the present paper (cf. [9]).

(1.11) DEFINITION. If A and B are two subsets of H we say that they are tangent (externaly) at the point u of $A \cap B$ if and only if

$$(-\partial^- I_A(u)) \cap \partial^- I_B(u) \neq \{0\} .$$

We shall say that A and B are not tangential (externaly) if there does not exist $u \in A \cap B$ which satisfies the above property.

In the sequel we consider a hypersurface M of class C^1 in W and we denote by $\nu(u)$ one of the normals to M at u with $\|\nu(u)\| = 1$.

(1.12) REMARKS. Let K be a convex subset of H and let $u \in K \cap M$. It is clear that:

a) K and M are not tangent at u if and only if there exist u^+ and u^- in K, such that $\langle \nu(u), u^+ - u\rangle > 0$ and $\langle \nu(u), u^- - u\rangle < 0$.

b) If K and M are not tangent at a point u of $K \cap M$, then there exists a neighbourhood U of u, such that K and M are not tangent at any point of $K \cap M \cap U$.

(1.13) THEOREM. Suppose that $f_o : W \to R \cup \{+\infty\}$ is lower semicontinuous of class $C(0,q_o)$, where q_o is a real valued continuous function defined on $D(f_o)$. If K and M are not tangent at any point, then:

a) For every u in $M \cap D(f_o)$ we have $\partial^-(f+I_M)(u) = \partial^- f_o(u) + \partial^- I_M(u)$ (hence if $\partial^-(f_o+I_M(u) \neq \emptyset$, then also $\partial^- f_o(u) \neq \emptyset$).

In particular, ("constrained minimum theorem"): if at $u \in M \cap D(f_o)$ the minimum of f_o on M is achieved, then there exists a λ in R, such that $\lambda\nu(u) \in \partial^- f_o(u)$. (In particular however: $\lambda\nu(u) \neq \operatorname{grad}^- f_o(u)$).

b) There exist two continuous functions C_1, C_2: $M \cap D(f_o) \to R$ such that for every $u \in M \cap D(f_o)$, if $\alpha \in \partial^-(f_o+I_M)(u)$ and $\lambda \in R$ is such that $\alpha - \lambda\nu(u) \in \partial^- f_o(u)$, then we have $|\lambda| \le C_1(u)\|\alpha\| + C_2(u)$.

c) Furthermore, if for some $u \in M \cap D(f_o)$ and $\alpha \in \partial^-(f_o+I_M)(u)$ there exists a unique $\lambda \in R$ such that $\alpha - \lambda\,\nu(u) \in \partial^- f_o(u)$, then λ depends continuously on these u and α.

d) If M is locally of class $C^{1,1}$, then the function $f = f_o + I_M$ is of class $C(p,q)$ where p and q appropriate continuous functions defined on $D(f)$.

The proof is based on the following lemmas.

(1.14) LEMMA. Let the functions $g: H \to R \cup \{+\infty\}$ be convex. Let V be a hyperplane in H of equation $\langle v,e_o\rangle = c$, where $e_o \in H$, $\|e_o\| = 1$ and $c \in R$. If $g + I_V$ has a minimum at the point u_o of V, then for

$$V^+ = \{v \in H \mid \langle v,e_o\rangle > c\}, \quad V^- = \{v \in H \mid \langle v,e_o\rangle < c\}$$

$$\lambda^+(u_o) = \inf_{v \in V^+} \frac{g(v)-g(u_o)}{\langle v-u_o,e_o\rangle}, \quad \lambda^-(u_o) = \sup_{v \in V^-} \frac{g(v)-g(u_o)}{\langle v-u_o,e_o\rangle}$$

we have

a) $$\lambda^-(u_o) = \limsup_{\substack{v \to u_o \\ v \in V^-}} \frac{g(v)-g(u_o)}{\langle v-u_o,e_o\rangle} \le \lambda^+(u_o) = \liminf_{\substack{v \to u_o \\ v \in V^+}} \frac{g(v)-g(u_o)}{\langle v-u_o,e_o\rangle}.$$

b) For any real λ, with $\lambda^-(u_o) \le \lambda \le \lambda^+(u_o)$, we have $\lambda e_o \in \partial^- g(u_o)$.

c) If $D(g)$ and V are not tangent at u_o, then $-\infty < \lambda^-(u_o) \le \lambda^+(u_o) < +\infty$ and hence there exists λ in R such that $\lambda e_o \in \partial^- g(u_o)$.

PROOF. To prove a) we take v^- in $V^- \cap D(g)$ and v^+ in $V^+ \cap D(g)$ (if

one of these sets is empty, then a) is obvious). Let $t_o \in]0,1[$ be such that $v_o = t_o v^- + (1-t_o)v^+ \in V$. Then

$$g(u_o) \le g(v_o) \le t_o g(v^-) + (1-t_o)g(v^+).$$

Moreover $0 = \langle v_o-u_o,e_o\rangle = t_o\langle v^- - u_o,e_o\rangle + (1-t_o)\langle v \pm u_o,e_o\rangle$. Hence

$$\frac{g(v^-)-g(u_o)}{\langle v^- - u_o,e_o\rangle} \le \frac{g(v^+)-g(u_o)}{\langle v^+ - u_o,e_o\rangle}$$

which implies $\lambda^-(u_o) \le \lambda^+(u_o)$. The other equalities in a) are obvious (using again the convexity of g). To prove b) it is sufficient to note that if, for example, for some λ in R we have $\lambda \le \lambda^+(u_o)$, then

$$g(v) - g(u_o) \ge \langle v-u_o,\lambda e_o\rangle \quad \forall v \in V^+.$$

Reasoning as in a), it is easy to verify that under the hypotheses of c) we have indeed $-\infty < \lambda^-(u_o)$ and $\lambda^+(u_o) < +\infty$.

(1.15) LEMMA. Let A and B be two subsets of H and let $u_o \in A \cap B \cap D(f)$. Let there exist a neoghbourhood U of u_o and a mapping

$$\psi: U \cap A \cap D(f) \to B \cap D(f)$$

such that

$$\lim_{\substack{v \to u_o \\ v \in A \cap D(f)}} \frac{\|v-\psi(v)\|}{\|v-u_o\|} = 0, \quad \liminf_{\substack{v \to u_o \\ v \in A \cap D(f)}} \frac{f(v)-f(\psi(v))}{\|v-u_o\|} \ge 0.$$

Then $\partial^-(f + I_B)(u_o) \subseteq \partial^-(f + I_A)(u_o)$.

The statement follows directly from the definition of subdifferential.

(1.16) <u>LEMMA</u>. If $g: H \to R \cup \{+\infty\}$ is convex, if $u_o \in M \cap D(g)$ and if M and D(g) are not tangent at u_o, then with $T = u_o + T_{u_o}(M)$ we have $\partial^-(g + I_M)(u_o) = \partial^-(g + I_T)(u_o)$.

<u>PROOF</u>. We shall prove the inclusion $\partial^-(g + I_M)(u_o) \subseteq \partial^-(g + I_T)(u_o)$. The other follows in a similar way.

To this end, we construct a mapping $\phi: U \cap T \cap D(g) \to M \cap D(g)$ defined on a suitable neighbourhood U of u_o, which satisfies the hypotheses of lemma (1.15). Since M and D(g) are not tangent at u_o, there exist u^+ and u^- in D(g) with

$$\langle u^+ - u_o, \nu(u)\rangle > 0 \quad \text{and} \quad \langle u^- - u_o, \nu(u)\rangle < 0$$

We can suppose that $u_o = 0$, and that in a ball B with center u_o, M is the graph of some function $\phi: B \cap T \to R$ of class C^1 with $\phi(0) = 0$ and $\phi'(0) = 0$, by identifying H with $T \times R$. furthermore, we can suppose that $u^+ = (x^+,y^+)$ and $u^- = (x^-,y^-)$ are in B and $y^+ > \phi(x^+)$, $y^- < \phi(x^-)$, $y^+ > 0$, $y^- < 0$.

Now for $\phi^+(x,t) = ty^+ - \phi(x+t(x^+-x))$

we have

$$\phi^+(0,0) = 0 \quad , \quad \frac{\partial\phi^+}{\partial t}(0,0) = y^+ > 0 \ .$$

Hence there exist a neighbourhood U of 0 and a function t^+ of class C^1 defined on $U \cap T$, such that $\phi^+(x,t^+(x)) = 0$ on $U \cap T$. in a similar manner, we can find a function t^- of class C^1 on $U \cap T$ (restricting U if necessary), such that $t^-(x)y^- - \phi(x+t^-(x)(x^- - x)) = 0$. Now let us define for x in $U \cap T$

$$\psi(x) = \begin{cases} (x+t^+(x)(x^+-x),\ t^+(x)y^+) & \text{if} \quad \phi(x) \geq 0 \\ (x+t^-(x)(x^--x),\ t^-(x)y^-) & \text{if} \quad \phi(x) < 0 \ . \end{cases}$$

Obviously for x in $T \cap U$ with $\phi(x) \geq 0$, $\psi(x)$ is on the line segment with endpoints $v = (x,0)$ and $u^+ = (x^+,y^+)$ and moreover

$$g(\psi(v)) \leq g(u^+)\frac{\|v-\psi(v)\|}{\|v - u^+\|} + g(v)\frac{\|\psi(v)-u^+\|}{\|v - u^+\|} \ .$$

Similar inequality is obtained (with u^- instead of u^+) when $\phi < 0$. Taking into account that

$$\lim_{v \to u_o} \frac{\|v-\psi(v)\|}{\|v - u_o\|} = 0$$

as can easily be seen, and applying lemma (1.15) we can infer that

$$\partial^-(g + I_M)(u_o) \subseteq \partial^-(g + I_T)(u_o) \ .$$

<u>PROOF (of Theorem (1.13))</u>. To prove a), let $u_o \in M \cap D(f_o)$ be fixed and let U be a fixed ball with center at u_o and such that $U \subseteq W$ and

$\sup_U q = \bar{q} < +\infty$. We shall prove that for each α in $\partial^-(f_o + I_M)(u_o)$ there exists a real number λ such that $\alpha + \lambda\nu(u) \in \partial^- f_o(u_o)$. Indeed, for $\alpha \in \partial^-(f_o + I_M)(u_o)$ let

$$g(v) = f_o(v) - \langle\alpha, v-u_o\rangle + \bar{q}\,\|v - u_o\|^2 \;;$$

g is of class C(0,0) on U as is evident. Since g is lower semi-continuous (on U), according to a) in (1.10) g is convex on U. Hence $D(f_o) \cap U$ is convex. To obtain the statement, we can suppose, without loss of generality, that g is defined on the whole space H (with values in $R \cup \{+\infty\}$), (restricting U and extending g with value $+\infty$ outside if necessary), and that it is convex and lower semi-continuous. Obviously $0 \in \partial^-(g + I_M)(u_o)$ and hence, according to lemma (1.16), with $T = u_o + T_{u_o}(M)$, it follows that $0 \in \partial^-(g + I_T)(u_o)$, whence $g + I_T$ has a minimum at u_o. According to (1.14) there exists a real number λ such that $\lambda\nu(u_o) \in \partial^- g(u_o)$. As a consequence $\alpha + \lambda\nu(u_o) \in \partial^- f_o(u_o)$.

To verify b) let us consider a u_o in $M \cap D(f_o)$. There exist u^+ and u^- in $D(f_o)$ with $\langle u^+ - u, \nu(u)\rangle > 0$ and $\langle u^- - u, \nu(u)\rangle < 0$ for all u in some appropriate neighbourhood U of u_o on M.
Now if $u \in U \cap D(f_o)$, $\alpha \in \partial^-(f_o + I_M)(u)$ and $\alpha - \lambda\nu(u) \in \partial^- f_o(u)$ for some $\lambda \in R$, then, since f_o is of class $C(0,q_o)$, obviously we have

$$-\lambda \geq \frac{1}{\langle u^- - u, \nu(u)\rangle}\left[f_o(u^-) - f_o(u) - \langle\alpha, u^- - u\rangle + q_o(u)\,\|u^- - u\|^2\right]$$

and

$$-\lambda \leq \frac{1}{\langle u^+ - u, \nu(u)\rangle}\left[f_o(u^+) - f_o(u) - \langle\alpha, u^+ - u\rangle + q_o(u)\,\|u^+ - u\|^2\right].$$

Taking into account that f_o is locally bounded from below, (and restricting U if necessary) we obtain $|\lambda| \leq \bar{C}_1\,\|\alpha\| + \bar{C}_2$ for $u \in U$ with some real constants $\bar{C}_1$ and $\bar{C}_2$. The rest is obvious (using a partition of unity).

Clearly c) follows from the inequalities in b) and from c) in (1.7).

At last, to prove d) let us note that if M is locally of class $C^{1,1}$, then there exists a continuous function $p_o : M \to R$ such that $\langle\nu(u), v-u\rangle \leq p_o(u)\|v-u\|^2$ for u and v in M. Let now u be in $M \cap D(f_o)$ and let $\alpha \in \partial^-(f_o + I_M)(u)$. According to b) there exists $\lambda \in R$, such that $\alpha + \lambda\nu(u) \in \partial^- f_o(u)$. Hence if $v \in M$, then:

$$f_o(v) - f_o(u) - \langle\alpha, v-u\rangle = f_o(v) - f_o(u) - \langle\alpha+\lambda\nu(u), v-u\rangle + \lambda\langle\nu(u), v-u\rangle \geq$$
$$\geq -q_o(u)\|v-u\|^2 - |\lambda|\,p_o(u)\|v-u\|^2 \;.$$

The statement now follows using b). //

§2. SOME ABSTRACT RESULTS ON THE EVOLUTION PROBLEM

In this section we recall some results from [6] , which will be useful in the sequel.

As in §1, H is a Hilbert space, W is an open subset of H and $f: W \to R \cup \{+\infty\}$ is a given function.

(2.1) DEFINITION. Let I be an interval in R with $\overset{\circ}{I} \neq \emptyset$ and let U be a mapping from I into H.

U is said to be a curve of maximal (descending) slope for f if:

a) U is absolutely continuous on compact subsets of I;

b) $U(t) \in D(f)$ and $-U'(t) \in \partial^- f(U(t))$ almost everywhere on I;

c) $f \circ U$ is non increasing on I.

Let us note that the definition given in (2.1) of [6] is slightly more general then the one above, since the property required now in a) is somewhat more restrictive.

The above definition however is sufficient for the scope of the present paper, since chosen a point u in D(f), one can prove (see theorem (2.4)), that there exists a U satisfying this more restrictive definition with u as the initial point.

(2.2) DEFINITION. Let ϕ be a real valued continuous function defined on $D(f) \times R \times R$.

The function f is said to be ϕ-convex if and only if for any u in D(f) with $\partial^- f(u) \neq \emptyset$ and any α in $\partial^- f(u)$ we have

$$f(v) \geq f(u) + \langle \alpha, v-u \rangle - \phi(u, f(u), \|\alpha\|) \|v-u\|^2 \qquad \forall v \in W$$

f is said to be ϕ-convex of order r $(r \geq 0)$ if for some appropriate real function ϕ_o defined on $D(f) \times R$ the inequality

$$\phi(u,s,t) \leq \phi_o(u,s)(1+t^r)$$

for t non negative holds.

(2.3) REMARK. The functions of class C(p,q) introduced in (1.6) are ϕ-convex of order 1.

(2.4) THEOREM. If f is lower semi-continuous and ϕ-convex of order 2 (in particular, if f is of class C(p,q)),then:

For every u_o in D(f) there exist $T > 0$ and an unique curve U of maximal slope for f defined on $[0,T[$ with $U(0) = u_o$.

Furthermore:

a) for every t in $]0,T[$ we have $\partial^- f(U(t)) \neq 0$ and

$$U'_+(t) = - \operatorname{grad}^- f(U(t))$$

($U'_+(t)$ is the right-hand derivative of U at t); moreover $t \to U'_+(t)$ is right continuous in $]0,T[$;

b) $f \circ U(t_2) - f \circ U(t_1) = -\int_{t_1}^{t_2} \|\text{grad}^- f(U(t))\|^2 dt \quad \forall t_1, t_2 \in [0,T[$;

c) if I is the maximal interval of existence of U with $0 \in I$ and if $\sup I < +\infty$, then at least one of the following holds

$$\lim_{t \to \sup I} f \circ U(t) = -\infty \ , \qquad \lim_{t \to \sup I} d(U(t), W^c) = 0$$

(if f is of class $C(p,q)$, then, according to e) in (1.7), the first alternative is excluded).

d) if $(u_h)_h$ with $\sup_h f(u_h) < +\infty$ is a sequence converging to $u \in D(f)$ and if $U:[0,T] \to H$ is a maximal slope curve for f with $U(0) = u$, then:

- the maximal slope curve U_h for f with $U_h(0) = u_h$ is defined on $[0,T]$ for h sufficiently big:
- U_h converges to U uniformly on $[0,T]$;
- $f \circ U_h(t)$ converges to $f \circ U(t)$ for every t in $]0,T]$.

PROOF: For a) see (3.2) of [6] , for b) and c) see (3.4) of [6] , for d) see (3.7) of [6] . //

(2.5) PROPOSITION. If f is lower semi-continuous and of class $C(p,q)$, and if the curve $U: I \to H$ is absolutely continuous on the compact subsets of I and such that $-U'(s) \in \partial^- f(U(s))$ almost everywhere on I, then:

a) $\|U'(t)\| \leq \|U'(s)\| \exp\left(2\int_s^t [q(U(r)) + p(U(r))\|U'(r)\|]\,dr\right)$ for almost all s and t in I with $s < t$.

b) U is a curve of maximal slope for f.

PROOF: a) can be deduced with standard argument from d) in (1.7). See also (3.5) of [6] .

To prove b) let us note that if $[a,b] \subseteq I$, then the inequality

$$f \circ U(s+h) \geq f \circ U(s) - \langle U'(s), U(s+h) - U(s)\rangle +$$
$$+ [p(U(s))\|U'(s)\| + q(U(s))]\|U(s+h) - U(s)\|^2$$

and a) above imply the existence of a real number c, such that

$$f \circ U(s+h) - f \circ U(s) \geq c|h|$$

for almost all s in $[a,b]$ and all h with $s + h$ in $[a,b]$.
Since f is lower semi-continuous, this inequality holds for all s in $[a,b]$. Hence $f \circ U$ is lipschitz continuous on $[a,b]$ with constant c. Moreover we have

$$(f \circ U)'(s) = -\|U'(s)\|^2$$

a.e. in I, as follows from the following simple remark. //

(2.6) REMARK. Given $U: I \to W$ and $t_0 \in \mathring{I}$, with $U(t_0) \in D(f)$ and $\partial^- f(U(t_0)) \neq \emptyset$ if $U'(t_0)$ and $(f \circ U)'(t_0)$ exist, then for all α in $\partial^- f(U(t_0))$ we have

$$(f \circ U)'(U(t_0)) = \langle \alpha, U'(t_0)\rangle \ .$$

§3. A FUNCTIONAL OF CLASS C(p,q)

In this section we first of all verify that the functionals f_o and f_1, defined respectively in (3.3) and (3.7) below, which are of frequent use in the analysis, are of class C(0,q) under the assumptions (3.1), which seem natural for this purpose. Herefrom we infere that under suitable conditions, the functional f defined in (3.7) is of class C(p,q).

One can consider more general assumptions that those in (3.1), (cf. for instance [7]) under which the functionals f_o and f_1 would be in the slightly wider class of ϕ-convex functions of order 0. We have not considered this case here, since in our opinion it would require a rather intricate and hard version of theorem (1.13), even though this would enable one to arrive nevertheless at the results we are interested in. in this case f would be ϕ-convex of order 1, and not necessarily of class C(p,q).

For a general theory concerning the ϕ-convex functions on certain constraints see [9] .

Let Ω be a bounded open set in R^n. We consider in the sequel a function $G: \Omega \times R \to R$ satisfying the following assumptions (all or some of them):

(3.1)

(G.1) for almost all x, G(x,t) is continuous with respect to t, and for every t it is measurable with respect to x; moreover, there exist a in L^1 and b in R, such that for almost all x

$$G(x,t) \geq -a(x) - bt^2$$

holds for all t;

(G.2) for almost all x, $\frac{\partial G}{\partial t} = g(x,t)$ exists for all t, g is continuous with respect to t for almost all x, and is measurable with respect to x for all t; moreover, there exists c in R such that for almost all x and every t_1 and t_2 in R

$$G(x,t_2) \geq G(x,t_1)+g(x,t_1)(t_2-t_1)-c(t_2-t_1)^2$$

(G.3) $G(\cdot,t) \in L^1$ for every t in R.

Obviously the validity of the assumptions (3.1) is ensured by the following:

(3.2) for almost all x, G is of class C^2 with respect to t; for all t it is measurable with respect to x and there exists a c_1 in R such that $G_{tt}(x,t) \geq -c_1$; moreover $G(\cdot,t) \in L^1$ for every t and $G_t(\cdot,0) \in L^2$.

On the space $L^2(\Omega)$ with the usual scalar product $\langle u,v\rangle = \int_\Omega u(x)v(x)dx$ and norm $\|u\| = (\langle u,u\rangle)^{\frac{1}{2}}$ we consider the functional $f_1: L^2(\Omega) \to R \cup \{+\infty\}$ defined by

$$(3.3)\quad \begin{cases} f_1(u) = \frac{1}{2}\int_\Omega |Du|^2dx + \int_\Omega G(x,u(x))dx & \text{if } u \in H_o^1(\Omega) \\ f_1(u) = +\infty & \text{if } u \in L^2(\Omega)\setminus H_o^1(\Omega) \end{cases}$$

$(Du = (D_{x_1}u, D_{x_2}u, \ldots, D_{x_n}u))$.

(3.4) REMARKS. It is obvious that:

a) under the assumptions (G.1) of (3.1), f_1 is well defined, lower semi-continuous and moreover

$$D(f_1) = \{u \in H_o^1 \mid G(\cdot,u(\cdot)) \in L^1(\Omega)\} ;$$

b) under the assumptions (G.1) and (G.2) of (3.1), the function $g(x,u)(v-u)$ is upper semi-integrable for every u and v of L^2 with $G(x,v) \in L^1$;

c) under the assumptions (3.1) the function $g(x,u)u$ is lower semi-integrable for every $u \in L^2$.

d) under the assumption (G.2) in (3.1), for any t_1, t_2 and τ in R with $0 \leq \tau \leq 1$ we have

$$G(x,t_1+\tau(t_2-t_1)) \leq (1-\tau)G(x,t_1) + \tau G(x,t_2)+\tau(1-\tau)c(t_2-t_1)^2 .$$

(3.5) <u>THEOREM</u>. Under the assumptions (G.1) and (G.2) of (3.1):

a) - $D(f_1)$ is convex and besides if u and v are in $D(f_1)$ and if $w \in H^1_o$ is such that $u \wedge v \leq w \leq u \vee v$ then $w \in D(f_1)$;

- if u and v are in $D(f_1)$, then for every τ in $[0,1]$

$$f_1((1-\tau)u+\tau v) \leq (1-\tau)f_1(u) + \tau f_1(v) + \tau(1-\tau)c \int_\Omega (v-u)^2 dx .$$

b) if u and v are in $D(f_1)$, then:

b_1) $$\lim_{t\to 0^+} \frac{f_1(u+t(v-u)-f_1(u)}{t} = \int_\Omega Du\, D(v-u)dx + \int_\Omega g(x,u)(v-u)dx$$

b_2) $$f_1(v) \geq f_1(u) + \int_\Omega Du\, D(v-u)dx + \int_\Omega g(x,u)(v-u)dx - \int_\Omega (v-u)^2 dx$$

c) if u and v are in $D(f_1)$ and $\partial^- f_1(u) \neq \emptyset$ then $g(x,u)(v-u) \in L^1$.

d) if $u \in D(f_1)$ and $\alpha \in L^2$, then $\alpha \in \partial^- f_1(u)$ if and only if

$$\int_\Omega Du\, D(v-u)dx + \int_\Omega g(x,u)(v-u)dx \geq \int_\Omega \alpha (v-u)dx \quad \forall v \in D(f_1)$$

e) if $u \in D(f_1)$ and $\alpha \in \partial^- f_1(u)$ then

$$f_1(v) \geq f_1(u) + \langle \alpha, v-u \rangle - c\, \|v-u\|^2 \qquad \forall x \in D(f_1)$$

i.e. f_1 is of class $C(0,c)$.

<u>PROOF</u>: a) The assumption (G.1) in (3.1) implies that $G(x,u)$ is lower semi-integrable for any u in H^1_o. It is enough then to use d) in (3.4).

b) if u and v are in $D(f_1)$, then $G(x,u)$ and $G(x,v)$ are integrable and (G.2) in (3.1) implies b_2). The inequality

$$G(x,u+t(v-u))-G(x,u) \leq t[G(x,v)-G(x,u)] + t(1-t)c(v-u)^2$$

and Fatou's lemma imply

$$\limsup_{t\to 0^+} \frac{f_1(u+t(v-u))-f_1(u)}{t} \leq \int_\Omega Du\, D(v-u)dx + \int_\Omega g(x,u)(v-u)dx$$

Now b_1) follows from b_2).

If u and v are in $D(f_1)$ and $\alpha \in \partial^- f_1(u)$, then obviously

$$\liminf_{t\to 0^+} \frac{f_1(u+t(v-u))-f_1(u)}{t} \geq \langle \alpha, v-u\rangle .$$

Now from b_2) and b) in (3.4) follow c) and d). The rest is obvious.//

(3.6) THEOREM. Under the assumptions (3.1), besides the conclusions of theorem (3.5), one has:

a) $D(f_1) \supseteq H_o^1 \cap L^\infty$;

b) if $u \in D(f_1)$, then

$$\partial^- f_1(u) \neq \emptyset \iff \begin{cases} g(x,u) \in L^1 \text{ and in the sense of} \\ \text{distributions: } -\Delta u + g(x,u) \in L^2 \end{cases}$$

(under the same conditions, if g does not depend on x, then $u \in H^2$ as has been proved in [1]). For the converse implication the assumptions (G.1) and (G.2) are sufficient: these ensure that $-\Delta u + g(x,u) \in \partial^- f_1(u)$). $\partial^- f_1(u)$ has the unique element: $\alpha = -\Delta u + g(x,u)$.

PROOF: a) follows from d) in (3.4).

Let us suppose now that $u \in D(f_1)$ and $\alpha \in \partial^- f_1(u)$.

From c) in (3.5) it follows that $g(x,u)u$ is in L^1 since $0 \in D(f_1)$ and hence $g(x,u)v$ is in L^1 for every $v \in D(f_1)$. From (G.2) of (3.1) follows that $g(x,u)v$ is upper semi-integrable for all $v \in L^2$ with $G(x,v) \in L^1$. Next setting $v = 1$ and $v = -1$ we obtain $g(x,u) \in L^1$. We deduce, from

condition d) of (3.5) considered for $v \in C_o^\infty$, that if $\alpha \in \partial^- f_1(u)$ than $\alpha = -\Delta u + g(x,u)$ in the sense of distributions (if $v \in C_o^\infty$ then also $tv \in C_o^\infty$ for every t).

Thus the implication from left to right in b) is proved. To complete b), let us suppose that $u \in D(f_1)$, that $g(x,u)$ is integrable and in the sense of distributions $\alpha = -\Delta u + g(x,u)$ is in L^2.

To prove that $\alpha \in \partial^- f_1(u)$, due to d) in (3.5) it is enough to show that

$$\int_\Omega Du\, D(v-u)dx + \int_\Omega g(x,u)(v-u)dx \geq \int_\Omega \alpha\,(v-u)dx \quad \forall v \in D(f_1)$$

(in virtue of b) in (3.4) this expression has sense).

According to the assumptions, the identity

$$\int_\Omega Du\, Dw\, dx + \int_\Omega g(x,u)w dx = \int_\Omega \alpha\, w dx$$

holds for $w \in C_o^\infty$. If now $w \in H_o^1 \cap L^\infty$ there exists a sequence $(w_k)_k$ in C_o^∞ converging to w in H_o^1 and almost everywhere, and such that $\sup_k \|w_k\|_{L^\infty} < +\infty$. This implies that the identity holds for every w in $H_o^1 \cap L^\infty$ also, since $|g(x,u)w_k| \leq |g(x,u)| \sup_k \|w_k\|_{L^\infty}$.

Let us consider now $w \in H_o^1$ such that $g(x,u)w$ is upper semi-integrable. Clearly there exists a sequence $(w_k)_k$ in $H_o^1 \cap L^\infty$ converging to w in H_o^1 and almost everywhere, and such that $w_k^+ \leq w^+$ and $w_k^- \leq w^-$, we have also

$$g(x,u)w_k \leq (g(x,u)w_k)^+ = (g(x,u))^+ w_k^+ + (g(x,u))^- w_k^- \leq$$

$$\leq (g(x,u))^+ w^+ + (g(x,u))^- w^- = (g(x,u)w)^+ .$$

Fatou's lemma now implies $\int_\Omega Du\, Dw\, dx + \int_\Omega g(x,u)w dx \geq \int_\Omega \alpha\, w dx$.
Since $g(x,u)(v-u)$ is upper semi-integrable for v in $D(f_1)$, this completes the proof. //

In the sequel we consider two measurable functions ϕ_1 and ϕ_2 from Ω in $\tilde{R} = R \cup \{-\infty,+\infty\}$ with $\phi_1 \leq \phi_2$ a.e. on Ω.

(3.7) DEFINITIONS. Let

$K = \{u \in L^2 \mid \phi_1 \leq u \leq \phi_2 \text{ a.e. } \Omega\}$ (K is convex);

$u_k = (\phi_1 \vee 0) \wedge \phi_2$; $S_\rho = \{u \in L^2 \mid \int_\Omega u^2(x)dx = \rho^2\}$;

$f_o = f_1 + I_K$, $f = f_o + I_{S_\rho}$ (cf. definition (1.2)).

(3.8) In the sequel we use the following condition:

(ϕ.1) $\phi_1, \phi_2 \in H^1$; $g(x,\phi_1),\ g(x,\phi_2) \in L^1$

There exists a $\bar{c} > 0$ such that for all $w \in C_o^\infty$ with $w \geq 0$

$$\int_\Omega D\phi_1\, Dw\, dx + \int_\Omega g(x,\phi_1)w dx \leq \bar{c}\, \|w\|_{L^2}$$

$$\int_\Omega D\phi_2\, Dw\, dx + \int_\Omega g(x,\phi_2)w dx \geq -\bar{c}\|w\|_{L^2} .$$

(The last two inequalities are of the type indicated in [2] and [15] and are satisfied, if $-\Delta\phi_1 + g(x,\phi_1)$ and $-\Delta\phi_2 + g(x,\phi_2)$ taken in the sense of distributions are in L^2).

(3.9) REMARK. If $u \in K$ and $\beta \in L^2$, then $\beta \in \partial^- I_K(u)$ if and only if:

$\beta(x) \leq 0$ for almost all x , such that $\phi_1(x) = u(x) < \phi_2(x)$

$\beta(x) = 0$ for almost all x , such that $\phi_1(x) < u(x) < \phi_2(x)$

$\beta(x) \geq 0$ for almost all x , such that $\phi_1(x) < u(x) = \phi_2(x)$

(cf. definition (1.2)). The proof is very simple.

(3.10) PROPOSITION. a) Under the conditions (G.1) and (G.2) in (3.1) the following hold:

a_1) the function $f_o = f_1 + I_K$ is of class $C(0,c)$ (where c is the number which appears in (G.2)) and $D(f_o) = D(f_1) \cap K$.

a_2) if $u,v \in D(f_o)$ and $\partial^- f_o(u) \neq \emptyset$, then $g(x,u)(v-u) \in L^1$.

a_3) if $u \in D(f_o)$ and $\alpha \in L^2$, then $\alpha \in \partial^- f_o(u)$ if and only if

$$\int_\Omega Du\, D(v-u)dx + \int_\Omega g(x,u)(v-u)dx \geq \int_\Omega \alpha(v-u)dx \quad \forall v \in D(f_o)$$

b) Under the assumptions (3.1), if ϕ_1 and ϕ_2 are in H^1, then

b_1) $D(f_o) \neq \emptyset \Leftrightarrow u_k \in D(f_1) \Leftrightarrow \phi_1^+ , -\phi_2^- \in D(f_1)$

b_2) $D(f_o) \neq \emptyset \Rightarrow D(f_o)$ is dense in K (with respect to L^2)

c) Under the assumptions (3.1) and (3.8), for every u in $D(f_o)$:

$$\partial^- f_o(u) = \partial^- f_1(u) + \partial^- I_K(u) .$$

PROOF: a_1) follows from the inequality

$$f_1(v) \geq f_1(u) + \liminf_{t \to 0^+} \frac{f_1(u+t(v-u))-f_1(u)}{t} - c\,\|v-u\|^2$$

(see b) in (3.5)) which is true for f_o also, since K is convex and from the inequality

$$\liminf_{t \to 0^+} \frac{f_o(u+t(v-u)-f_o(u)}{t} \geq \langle \alpha, v-u\rangle$$

for every u, v in $D(f_o)$ and every α in $\partial^- f_o(u)$.
Also a_3) follows from this last inequality and b) in (3.5).

Finally a_2) follows from a_3) and b) in (3.4).

b_1) follows from a) in (3.5).

To prove b), let u be an element of K and let $(u_h)_h$ be a sequence in $H_o^1 \cap L^\infty$ converging to u in L^2. As we already know $u_h \in D(f_1)$ and $v_h = (u_k \vee \phi_1) \wedge \phi_2$ is still in $D(f_1)$ since $u_h \wedge (-\phi_2) \leq v_h \leq u_h \vee \phi_1$; besides $(v_h)_h$ also converges to u since $u \in K$.

Finally to prove c), let us note that if $\alpha \in \partial^- f_o(u)$, then the inequality in a_3) holds. Then, as we shall see subsequently, (see (3.6) in

[3]), $g(x,u) \in L^1$ and $\alpha_1 = -\Delta u + g(x,u) \in L^2$ in the sense of distributions. Let us note that this kind of regularity result cen be inferred from theorem II.1 in [2] if g satisfies certain growth restrictions.

Hence $\alpha_1 \in \partial^- f_1(u)$ according to theorem (3.6). From the inequality a_3) it follows that $\alpha - \alpha_1 \in \partial^- I_K(u)$. //

In the variational inequality in a_3) of (3.10) the set $D(f_o)$ of the test functions depends on G. As in the case of f_1 one can substitute this set with another one that does not depend on G. This will turn useful, for example, in (4.2).

(3.11) REMARK. Let $K_o = \{(\phi_1 \vee w) \wedge \phi_2 \mid w \in C_o^\infty(\Omega)\}$.
Suppose that the assumptions (3.1) hold and that ϕ_1 and ϕ_2 are in H^1. Then

a) If $D(f_o) \neq \emptyset$ then $K_o \subseteq D(f_o)$ and moreover for every v in $D(f_o)$ there exists a sequence $(v_h)_h$ in K_o converging to v in L^2 and such that also $(f_o(v_h))_h$ tends to $f_o(v)$.

b) If $u \in D(f_o)$ and if $\alpha \in L^2$, then $\alpha \in \partial^- f_o(u)$ if and only if

$$\int_\Omega Du\, D(v-u)dx + \int_\Omega g(x,u)(v-u)dx \geq \int_\Omega \alpha\,(v-u)dx \qquad \forall v \in K_o .$$

PROOF: a) If $D(f_o) \neq \emptyset$ then ϕ_1^+ and $-\phi_2^-$ are in $D(f_1)$.
For any $w \in C_o^\infty$ one has $w \wedge (-\phi_2^-) \leq (\phi_1 \vee w) \wedge \phi_2 \leq w \vee \phi_1^+$. Since $C_o^\infty \subseteq D(f_1)$, according to a) in (3.5), $(\phi_1 \vee w) \wedge \phi_2 \in D(f_1)$.
Let now $v \in D(f_o)$. Let us suppose first that $v = (\phi_1 \vee w) \wedge \phi_2$ with $w \in H_o^1 \cap L^\infty$.

Then there exists a sequence $(w_h)_h$ in C_o converging to w in H^1 with $\sup_h \|w_h\| \le s < +\infty$. Clearly $v_h = (\phi_1 \vee w_h) \wedge \phi_2 \in K_o$ and $(v_h)_h$ tends to v in H^1. In order to see that

$$\lim_{h\to\infty} \int_\Omega G(x,v_h)dx = \int_\Omega G(x,v)dx$$

let us note that $(-s)\wedge(-\phi_2^-) \le v_h \le s\vee\phi_1^+$ and besides $G(x,(-s)\wedge(-\phi_2^-))$ and $G(x,s\vee\phi_1^+)$ obviously are integrable. The conclusion now follows from d) in (3.4).

In the general case, given $v \in D(f_o)$ one can consider $v_h = (\phi_1 \vee w_h) \wedge \phi_2$ with $w_h = (-h \vee v) \wedge h \in H_o^1 \cap L^\infty$. Then obviously $(v_h)_h$ converges in H^1 to v and moreover $\phi_2 \wedge v \vee 0 \le v_h \le \phi_1 \vee v \vee 0$.

One can deduce as before that $(f_o(v_h))_h$ tends to $f_o(v)$.

b) It is enough to use a) taking also into account (G.2). //

As regards the functional $f = f_o + I_{S_\rho}$, theorem (1.13) allows easily to conclude that f is of class $C(p,q)$ provided that $D(f_o)$ and S_ρ are not tangent at any point, as now we proof in (3.13).

(3.12) <u>REMARK</u>. Suppose that all the assumptions (3.1) hold, that ϕ_1 and ϕ_2 are in H^1, and that $u \in D(f_o) \cap S_\rho$. Then the following facts are equivalent:

a) $D(f_o)$ and S_ρ are tangent at u.

b) K and S_ρ are tangent at u.

c) $u = u_k$ (i.e. $-u \in \partial^- I_K(w)$) or

meas $(\{x: \phi_1(x) < u(x) < 0\} \cup \{0 < u(x) < \phi_2(x)\}) = 0$

(i.e. $u \in \partial^- I_K(u)$).

The simple verifaction follows from the fact that $D(f_o)$, if nonempty, is dense in K and from the characterization of $\partial^- I_K(u)$ given in (3.9).

An analysis of the non tangency between $D(f_o)$ and S_ρ in terms of ϕ_1, ϕ_2 and λ is carried out in section 2 of [3] .

We now state the following theorem which can easily be obtained from theorem (3.10) using of theorem (1.13).

(3.13) <u>THEOREM</u>. a) If the assumptions (G.1) and (G.2) in (3.1) hold and if $D(f_o)$ and S_ρ are not tangent at any point, then

a_1) $f = f_o + I_{S_\rho}$ is of class $C(p,q)$ with p and q (suitable) appropriate continuous functions on $D(f) = D(f_o) \cap S_\rho$;

a_2) for any u in $D(f)$

$$\partial^- f(u) = \partial^- f_o(u) + \{\lambda u: \lambda \in \mathbb{R}\}$$

a_3) if $u \in D(f)$ and $\alpha \in L^2$, then $\alpha \in \partial^- f(u)$ if and only if there exists λ in $\mathbb{R}$ such that:

$$\int_\Omega Dx\, D(v-u)dx + \int_\Omega g(x,u)(v-u)dx + \lambda \int_\Omega u(v-u)dx \ge \int_\Omega \alpha\,(v-u)dx \quad \forall v \in D(f_o)$$

This variational inequality is equivalent to $\alpha - \lambda u \in \partial^- f_o(u)$. (The implication $\Leftarrow$ is also valid when $D(f_o)$ and S_ρ are tangent at u).

b) If (3.1) and (ϕ.1) in (3.8) hold and if $D(f_o)$ and S_ρ are not tangent at any point, then for every u in D(f):

b_1) $\partial^- f(u) = \partial^- f_1(u) + \partial^- I_K(u) + \{\lambda u : \lambda \in R\}$

b_2) $\partial^- f(u) \neq \emptyset \Leftrightarrow$ $\begin{cases} g(x,u) \in L^1 \text{ and } -\Delta u + g(x,u) \text{ taken in the} \\ \text{sense of distributions belong to } L^2. \end{cases}$

(3.14) <u>REMARK</u>. Suppose that the conditions (3.1) hold and that ϕ_1 and ϕ_2 belong to H^1. Let $u \in D(f)$ such that:

$$\text{meas } (\{x \in \Omega : u(x) \neq 0,\ \phi_1(x) < u(x) < \phi_2(x)\}) > 0$$

(in particular $D(f_o)$ and S_ρ are not tangent at u according to (3.12)). Then for every α in $\partial^- f(u)$ the pair (α_o, λ) with $\alpha_o \in \partial^- f_o(u)$ and $\lambda \in R$ such that $\alpha = \alpha_o + \lambda u$ is unique.

<u>PROOF</u>: The uniqueness of λ follows from the variational inequality a_3) in (3.13). Now we have to find a w in H^1_o with the properties

$$u + w,\ u - w \in D(f_o) \quad \text{and} \quad \int_\Omega u(x)w(x)dx \neq 0.$$

If for example

$$\text{meas } (\{x \in \Omega : u(x) \neq 0,\ \phi_1(x) < u(x) < \phi_2(x)\}) > 0 ,$$

given positive integer k, let us consider $\bar{u} = u^+ \wedge k$, $\bar{\phi}_2 = \phi_2^+ \wedge k$, $\bar{\phi}_1 = \phi_1^+ \wedge k$ and $w = (\bar{u} - \bar{\phi}_1) \wedge (\bar{\phi}_2 - \bar{u})$.
It is clear that $w \in H^1_o$ and $w \geq 0$; if $u(x) < 0$ of $u(x) > k$ then $w(x) = 0$ and moreover if $0 \leq w(x) \leq k$ then $\phi_1^+ \leq u - w \leq u + w \leq \phi_2 \wedge k$. Hence u + w and u - w are in K.

Finally u + w and u - w are in $D(f_o)$ since $G(x,u + w)$ and $G(x,u - w)$ are integrable on Ω by (G.3) and the above inequalities.
It is also evident that $\int_\Omega uwdx \neq 0$ if k is big enough. //

(3.14) <u>REMARK</u>. Assume the hypothesis of (3.14) and let $u \in D(f)$. The condition that

$$\text{meas } (\{x \in \Omega : u(x) \neq 0,\ \phi_1(x) < u(x) < \phi_2(x)\}) > 0$$

is equivalent, as is evident, to saying that $D(f_o)$ and S_ρ are not tangential at u under the additional assumption that $\phi_1 \leq 0 \leq \phi_2$ on Ω

§4. AN EXISTENCE THEOREM

In this section we present an existence theorem for the problem (Pe) stated below after the assumptions (4.1), the solutions of the problem being interpreted as curves of maximal slope for the functional f considered in section 3. Crucial for obtaining the existence of solutions to (Pe) is the hypothesis of "non tangency" between K_g and S_ρ : see (1.11) and (3.12). This hypothesis is completely characterized in terms of g, ϕ_1, ϕ_2 and ρ only in section 2 of [3] .

For the existence in the "strong sense" we resort instead to the "regularity" of ϕ_1 and ϕ_2 (by means of the regularity theorem (3.6) in [3]).

In this section we also consider a bounded open subset Ω of R^n, the function $g: \Omega \times R \to R$, the measurable functions $\phi_1, \phi_2: \Omega \to R$ with $\phi_1 \le \phi_2$, and a number $\rho > 0$.

Let us recall that:

for $v \in L^2$ we denote by $\|v\|$ the standard norm in L^2;

$$K = \{ v \in L^2 | \phi_1 \le v \le \phi_2 \quad \text{a.e. on } \Omega \} \quad ;$$

$$u_K = \phi_1 \vee 0 + \phi_2 \wedge 0; \qquad S_\rho = \{ v \in L^2 | \|v\| = \rho \} \quad .$$

For $G(x,t) = \int_0^t g(x,s)ds$ we consider also the convex set

$$K_g = \{ v \in H^1_o \cap K \mid G(x,v) \in L^1 \}$$

In the statements below we suppose that g, ϕ_1, ϕ_2 and ρ satisfy some of the following assumptions

(4.1)

(g.1) for almost all x, g is continuous with respect to t, and for every t it is measurable in x; moreover there are $a \in L^1$ and $b \in R$, such that for almost all x:

$$\int_0^t g(x,s)ds \ge - a(x) - bt^2 \qquad \forall t \in R$$

(g.2) There exists $c_1 \in R$ such that for almost all x and every t_1 and t_2 in R with $t_1 \ne t_2$:

$$\frac{g(x,t_1) - g(x,t_2)}{t_1 - t_2} \ge - c_1$$

(g.3) g is integrable in $\Omega \times [a,b]$ for $a, b \in R$

(ϕ.1) ϕ_1 and ϕ_2 are in H^1; ϕ_1^+ and ϕ_2^- are in H^1_o; $g(x,\phi_1)$ and $g(x,\phi_2)$ are in L^1; $-\Delta\phi_1 + g(x,\phi_1)$ and $-\Delta\phi_2 + g(x,\phi_2)$ are in L^2 in the sense of distributions (or more generally (ϕ.1) in (3.8) holds).

(ϕ.2) K_g and S_ρ are not tangent at any point of $K_g \cap S_\rho$ (see the following remarks and the remark (3.12)).

It should be noticed that (ϕ.2) is completely characterized by (2.5) of [3] ($K_g = D(f_o)$). In view of this a rather comprehensive criterion for the validity of (ϕ.2) <u>for any</u> $\rho > \|u_K\|$ is the following: the hypothesis (g.1), (g.2) and (g.3) hold for the function G, and moreover ϕ_1 and ϕ_2 are in $H^1\ C(\bar{\Omega})$ and there is no open subset Ω' of Ω such that at least one of the following holds:

$$\phi_2 > 0 \quad \text{on} \quad \Omega' \quad \text{and} \quad \phi_2 \in H^1_o(\Omega')$$

$$\phi_1 < 0 \quad \text{on} \quad \Omega' \quad \text{and} \quad \phi_1 \in H^1_o(\Omega')$$

We study the following problem.

<u>EVOLUTION PROBLEM</u>: Given u_o in $K_g \cap S_\rho$ find an interval I in R with $0 = \min I$, a function $U: I \to L^2(\Omega)$ which is absolutely continuous with $U(0) = u_o$ and a $\Lambda : I \to R$ with the following properties:

(Pe)

$$\left|\begin{array}{l} U(s) \in K_g \cap S \qquad \forall s \in I \\ \text{and moreover, for almost all } s \text{ and with } u = U(s) \\ g(x,u)(v-u) \in L^1 \qquad \forall v \in K_g \\ \text{and} \\ \displaystyle\int_\Omega U'(s)(v-u)dx + \int_\Omega Du\, D(v-u)dx + \int_\Omega g(x,u)(v-u)dx + \\ \qquad + \Lambda(s) \displaystyle\int_\Omega u(v-u)dx \geq 0 \qquad \forall v \in K_g \end{array}\right.$$

(4.2) <u>REMARK</u>. a) Note that, in the variational inequality in (Pe), the test function v varies in the convex set K_g, which depends on g. If, however, one assumes (g.1), (g.2) and (g.3), and if ϕ_1 and ϕ_2 are in H^1, then we obtain a problem equivalent to (Pe) by taking the test function v in the convex set K_o defined by $K_o = \{(\phi_1 \vee w) \wedge \phi_2 \mid w \in C^\infty_o(\Omega)\}$ instead in K_g.

b) If for example g satisfies the restriction

$$|g(x,t)| \leq a_1(x) + b_1 |t|^{2^*-1}$$

with $a_1 \in L^{\frac{2^*}{2^*-1}}$, $b_1 \in R$, and $n > 2$, then obviously $K_g = K \cap H^1_o$.

The proof consists of an easy consequence of the Remark (3.11) after taking into consideration the condition b) of (3.4) and a_3) of (3.10).

In the existence theorem (4.5), also solutions of (Pe) "in the strong sense" are considered. In view of this we give the following definition:

(4.3) <u>DEFINITION</u>. Let $D(A) = \{u \in K_g \mid g(x,u) \in L^1,\ -\Delta u + g(x,u) \in L^2$ in the sense of distributions$\}$. For u in D(A) and $\lambda \in R$ let

$$A(\lambda,u) = \left\{\begin{array}{lll} (\Delta u - g(x,u) - \lambda u) \vee 0 & \text{a.e.} & \text{on } \{x \in \Omega: \phi_1(x) = u(x) < \phi_2(x)\} \\ \Delta u - g(x,u) - \lambda u & \text{a.e.} & \text{on } \{x \in \Omega: \phi_1(x) < u(x) < \phi_2(x)\} \\ (\Delta u - g(x,u) - \lambda u) \wedge 0 & \text{a.e.} & \text{on } \{x \in \Omega: \phi_1(x) < u(x) = \phi_2(x)\} \\ 0 & \text{a.e.} & \text{on } \{x \in \Omega: \phi_1(x) = u(x) = \phi_2(x)\} \end{array}\right.$$

(4.4) PROPOSITION. let $U\colon I \to S_\rho$ be an absolutely continuous function (in the L^2-norm) and a $A\colon I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$.

Under the assumptions (g.1) and (g.2), if $U(s) \in D(A)$ and $U'(s) = A(A(s),U(s))$ for almost all s in I, then (U,A) satisfies (Pe)

The inverse is also true under the assumptions (g.1), (g.2) and (g.3), and (ϕ.1).

The proof is given in (4.7).

(4.5) THEOREM (EXISTENCE). Let ρ be a given number, such that $K_g \cap S_\rho \neq \emptyset$ and $\rho > \|\phi_1 \vee 0 + \phi_2 \wedge 0\|$.

a) suppose that (g.1), (g.2) and (g.3) hold, and let $\phi_1, \phi_2 \in H^1$. Then:

a_1) for every u_o in $K_g \cap S_\rho$ with

$$\text{meas}\ (\{x \in \Omega : \phi_1(x) < u_o(x) < 0\} \cup \{x \in \Omega : 0 < u_o(x) < \phi_2(x)\}) > 0$$

there exist T in $]0,+\infty]$ and a unique absolutely continuous $U\colon [0,T[\to L^2$ with $U(0) = u_o$, such that (Pe) holds with an appropriate function $A\colon [0,T[\to \mathbb{R}$. Moreover the energy $\int_\Omega |D_x U(s)|^2 dx$ and $\int_\Omega G(x,U(s)(x))dx$ are continuous and bounded in $[0,T[$;

if $(u_k)_k$ is a sequence in $K_g \cap S_\rho$ converging to u_o and if the sequence

$$\left(\int_\Omega |Du_k|^2 dx\right)_k \quad \text{and} \quad \left(\int_\Omega G(x,u_k)dx\right)_k$$

are bounded, then for any T' with $0 < T' < T$, there exist $U_k\colon [0,T'[\to L^2$ with $U_k(0) = u_k$, satisfying the above conditions and converging to U.

a_2) If (ϕ.2) also holds then, for every u_o in $K_g \cap S_\rho$, there exist U and A defined in $[0,+\infty[$ with the properties stated in a_1).

b) If the assumptions (g.1), (g.2), (g.3) and (ϕ.1) hold, then (U,A) in a_1) or a_2) is such that $U(s) \in D(A) \cap S_\rho$ and $U'_+(s) = A(A(s),U(s))$ for every $s > 0$.

For the uniqueness of A see (2.6) in [3] .

The proof of the theorem is concluded in (4.10).

To prove (4.4) and (4.5) we consider again the functionals f_1 (see (3.3)), f_o and f (see (3.7)) defined by means of the functions G, ϕ_1 and ϕ_2, and the number $\rho > 0$ considered above in this section. Note that $K_g = D(f_o)$ and $K_g \cap S_\rho = D(f)$.

(4.6) PROPOSITION. (Interrelation between (Pe) and f). Suppose that (g.1), (g.2) and (ϕ.2) hold. Let $U\colon I \to L^2$, where I is an interval in R be an absolutely continuous curve. Then there exists a $A\colon I \to \mathbb{R}$ such that (Pe) holds for (U,A) if and only if U is a curve of maximal slope for f.

The condition (ϕ.2) can be substituted with the following two:

$\rho > \|\phi_1 \vee 0 + \phi_2 \wedge 0\|$ and

$$\text{meas}\ (\{x : \phi_1(x) < U(s)(x) < 0\} \cup \{x : 0 < U(s)(x) < \phi_2(x)\}) > 0 .$$

PROOF: If (U,Λ) resolves (Pe), then $-\Lambda(s)U(s) - U'(s) \in \partial^- f_o(U(s))$ for almost all s according to $a_3)$ in (3.10) and hence $-U'(s) \in \partial^- f_1(U(s))$. On the other hand f is of class C(p,q) according to theorem (3.13) and hence the assertion follows from (2.5).

The inverse implication can be easily obtained from the identity $\partial^- f(u) = \partial^- f_o(u) + \partial^- I_{S_\rho}(u)$, which holds in view of the assumption of non tangency (see (3.13) and the characterization of $\partial^- f_o(u)$ by means of the variational inequality (3.10). //

(4.7) PROOF OF (4.4): Let us first of all note that for every s such that $U'(s) = A(\Lambda(s),U(s))$ we have $g(x,U(s))(v-U(s)) \in L^1$ for every v in $D(f_1)$ since by (3.6) we have $\partial^- f_1(U(s)) \neq \emptyset$ and besides (3.5) holds. We obtained the first part of the assertion by multiplying the expression $A(\Lambda(s),U(s) - [\Delta U(s)-g(x,U(s))-\Lambda(s)U(s)]$ by $v-U(s)$ for v in K_g. If now, on the contrary, (U,Λ) satisfies (Pe), then according to $a_3)$ and c) of (3.10) we obtain, for almost all s, $-U'(s)-\Lambda(s)U(s) \in \partial^- f_o(U(s))$ and $\partial^- f_o(U(s)) = \partial^- f_1(U(s)) + \partial^- I_K(U(s))$. It then follows, from b) of (3.6), that $U(s) \in D(A)$ and moreover $-U'(s)+\Delta U(s)-g(x,U(s))-\Lambda(s)U(s) \in \partial^- I_K(U(s))$ for almost all s.

The assertion follows in view of the following simple remark. //

(4.8) REMARK. If $V: I \to K$ is differentiable (in L^2) at some $s_o \in \mathring{I}$, then

$$V'(s_o)(x) = 0 \quad \text{a.e. on } \{x \in \Omega : V(s_o)(x) = \phi_2(x)\} \cup \{x \in \Omega : V(s_o)(x) = \phi_1(x)\}.$$

PROOF: Apply (2.6) for the curve V and the function I_K. //

(4.9) PROPOSITION. Let all the assumptions (4.1) hold. Then for every $u \in D(f) = K_g \cap S$ we have $\partial^- f(u) \neq \emptyset$ if and only if $u \in D(A)$; if $\partial^- f(u) \neq \emptyset$, then there exists a $\lambda \in R$ such that $-\text{grad}^- f(u) = A(\lambda,u)$.

The condition $(\phi.2)$ can be substituted by $\rho > \|\phi_1 \vee 0 + \phi_2 \wedge 0\|$ and meas $(\{x \in \Omega : \phi_1(x) < u(x) < 0\} \cup \{x \in \Omega : 0 < u(x) < \phi_2(x)\}) > 0$.

PROOF: The equivalence is already clear (see $b_2)$ of (3.13)).

If $\partial f(u) \neq \emptyset$, then from the relation $\partial^- f(u) = \partial^- f_1(u) + \partial^- I_K(u) + \partial^- I_{S_\rho}(u)$ (see (3.13)) the existence of λ in R and β in $\partial^- I_K(u)$, such that $-\text{grad}^- f(u) = \Delta u - g(x,u) - \lambda u - \beta$ follows.
For this λ the β has to minimize the L^2 norm of the expression on the right-hand side. Hence the assertion follows.

(4.10) PROOF OF THEOREM (4.5): Under our assumptions the functional f is of class C(p,q) in a neighbourhood of u_o according to theorem (3.13) and remark (3.12): $D(f_o)$ and S_ρ are not tangential in a neighbourhood of u_o, because $D(f_o)$ is dense in K (see $b_2)$ of (3.10)) and K and S_ρ are not tangential at u_o by assumption. If, moreover, $(\phi.2)$ holds, then f is of class C(p,q).
Let us then make this assumption.

From the existence theorem (2.4) it follows that given u_o in $D(f) = K_g \cap S_\rho$, there exists a unique absolutely continuous $U:[0,+\infty[\to L^2$ with $U(0) = u_o$ which is a curve of maximal slope for f.

From b) in (4.6) it follows that there exists one and only one solution of (Pe) with $U(0) = u_o$. Since $f \circ U$ is nonincreasing (on $[0,+\infty[$) and since $\int_\Omega G(x,u(x))dx$ is bounded from below on S_ρ, both functions

$$s \to \int_\Omega |D_x U(s)|^2 dx \qquad \text{and} \qquad s \to \int_\Omega G(x,U(s))dx$$

are bounded on $[0,+\infty[$. Moreover, obviously these two functions of s are lower semi-continuous and the function $f \circ U$ is continuous according to (2.4). Whence both addenda are continuous. Now the dependence on the initial data is an immediate consequence from theorem (2.4).

b) follows from (4.4). //

The following remark is obtained from the properties of the curves of maximal slope, stated in theorem (2.4).

(4.11) REMARK. Let (U,A) with $U:[0,+\infty[\to L^2$ absolutely continuous and $A:[0,+\infty[\to R$ be a solution of (Pe). Under the assumptions (g.1), (g.2) and (ϕ.2):

a) $f \circ U$ is bounded;

b) $f \circ U(s_2) - f \circ U(s_1) = -\int_{s_1}^{s_2} \|U'(s)\|^2 ds$ for $0 \le s_1 \le s_2$;

c) $\|U'(s)\|_{L^2}$ is bounded on $[\bar{s},+\infty[$ for every $\bar{s} > 0$.

If (ϕ.2) is substituted with the following two of conditions,

$\rho \ge \|\phi_1 \vee 0 + \phi_2 \wedge 0\|$

$\text{meas}\,(\{x:\phi_1(x) < U(0)(x) < 0\} \cup \{x:0 < U(0)(x) < \phi_2(x)\}) > 0$

then the above properties hold in a neighbourhood of 0.

Finally, if (4.1) hold, then for every $s > 0$, $U(s) \in D(A)$ and moreover,

the mapping $s \to A(A(s),U(s))$ is right-continuous and bounded on $[\bar{s},+\infty[$ for every $\bar{s} > 0$.

PROOF: Taking into account that f is proved to be of class $C(p,q)$, it follows from (4.6) that U is a curve of maximal slope for f.

Then a) follows from the inequality $f(v) \ge -a - b\|v\|^2$ (see e) in (1.7)) and the fact tha $f \circ U(t) \le f \circ U(0) < +\infty$

b) follows from the general properties stated in (2.4).

c) To prove c), it is necessary to note that f is of class $C(p,q)$ and that on the compact set $\{v:f(v) \le c\}$ we have $\sup p = \bar{p} < +\infty$ and $\sup q = \bar{q} < +\infty$. The assertion then follows from a) in (2.5).

If all the assumptions (4.1) hold, then from a) in (2.4) and (4.9) it follows that $U(s) \in D(A)$ and $U'(s) = A(A(s),U(s))$ for every $s > 0$.

The right-continuity of $s \to U'_+(s)$ and its boundedness on $[\bar{s},+\infty[$ for every $\bar{s} > 0$ follow from (2.5).//

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