INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392-1

SMR 281/22

COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS
(11 January - 5 February 1988)

NONLINEAR SUBDIFFERENTIAL ANALYSIS
PART II

Curves of Maximal Slope and Parabolic Variational Inequalities on Non Convex Constraints

A. MARINO
Dipartimento di Matematica
Universita degli Studi di Pisa
PISA
ITALY

CURVES OF MAXIMAL SLOPE AND PARABOLIC VARIATIONAL INEQUALITIES ON NON CONVEX CONSTRAINTS

ANTONIO MARINO(*) - CLAUDIO SACCON(*) - MARIO TOSQUES(**)

(*) Dipartimento di Matematica, Via Buonarroti, PISA
(**) Facolta' di Ingegneria, Via delle Brecce Bianche, ANCONA

Lavoro inviato a *Annali della Scuola Normale Superiore di Pisa*

Introduction.

In this paper we deal with some classes of "evolution equations of variational type"; by this expression we mean those equations whose unknown may be seen as a curve, with values in a suitable space, along which a given function decreases as fast as possible.

In this work we have developed a theoretical framework proposed in the paper [6], where the "curves of maximal slope" for a function f have been introduced.

We recall that during these years, following the general ideas proposed in [6], the theory of Φ - convex functions has also been developed (see [5] and [11]). In this theory the compactness hypotheses, which are required throughout this paper, are not assumed, but stronger conditions on the behaviour of the functions are imposed. These conditions ensure not only the existence but also the uniqueness of the solution of the evolution equation, with a given initial data, and the continuous dependence on the data.

On the contrary the hypotheses made in this paper enable to obtain, in general, only existence theorems.

It was also felt worthwhile to recall briefly, in section 7 below, some equations which have been solved during these years, following the general ideas of [6], and to show how they are covered by the results proved in this paper.

In (7.1) we recall the evolution problem associated with "geodesics with respect to an obstacle". This subject was studied in [16] and a multiplicity result for such geodesics was obtained, by means of an existence theorem for the curves of maximal slope which is proved in this paper.

In (7.2) the evolution problem associated with "eigenvalues of the Laplace operator with respect to an obstacle"is presented, which was studied in [3] and [4].

In (7.3) another parabolic equation on a non convex contraint is described.

We recall that we generalize, using the notion of curve of maximal slope for a function f, the usual evolution equation of variational type:

(1)
$$\mathcal{U}'(t) + \operatorname{grad}_V f(\mathcal{U}(t)) = 0$$

where f is a differentiable function, defined, for instance, on a Hilbert space H, and "the constraint" V is a smooth submanifold of H. The equation (1) has been the object of several extensions, having different goals.

It is useful to recall now some key ideas of the theory of "maximal monotone operators", which have been very important to frame and to solve many differential equations of parabolic type.

In this theory one introduces, first of all, the notion of "subdifferential " ∂h of a convex function h, defined on a Hilbert space H. Then, if f_o and f_1 are two functions defined on H such that f_o is convex and lower semicontinuous, $f_1 \in C^{1,1}$, if V is a closed and convex subset of H, introducing the function $I_E : H \to \mathbb{R} \cup \{+\infty\}$ defined by: $I_V(u) = \begin{cases} 0, & \text{if } u \in V \\ +\infty, & \text{if } u \in H \setminus V \end{cases}$, for each $E \subset H$, the equation:

(2)
$$-\mathcal{U}'(t) \in \partial (f_o + I_V)(\mathcal{U}(t)) + \operatorname{grad} f_1(\mathcal{U}(t))$$

generalizes (1) to the case where the "constraint" V is a closed and convex subset of H and $f = f_o + f_1$. We can also say that (2) stands for the evolution equation associated with the function $f_o + I_V + f_1$, on the whole space H.

In the context of this theory existence, uniqueness and regularity results hold for the solution of the equation (2) satisfying a given initial condition.

We want to consider some cases where not convex contraints are involved. Then, if f_o and f_1 are of the type mentioned above and V is a subset of H (possibly non convex), we study for instance the problem:

(3)
$$-\mathcal{U}'(t) \in \partial^-(f_o + f_1 + I_V)\mathcal{U}(t)$$

having defined the "subdifferential" $\partial^- h$ of a general h (see definition (1.6)) as a natural extension of the subdifferential in the convex case.

With this goal, in this paper we prove and extend some existence and regularity theorems which were announced, without proofs, in [6] and we also prove some new result which enlarge the framework given in [6].

We wish to point out that we consider two possible extensions of the equation (1).

The definition (1.2) introduces the "curves of maximal slope" in a metric space (using just the metric structure). This approach enables us to get existence theorems (see for instance (4.10)) by a sufficently elementary procedure which points out, in a natural way, some key hypotheses.

The definition (1.8) introduces the "strong evolution curves" in a Hilbert space, by precising (3): in many problems such a definition gives easily the concrete expression of the equation that one solves.

In section 1 we also point out that a curve of maximal slope is a strong evolution curve, if the function satisfies the key property (1.16) (the converse is always true).

Some classes of functions which satisfy this property and also the the hypotheses of the existence theorems, are introduced in section 5.

The sections 2,3 and 4 are devoted to the regularity and the existence theorems for the curves of maximal slope in metric spaces; the sections 5 and 6 contain analogous results for the strong evolution curves in Hilbert spaces.

We list now some of the main notations which will be used throughout this paper.

If X is a metric space, with metric d, if $R > 0$, $u \in$ X, we set:

$$B(u, R) = \{v \in \mathrm{X} \mid d(u, v) < R\}.$$

If $f : \mathrm{X} \to \mathbb{R} \cup \{+\infty\}$ is a function we say that f is "locally bounded from below at u", if there exists $R > 0$ such that f is bounded from below on $B(u, R)$; we say that f is "locally bounded from below on X, if it is locally bounded from below at every u in X.

Let I be an interval in $\mathbb{R}$ and $\mathcal{U} : I \to \mathrm{X}$ be a map. We say that $\mathcal{U}$ is absolutely continuous on I, if it is absolutely continuous (in the usual sense) on any compact interval contained in I.

If $t \in \mathring{I}$, we set:

$$|\delta_+ \mathcal{U}(t)| = \liminf_{h \to 0^+} \frac{d(\mathcal{U}(t+h), \mathcal{U}(t))}{h} \quad , \quad |\delta^+ \mathcal{U}(t)| = \limsup_{h \to 0^+} \frac{d(\mathcal{U}(t+h), \mathcal{U}(t))}{h}$$

$$|\mathcal{U}'_+(t)| = \lim_{h \to 0^+} \frac{d(\mathcal{U}(t+h), \mathcal{U}(t))}{h} \quad , \quad |\mathcal{U}'(t)| = \lim_{h \to 0} \frac{d(\mathcal{U}(t+h), \mathcal{U}(t))}{h}$$

If H is a Hilbert space and $\mathrm{X} \subset H$, we set:

$$\mathcal{U}'_+(t) = \lim_{h \to 0^+} \frac{\mathcal{U}(t+h) - \mathcal{U}(t)}{h}.$$

If $g : I \to \mathbb{R} \cup \{+\infty\}$ is a function and $g(t) < +\infty$, we set:

$$\mathrm{D}_{\pm} g(t) = \liminf_{h \to 0^{\pm}} \frac{g(t+h) - g(t)}{h} \quad , \quad \mathrm{D}^{\pm} g(t) = \limsup_{h \to 0^{\pm}} \frac{g(t+h) - g(t)}{h}.$$

Finally we denote by $\mathbb{R}^+$ the set $\{r \in \mathbb{R} \mid r \geq 0\}$ and, if $A, B \in \mathbb{R}^n$, by $A \setminus B$ the set $\{x \in A \mid x \notin B\}$.

§1. Curves of maximal slope and strong evolution curves.

In this section we wish to present two possible definitions of curves of steepest descent for a function f (see (1.2) and (1.8)). The first one is more general, the second however is closer to the usual notion of strong solution of the evolution equation associated with f. We shall also show that these definitions are equivalent under suitable assumptions.

Let X be a metric space with metric d and $f : \mathrm{X} \to \mathbb{R} \cup \{+\infty\}$ be a function. We define the "domain" of f by $D(f) = \{u \in \mathrm{X} \mid f(u) < +\infty\}$. Let us recall the notion of slope (see definition (1.1) of [6]).

(1.1) Definition

If $u \in D(f)$, $\rho \geq 0$, we set:

$$\chi_u(\rho) = \inf\{f(v) \mid d(u, v) \leq \rho\}$$

and we define $|\nabla f| : D(f) \to \mathbb{R}^+ \cup \{+\infty\}$ in the following way:

$$|\nabla f|(u) = -\liminf_{\rho \to 0^+} \frac{\chi_u(\rho) - \chi_u(0)}{\rho}.$$

$|\nabla f|$ will be called the "slope of f at u".

We introduce now a notion of curve of steepest descent for f which is slightly more general than the one already given in [6].

(1.2) Definition

Let I be an interval in $\mathbb{R}$ with $\mathring{I} \neq \emptyset$ and $\mathcal{U} : I \to \mathrm{X}$ be a curve. We say that $\mathcal{U}$ is a "curve of maximal slope almost everywhere for f" if there exists a negligible set E contained in I such that:

a) *$\mathcal{U}$ is continuous on I;*

b) $f \circ \mathcal{U}(t) < +\infty$ $\quad \forall t$ *in* $I \setminus E$,
$f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(\min I)$ $\quad \forall t$ *in* $I \setminus E$ *if I has minimum;*

c) $$d(\mathcal{U}(t_2), \mathcal{U}(t_1)) \leq \underline{\int_{t_1}^{t_2}} |\nabla f| \circ \mathcal{U}(t)\, dt \qquad \forall t_1, t_2 \text{ in } I \text{ with } t_1 \leq t_2;$$

d) $$f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) \leq -\overline{\int_{t_1}^{t_2}} \left(|\nabla f| \circ \mathcal{U}(t)\right)^2 dt \qquad \forall t_1, t_2 \text{ in } I \setminus E \text{ with } t_1 \leq t_2.$$

If in particular $f \circ \mathcal{U}$ is non increasing, we say that $\mathcal{U}$ is a "curve of maximal slope for f" (see definition (1.6) of [6]).

The proposition (1.4), which will be proved later, ensures the measurability of $|\nabla f| \circ \mathcal{U}$; therefore we can replace the upper and lower integrals, in c) and d), with the integrals (which clearly may be equal to $+\infty$).

(1.3) Remark

If $\mathcal{U}_1 : [a,b] \to X$ and $\mathcal{U}_2 : [b,c] \to X$ are two curves of maximal slope almost everywhere for f such that $\mathcal{U}_1(b) = \mathcal{U}_2(b)$ and $f \circ \mathcal{U}_1(t) \geq f \circ \mathcal{U}_1(b)$ for almost every t in $[a,b]$ (for instance if $f \circ \mathcal{U}_1$ is lower semicontinuous), then the curve $\mathcal{U} : [a,c] \to X$, which is equal to $\mathcal{U}_1$ on $[a,b]$ and to $\mathcal{U}_2$ on $[b,c]$, is a curve of maximal slope almost everywhere for f.

(1.4) Proposition

Let $\mathcal{U} : I \to X$ be a curve of maximal slope almost everywhere for f. Then:

a) *$|\nabla f| \circ \mathcal{U}$ is measurable and $|\nabla f| \circ \mathcal{U}(t) < +\infty$ almost everywhere on I;*

b) *$\mathcal{U}$ is absolutely continuous on $I \setminus \{\inf I\}$ (on I if I has minimum and $f \circ \mathcal{U}(\min I) < +\infty$) and:*

$$|\mathcal{U}'(t)| = |\nabla f| \circ \mathcal{U}(t) \qquad \textit{almost everywhere on } I;$$

c) *there exists a non increasing function $g : I \to \mathbb{R} \cup \{+\infty\}$, which is almost everywhere equal to $f \circ \mathcal{U}$ such that:*

$$g'(t) = -\left(|\nabla f| \circ \mathcal{U}(t)\right)^2 \qquad \textit{almost everywhere on } I.$$

If $\mathcal{U}$ is a curve of maximal slope for f, then we can take $g = f \circ \mathcal{U}$.

Proof

Let E be as in definition (1.2) and g be any non increasing function which is equal to $f \circ \mathcal{U}$ outside of E. Then we have:

$$g(t) < +\infty \qquad \forall t \text{ in } I \text{ with } t > \inf I,$$

$$g(t_2) - g(t_1) \leq -\int_{t_1}^{t_2} \left(|\nabla f| \circ \mathcal{U}(t)\right)^2 \, dt \qquad \forall t_1, t_2 \text{ in } I \text{ with } t_1 \leq t_2,$$

which implies $g'(t) \leq -\left(|\nabla f| \circ \mathcal{U}(t)\right)^2$ almost everywhere on I. Furthermore, by c) of (1.2), we get: $|\delta^+ \mathcal{U}(t)| \leq |\nabla f| \circ \mathcal{U}(t)$ almost everywhere on I.

Since, for almost every t in I, it is:

$$g'(t) \geq \limsup_{\substack{s \to t^+ \\ s \notin E}} \frac{f \circ \mathcal{U}(s) - f \circ \mathcal{U}(t)}{s - t} \geq -\left(|\nabla f| \circ \mathcal{U}(t)\right) |\delta_+ \mathcal{U}(t)|,$$

we have, for almost every t in I:

$$g'(t) \leq -\left(|\nabla f| \circ \mathcal{U}(t)\right)^2 \leq -\left(|\nabla f| \circ \mathcal{U}(t)\right) |\delta^+ \mathcal{U}(t)| \leq \left(-|\nabla f| \circ \mathcal{U}(t)\right) |\delta_+ \mathcal{U}(t)| \leq g'(t).$$

Therefore, for almost every t in I:

$$|\mathcal{U}'(t)| = |\nabla f| \circ \mathcal{U}(t) \quad , \quad g'(t) = -\left(|\nabla f| \circ \mathcal{U}(t)\right)^2.$$

a) and c) follow from the last equality. Since $|\nabla f| \circ \mathcal{U}$ is square integrable on the compact subsets of $I \setminus \{\inf I\}$ (by d) of (1.2)) then $\mathcal{U}$ is absolutely continuous on $I \setminus \{\inf I\}$ (by c) of (1.2)); therefore b) is proved.

The following proposition characterizes the curves of maximal slope.

(1.5) Proposition

Let I be an interval in $\mathbb{R}$ with $\mathring{I} \neq \emptyset$ and $\mathcal{U} : I \to X$ be a continuous curve. Then $\mathcal{U}$ is a curve of maximal slope almost everywhere for f if and only if:

a) *$\mathcal{U}$ is absolutely continuous on $I \setminus \{\inf I\}$ and*

$$|\mathcal{U}'(t)| \leq |\nabla f| \circ \mathcal{U}(t) \qquad \textit{almost everywhere on } I;$$

b) *there exists a non increasing function $g : I \to \mathbb{R} \cup \{+\infty\}$ which is almost everywhere equal to $f \circ \mathcal{U}$ such that:*

$$g(t) < +\infty \qquad \forall t \textit{ in } I \textit{ with } t > \inf I,$$
$$g(\min I) \leq f \circ \mathcal{U}(\min I) \qquad \textit{if } I \textit{ has minimum},$$
$$g'(t) \leq -\left(|\nabla f| \circ \mathcal{U}(t)\right)^2 \qquad \textit{almost everywhere on } I.$$

Furthermore $\mathcal{U}$ is a curve of maximal slope for f if and only if a) and b) hold with $g = f \circ \mathcal{U}$.

Proof

Clearly a) and b) are necessary, as we have seen in proposition (1.4). We prove now that they are sufficent.

b) of (1.2) follows immediately from the first two conditions on g.

Since $\mathcal{U}$ is continuous on I and absolutely continuous on $I \setminus \{\inf I\}$, we have:

$$d(\mathcal{U}(t_2), \mathcal{U}(t_1)) \leq \int_{t_1}^{t_2} |\mathcal{U}'(t)| \, dt \leq \int_{t_1}^{t_2} |\nabla f| \circ \mathcal{U}(t) \, dt \qquad \forall t_1, t_2 \text{ in } I \text{ with } t_1 \leq t_2,$$

which implies c) of (1.2).

Since g is monotone:

$$g(t_2) - g(t_1) \leq \int_{t_1}^{t_2} g'(t) \, dt \leq -\int_{t_1}^{t_2} \left(|\nabla f| \circ \mathcal{U}(t)\right)^2 \, dt \qquad \forall t_1, t_2 \text{ in } I \text{ with } t_1 \leq t_2,$$

which implies d) of (1.2), being $g(t) = f \circ \mathcal{U}(t)$ almost everywhere on I.

We want to show now the meaning of the definitions (1.1) and (1.2) in the case the space X has also a vectorial structure. We shall consider, in this paper, only Hilbert spaces, since we think they play a meaningful role in this kind of problems. Analoguous definitions and statements may be given in suitable classes of Banach spaces (see §4 of [6]).

Let H be a Hilbert space. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and the norm in H. Let W be a subset of H and $f : W \to \mathbb{R} \cup \{+\infty\}$ be a function. We recall now the notions of subdifferential and subgradient (see §4 of [6]).

(1.6) Definition

If $u \in D(f)$, we call "subdifferential of f at u" the set:

$$\partial^- f(u) = \left\{ \alpha \in H \mid \liminf_{v \to u} \frac{f(v) - f(u) - \langle \alpha, v - u \rangle}{\|v - u\|} \geq 0 \right\}.$$

t is easy to see that $\partial^- f(u)$ is closed and convex. If $\partial^- f(u) \neq \emptyset$, we say that f is "subdifferentiable at u" nd we denote by $\mathrm{grad}^- f(u)$ the element of minimal norm in $\partial^- f(u)$, which will be called the "subgradient f f at u".

(1.7) Remark

a) If $u \in D(f)$ and $\partial^- f(u) \neq \emptyset$, then:

$$|\nabla f|(u) < +\infty \qquad \textit{and} \qquad |\nabla f|(u) \leq \|\alpha\| \qquad \forall \alpha \textit{ in } \partial^- f(u),$$

and, in particular, $|\nabla f|(u) \leq \|\mathrm{grad}^- f(u)\|$.

b) If f is lower semicontinuous and convex, or more generally Φ-convex (see definition (1.16) of [11]), then the following property holds:

$$\forall u \textit{ in } D(f): \qquad |\nabla f|(u) < +\infty \Rightarrow \partial^- f(u) \neq \emptyset \textit{ and } |\nabla f|(u) = \|\mathrm{grad}^- f(u)\|$$

(see theorem (1.15) of [11]).

c) If f is lower semicontinuous, then the set: $\{u \in D(f) \mid \partial^- f(u) \neq \emptyset\}$ is dense in $D(f)$ (see proposition (1.2) of [11]).

In §5 we consider another important class of functions which verify the property stated in b) (see a) of theorem (5.4)). We will verify in theorem (1.11) that, if this property holds, then the curves of maximal slope for f solve an evolution equation analogous to the classical one.

For this reason we introduce the following definition.

(1.8) Definition

Let I be an interval in $\mathbb{R}$ with $\mathring{I} \neq \emptyset$ and $\mathcal{U} : I \to W$ be a curve. We say that $\mathcal{U}$ is a strong evolution curve almost everywhere for f, if there exists a negligible subset E in I such that:

a) $\mathcal{U}$ *is continuous on I and absolutely continuous on $I \setminus \{\inf I\}$;*

b) $f \circ \mathcal{U}(t) < +\infty \qquad \forall t$ *in* $I \setminus E$
$f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(\min I) \qquad \forall t$ *in* $I \setminus E$ *if I has minimum;*

c) $\partial^- f(\mathcal{U}(t)) \neq \emptyset$ *and* $-\mathcal{U}'(t) \in \partial^- f(\mathcal{U}(t)) \qquad \forall t$ *in* $I \setminus E$;

d) $f \circ \mathcal{U}$ *is non increasing on $I \setminus E$.*

f, in particular, $f \circ \mathcal{U}$ is non increasing on I, we say that $\mathcal{U}$ is a strong evolution curve for f.

Totally elementary examples, even in the case $H = \mathbb{R}$ and $E = \emptyset$, show that the conditions a), b) and c) do not ensure, in general, that d) holds.

We shall see now that every strong evolution curve almost everywhere for f is a curve of maximal slope almost everywhere for f; the converse is true only under suitable assumptions, so that definition (1.2) is more general than definition (1.8).

(1.9) Proposition

If $\mathcal{U} : I \to W$ is a strong evolution curve almost everywhere for f, then the following facts hold:

a) for almost every t in I it is:

$$\partial^- f(\mathcal{U}(t)) \neq \emptyset \quad , \quad \mathcal{U}'(t) = -\mathrm{grad}^- f(\mathcal{U}(t));$$

there is a non increasing function $g : I \to \mathbb{R} \cup \{+\infty\}$, almost everywhere equal to $f \circ \mathcal{U}$ such that:

$$g'(t) = -\|\mathrm{grad}^- f(\mathcal{U}(t))\|^2 \qquad \textit{almost everywhere on } I.$$

If $\mathcal{U}$ is a strong evolution curve for f, we can take $g = f \circ \mathcal{U}$.

b) $\mathcal{U}$ is a curve of maximal slope almost everywhere for f and:

$$|\nabla f| \circ \mathcal{U}(t) = \|\mathrm{grad}^- f(\mathcal{U}(t))\| \qquad \textit{almost everywhere on } I$$

If $\mathcal{U}$ is a strong evolution curve for f, then $\mathcal{U}$ is a curve of maximal slope for f.

Proof

Let E be as in (1.8). First of all we can enlarge E in such a way that E is still negligible and that the derivative $\left(f \circ \mathcal{U}|_{I \setminus E}\right)'(t)$ exists for every t in $I \setminus E$.

For t in $I \setminus E$ we have

$$-|\nabla f| \circ \mathcal{U}(t)\, \|\mathcal{U}'(t)\| \leq \left(f \circ \mathcal{U}|_{I \setminus E}\right)'(t) = -\|\mathcal{U}'(t)\|^2$$

where the last equality is a consequence of the following lemma (1.10). Then:

$$\|\mathcal{U}'(t)\| \leq |\nabla f| \circ \mathcal{U}(t) \qquad \forall t \text{ in } I \setminus E.$$

Since $-\mathcal{U}'(t) \in \partial^- f(\mathcal{U}(t))$, if $t \in I \setminus E$, we have:

$$\|\mathcal{U}'(t)\| = |\nabla f| \circ \mathcal{U}(t) \quad , \quad -\mathcal{U}'(t) = \mathrm{grad}^- f(\mathcal{U}(t)) \qquad \forall t \text{ in } I \setminus E.$$

Therefore, by the first inequality written above, we obtain that:

$$\left(f \circ \mathcal{U}|_{I \setminus E}\right)'(t) = -\|\mathrm{grad}^- f(\mathcal{U}(t))\|^2 \qquad \forall t \text{ in } I \setminus E.$$

By means of proposition (1.5) we conclude the proof (taking as g any monotone extension of $f \circ \mathcal{U}|_{I \setminus E}$).

The following lemma has been already used in [11].

(1.10) Lemma

Suppose that $D \subset \mathbb{R}$, $t \in D$ and that t is an accumulation point for D. Let $\mathcal{U} : D \to W$ be a map which is differentiable at t. Then, if $\mathcal{U}(t) \in \mathcal{D}(f)$, $\partial^- f(\mathcal{U}(t)) \neq \emptyset$, we have:

$$(\alpha, \mathcal{U}'(t)) \in \partial^-(f \circ \mathcal{U})(t) \qquad \forall \alpha \text{ in } \partial^- f(\mathcal{U}(t)).$$

Therefore, if, in particular, $f \circ \mathcal{U}$ is differentiable at t and t is an accumulation point for D from the right and from the left, then:

$$(\alpha, \mathcal{U}'(t)) = (f \circ \mathcal{U})'(t) \qquad \forall \alpha \text{ in } \partial^- f(\mathcal{U}(t)).$$

Proof

Let $\alpha \in \partial^- f(\mathcal{U}(t))$, from the inequality:

$$f \circ \mathcal{U}(t+h) - f \circ \mathcal{U}(t) \geq (\alpha, \mathcal{U}(t+h) - \mathcal{U}(t)) - o(\|\mathcal{U}(t+h) - \mathcal{U}(t)\|),$$

where $\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0$, we get, if for instance t is an accumulation point from the right and from the left for D, that:

$$D^-(f \circ \mathcal{U})(t) \leq (\alpha, \mathcal{U}'(t)) \leq D_+(f \circ \mathcal{U})(t);$$

therefore $(\alpha, \mathcal{U}'(t)) \in \partial^-(f \circ \mathcal{U})(t)$.

Now we want to verify that, if $\mathcal{U}$ is a curve of maximal slope almost everywhere for f, then the condition $|\nabla f| \circ \mathcal{U}(t) = \|\text{grad}^- f(\mathcal{U}(t))\|$ almost everywhere on I, which was found in (1.9), is also sufficent to ensure that $\mathcal{U}$ is a strong evolution curve almost everywhere for f. Precisely the following theorem holds.

(1.11) Theorem

Let I be an interval in $\mathbb{R}$ with $\mathring{I} \neq \emptyset$ and $\mathcal{U} : I \to W$ be a curve. Then the following facts are equivalent.

a) $\mathcal{U}$ is a strong evolution curve (almost everywhere) for f;

b) $\mathcal{U}$ is a curve of maximal slope (almost everywhere) for f such that:

$$\partial^- f(\mathcal{U}(t)) \neq \emptyset \quad \text{and} \quad |\nabla f| \circ \mathcal{U}(t) = \|\text{grad}^- f(\mathcal{U}(t))\| \qquad \text{almost everywhere on } I.$$

For the proof (see (1.15)) we need the following two lemmas.

(1.12) Lemma

Suppose that $D \subset \mathbb{R}$, $t \in D$ and t is an accumulation point from the right for D. Let $\mathcal{U} : D \to W$ be a map such that:

$$f \circ \mathcal{U}(t) < +\infty \quad \text{and} \quad \partial^- f(\mathcal{U}(t)) \neq \emptyset,$$
$$|\delta^+ \mathcal{U}(t)| \leq |\nabla f| \circ \mathcal{U}(t),$$
$$D^+(f \circ \mathcal{U})(t) \leq -\left(|\nabla f| \circ \mathcal{U}(t)\right)^2.$$

If at the point $u = \mathcal{U}(t)$ it is:

$$|\nabla f|(u) = \|\text{grad}^- f(u)\|,$$

then there exist $\mathcal{U}'_+(t)$, $(f \circ \mathcal{U})'_+(t)$, and we have:

$$\mathcal{U}'_+(t) = -\text{grad}^- f(\mathcal{U}(t)),$$
$$(f \circ \mathcal{U})'_+(t) = -\|\text{grad}^- f(\mathcal{U}(t))\|^2.$$

Proof

Set $D_0 = \{h \in \mathbb{R} \mid t + h \in D\}$ and define $\mathcal{V} : D_0 \mapsto H$ by:

$$\mathcal{V}(h) = \frac{\mathcal{U}(t+h) - \mathcal{U}(t)}{h}.$$

If $\alpha = \text{grad}^- f(\mathcal{U}(t))$, then we have, by the hypotheses:

$$\limsup_{h \to 0^+} \|\mathcal{V}(h)\| \leq |\nabla f| \circ \mathcal{U}(t) = \|\alpha\|.$$

Furthermore the following relation is evident:

(1.13) $$f \circ \mathcal{U}(t+h) \geq f \circ \mathcal{U}(t) + (\alpha, \mathcal{U}(t+h) - \mathcal{U}(t)) - o(\|\mathcal{U}(t+h) - \mathcal{U}(t)\|)$$

where $\lim_{h \to 0^+} \frac{o(\epsilon)}{\epsilon} = 0$.

Since $D^+(f \circ \mathcal{U})(t) \leq -\|\alpha\|^2$, we get, by the hypotheses:

$$\limsup_{h \to 0^+} (\alpha, \mathcal{V}(h)) \leq -\|\alpha\|^2.$$

By lemma (1.14) which follows, we have that:

$$\lim_{h \to 0^+} \mathcal{V}(h) = -\text{grad}^- f(\mathcal{U}(t)).$$

Finally, by (1.13), we obtain also that:

$$D_+(f \circ \mathcal{U})(t) \geq -\|\alpha\| \limsup_{h \to 0^+} \|\mathcal{V}(h)\| \geq -\|\alpha\|^2,$$

and the proof is over.

(1.14) Lemma

Suppose that $D_0 \subset \mathbb{R}$, $0 \in D_0$ and 0 is an accumulation point from the right for D_0. Let $\mathcal{V} : D_0 \to H$ be a map, α in H with $\alpha \neq 0$ and b in $\mathbb{R}$ be such that:

$$\limsup_{h \to 0^+} \|\mathcal{V}(h)\| \leq b \quad , \quad \limsup_{h \to 0^+} (\alpha, \mathcal{V}(h)) \leq -b\|\alpha\|.$$

Then we have:

$$\lim_{h \to 0^+} \mathcal{V}(h) = -\frac{b}{\|\alpha\|}\alpha.$$

Proof

It suffices to prove that, for any sequence $(h_k)_k$ in $D_\circ$ such that $\lim_{k\to\infty} h_k = 0$ and $(\mathcal{V}(h_k))_k$ converges weakly in H to an element $v_\circ$ in H, we have that $(\mathcal{V}(h_k))_k$ converges strongly to $v_\circ$ and $v_\circ = -\frac{b}{\|\alpha\|}\alpha$.

In fact, if this is the case, we have:

$$\|v_\circ\| \le \liminf_{k\to\infty} \|\mathcal{V}(h_k)\| \le \limsup_{k\to\infty} \|\mathcal{V}(h_k)\| \le b$$

and, by the hypotheses, it follows also $(\alpha, v_\circ) \le -b\|\alpha\|$, which together imply:

$$\|v_\circ\| = b \quad \text{and} \quad (\alpha, v_\circ) = -\|v_\circ\|\|\alpha\|.$$

Therefore $v_\circ = -\frac{b}{\|\alpha\|}\alpha$ and $(\mathcal{V}(h_k))_k$ converges strongly to $v_\circ$, since it converges weakly to $v_\circ$ and moreover $\lim_{h\to 0^+} \|\mathcal{V}(h)\| = \|v_\circ\|$.

(1.15) **Proof of (1.11)**

a) implies b) by proposition (1.9).Conversely suppose that $\mathcal{U} : I \to W$ is a curve of maximal slope (almost everywhere) for f. By (1.4) $\mathcal{U}$ is absolutely continuous on $I \setminus \{\inf I\}$ and there exists a negligible subset E in I such that, if $D = I \setminus E$, then the assumptions of lemma (1.12) hold for every t in D. By lemma (1.12) the theorem is proved.

To conclude this section we can say that the problem of the existence of a strong evolution curve $\mathcal{U}$ (almost everywhere) for f, which verifies an assigned initial condition, may be splitted in two steps:

a) to show that there exists a curve $\mathcal{U}$ of maximal slope (almost everywhere) for f, verifying the initial condition;
b) to verify that $\mathcal{U}$ is a strong evolution curve (almost everywhere) for f, by means of theorem (1.11).

For what concerns step a), we give in (4.4) a constructive procedure to find a curve of maximal slope almost everywhere for f, verifying a given initial condition.

For what concerns step b), in §5 we introduce some suitable classes of functions which verify the property:

(1.16) $\quad \forall u$ in $D(f)$: $\quad |\nabla f|(u) < +\infty \Rightarrow \partial^- f(u) \neq \emptyset$ and $|\nabla f|(u) = \|\text{grad}^- f(u)\|$.

For such functions any curve of maximal slope (almost everywhere) for f is a strong evolution curve (almost everywhere) for f, by theorem (1.11).

§2.Some classes of functions defined in metric spaces.

As we said in the introduction, in this paper we want to give a contribution to develop the well known theory of the evolution equations associated with functions of the type $f_\circ + f_1$, where $f_\circ$ is convex and $f_1 \in C^{1,1}$, in such a way to get out, as much as possible, from convexity conditions. For instance we are interested in studying functions of the previous kind restricted to some non convex constraint: such a situation does not fit anymore in the previous framework. In some other cases the function itself is far from the type $f_\circ + f_1$.

Problems of this type, which are recalled in §7, are considered , for instance,in [16], [3], [21] and [22]. They are faced by the theorems proved in this paper, which were partially announced, without proofs, in [6].

With this goal we introduce now some classes of functions which contain the functions involved in the previous papers.

Let X be a metric space with metric d and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function.

(2.1) **Definition**

If $u, v \in D(f)$, $|\nabla f|(u) < +\infty$, we set:

$$\mathcal{R}_f(u,v) = f(v) - \left[f(u) - |\nabla f|(u) d(u,v)\right].$$

Let r and s be two numbers such that:

$$0 \le r \le +\infty \quad , \quad 1 \le s < +\infty,$$

We define the class $K(X; r, s)$ in the following way:

a) if $0 \le r < +\infty$, $1 < s$, we say that $f \in K(X; r, s)$, if the following inequality holds:

$$\mathcal{R}_f(u,v) \ge -\Psi(u, v, |f(u)|, |f(v)|)\left(1 + (|\nabla f|(u))^r\right)(d(u,v))^s \quad \forall u, v \text{ in } D(f) \text{ with } |\nabla f|(u) < +\infty$$

where $\Psi : (D(f))^2 \times (\mathbb{R}^+)^2 \to \mathbb{R}^+$ is a function which is non decreasing in its real arguments and such that $(u,v) \mapsto \Psi(u, v, C_1, C_2)$ is continuous on $\{w \mid |f(w)| \le C\}^2$ for every C_1, C_2 and C in $\mathbb{R}^+$;

b) if $0 \le r < +\infty$, $1 = s$, we say that $f \in K(X; r, 1)$, if the inequality of case a) holds with $s = 1$ and Ψ has the additional property:

$$\Psi(u, u, C_1, C_2) = 0 \qquad \forall u \text{ in } D(f), \forall C_1, C_2 \text{ in } \mathbb{R}^+.$$

c) if $r = +\infty$, $1 < s$, we say that $f \in K(X; \infty, s)$, if the following inequality holds:

$$\mathcal{R}_f(u,v) \ge -\Phi(u, v, |f(u)|, |f(v)|, |\nabla f|(u))(d(u,v))^s \qquad \forall u, v \text{ in } D(f) \text{ with } |\nabla f|(u) < +\infty$$

where $\Phi : (D(f))^2 \times (\mathbb{R}^+)^3 \to \mathbb{R}^+$ is a function which is non decreasing in its real arguments and such that $(u,v) \mapsto \Phi(u, v, C_1, C_2, p)$ is continuous on $\{w \mid |f(w)| \le C\}^2$ for every C_1, C_2, p and C in $\mathbb{R}^+$;

d) if $r = +\infty$, $l = s$, we say that $f \in K(X; \infty, 1)$, if the inequality of case c) holds with $s = 1$ and Φ has the additional property:

$$\Phi(u, u, C_1, C_2, p) = 0 \qquad \forall u \text{ in } D(f), \forall C_1, C_2, p \text{ in } \mathbb{R}^+.$$

(2.2) **Remark**

Let X be a Hilbert space and f be a lower semicontinuous function.

a) If f is convex, then $R_f \geq 0$.

b) If $f = f_o + f_1$ where f_o is convex and $f_1 \in C^{1,\epsilon}_{loc}$ with $\epsilon > 0$ (or C^1), then $f \in K(X; 0, 1+\epsilon)$ (or $f \in K(X; 0, 1)$).

c) If f is (p,q)-convex (see definition (1.1) and theorem (2.5) of [7] and see [9], [10]), then $f \in K(X; 1, 2)$.

d) If $f \in C(p, q)$ (see definition (1.6) of [3]), then $f \in K(X; 1, 2)$.

e) If f is ϕ-convex of order r (see definition (4.1) of [15]), then $f \in K(X; r, 2)$.

f) If f is ϕ-convex (see definition (1.16) of [11]), then $f \in K(X; \infty, 2)$.

In fact, for all these functions, b) of (1.7) holds.

In §7 we expose some solved problems where functions of the previous classes are involved.
For the following it is useful to point out some properties of such functions.

(2.3) **Proposition**

a) If $f \in K(X; \infty, 1)$, then:

(2.4)
$$\limsup_{\substack{v \to u \\ f(v) \leq C, |\nabla f|(v) \leq C}} f(v) \leq f(u) \qquad \forall u \text{ in } D(f), \forall C \text{ in } \mathbb{R}.$$

b) If $f \in K(X; \infty, 1)$, if $Y \subset X$ and if $f_Y : Y \to \mathbb{R} \cup \{+\infty\}$ (defined by $f_Y(v) = f(v)\ \forall v$ in Y) is lower semicontinuous (with respect to the metric induced by X), then for any u in $Y \cap D(f)$ such that f is locally bounded from below at u we have:

(2.5)
$$\liminf_{\substack{v \to u \\ v \in Y, f(v) \leq C}} |\nabla f|(v) \geq |\nabla f|(u) \qquad \forall C \text{ in } \mathbb{R}.$$

c) If $f \in K(X; r, s)$ with $r \leq s$, then:

(2.6)
$$\limsup_{\substack{v \to u \\ f(v) \leq C \\ |\nabla f|(v) d(u,v) \to 0}} f(v) \leq f(u) \qquad \forall u \text{ in } D(f), \forall C \text{ in } \mathbb{R}.$$

Proof

a) and c) are evident. Let us prove b). Let $(u_k)_k$ be a sequence in $D(f) \cap Y$ which converges to u with $f(u_k) \leq C$ and $|\nabla f|(u_k) \leq p$, for any fixed C, p in $\mathbb{R}$. Since f_Y is lower semicontinuous, we can suppose that $|f(u_k)| \leq C$. By hypotheses we have that:

$$f(v) \geq f(u_k) - p d(u_k, v) - \Phi(u_k, v, C, |f(v)|, p) d(u_k, v) \qquad \forall v \text{ in } D(f), \forall k \text{ in } \mathbb{N}$$

(using the notation of definition (2.1) b)). Therefore for all v in $D(f)$:

$$f(v) \geq f(u) - p d(u, v) - \Phi(u, v, C, |f(v)|, p) d(u, v),$$

since f_Y is lower semicontinuous. The result follows by the properties of Φ , since f is locally bounded from below at u.

§3. Some regularity properties for the curves of maximal slope.

As well known, if X is a Hilbert space and if $f = f_o + f_1$, where f_o is a convex,lower semicontinuous function and $f_1 \in C^{1,1}$, then the solutions $U : I \to X$ of the equation:

$$-U'(t) \in \partial^- f(U(t))$$

are such that:

$f \circ U$ is continuous, even if, in general, f is not continuous;

$\partial^- f(U(t)) \neq \emptyset$ for all $t > \inf I$, even if, in general, $\partial^- f(u)$ may be empty for the u's in a dense subset of X;

$U'_+(t) = -\mathrm{grad}^- f(U(t)) \quad \forall t > \inf I$;

$\mathrm{grad}^- f(U(\cdot))$ is right continuous and its norm verifies some a priori estimates.

These properties are very important, for instance, in many evolution problems for partial differential equations: there, usually, X is a space of functions (for instance $L^2(\Omega)$) and the fact that $\partial^- f(u) \neq \emptyset$ for some u means that u is regular (for instance $u \in H^2(\Omega)$)and $\|\mathrm{grad}^- f(u)\|$ is a "strong norm" of u (for instance the norm in $H^2(\Omega)$).

Therefore the second property written above means that the solution $U(t)$ "regularizes" as soon as t is bigger than the initial time.

In this section we try to point out the properties of f that ensure that the statements written above hold for any curve of maximal slope for f, from the metric point of view. We shall show that such properties are verified by a class of functions sufficently large, which includes those introduced in §2, and therefore the functions involved in the problems described in §7. On the other hand it is clear that the functions considered in §7 are not the sum of a convex function and a regular one, since the "constraint" $\{u \mid f(u) < +\infty\}$. is neither convex nor locally convex.

From the vector spaces point of view, this analysis is carried out in §6.

As before let X be a metric space, with metric d and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function.

The main theorems stated in this section are the following ones.

(3.1) Theorem

Let $\mathcal{U} : I \to X$ be a curve of maximal slope almost everywhere for f. Suppose that f is locally bounded from below on X and that $f \circ \mathcal{U}$ is lower semicontinuous.

a) If $f \in K(X; r, s)$, with $r \le s$, then:

a1) $\mathcal{U}$ is a curve of maximal slope for f;

a2) $f \circ \mathcal{U}$ is continuous and non increasing;

a3) $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous on $I \setminus \{\inf I\}$ (on I if I has minimum and $f \circ \mathcal{U}(\min I) < +\infty$) and:

$$(3.2) \qquad \begin{cases} |\mathcal{U}'_+(t)| = |\nabla f| \circ \mathcal{U}(t) & \forall t \text{ in } I \text{ with } |\nabla f| \circ \mathcal{U}(t) < +\infty, \\ & \text{therefore almost everywhere on } I \text{ (see (1.4))}, \\ (f \circ \mathcal{U})'_+(t) = -\left(|\nabla f| \circ \mathcal{U}(t)\right)^2 & \forall t \text{ in } I. \end{cases}$$

b) If $f \in K(X; r, s)$ with $r \le s$ and $s > 1$, then, in addition to a1),a2),a3) the following properties hold:

b1) $$f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) = -\int_{t_1}^{t_2} \left(|\nabla f| \circ \mathcal{U}(t)\right)^2 dt \qquad \forall t_1, t_2 \text{ in } I;$$

b2) $|\nabla f| \circ \mathcal{U}(t) < +\infty \quad \forall t$ in $I \setminus \{\inf I\}$ and $|\nabla f| \circ \mathcal{U}$ is right continuous on $I \setminus \{\inf I\}$ (on I if I has minimum and $f \circ \mathcal{U}(\min I) < +\infty$);

b3) if $[t, T] \subset I$ with $t < T$, if $f \circ \mathcal{U}(t) < +\infty$ and if $|\nabla f| \circ \mathcal{U}(t) < +\infty$, then $|\nabla f| \circ \mathcal{U}$ is bounded on $[t, T]$ therefore $\mathcal{U}$ and $f \circ \mathcal{U}$ are Lipschitz continuous on $[t, T]$.

The proof is in (3.20).

3.3) Theorem

Let $\mathcal{U}$ be a curve of maximal slope almost everywhere for f. Suppose that f is locally bounded from below on X and that $f \circ \mathcal{U}$ is lower semicontinuous. Suppose that $f \in K(X; \infty, s)$ with $s > 1$. Then for every t_o in $I \setminus \{\sup I\}$ with $f \circ \mathcal{U}(t_o) < +\infty$, $|\nabla f| \circ \mathcal{U}(t_o) < +\infty$ and with $f \circ \mathcal{U}(t) \le f \circ \mathcal{U}(t_o)$ for almost every $t \ge t_o$, there exist $\delta > 0$ such that the following properties hold on $[t_o, t_o + \delta]$:

a) $\mathcal{U}$ is a curve of maximal slope for f;

b) $\mathcal{U}$ and $f \circ \mathcal{U}$ are Lipschitz continuous and:

$$f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) = -\int_{t_1}^{t_2} \left(|\nabla f| \circ \mathcal{U}(t)\right)^2 dt \qquad \forall t_1, t_2 \text{ in } [t_o, t_o + \delta]$$

c) $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous, right continuous, bounded and we have:

$$(3.4) \qquad |\mathcal{U}'(t)| = |\nabla f| \circ \mathcal{U}(t) < +\infty \quad , \quad (f \circ \mathcal{U}(t))'_+ = -\left(|\nabla f| \circ \mathcal{U}(t)\right)^2 \qquad \forall t \text{ in } [t_o, t_o + \delta[.$$

The proof is in (3.23).
Some elementary counterexamples are presented in (3.24) e (3.25).

The following statement may be useful.

(3.5) Proposition

Suppose that f is locally bounded from below on X, $f \in K(X; \infty, 1)$. Let $\mathcal{U} : I \to X$ be a curve of maximal slope for f, such that $f \circ \mathcal{U}$ is lower semicontinuous. Then $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous on $I \setminus \{\inf I\}$ (on I if I has minimum and $f \circ \mathcal{U}(\min I) < +\infty$), and (3.2) hold.

The proof is in (3.8).

For sake of completeness we recall a result, proved in [18] (see (1.3) and 2.4) of [18]), which will be used in the following.

(3.6) Proposition

Let $\mathcal{U} : I \to X$ be a curve of maximal slope for f. Suppose that $|\nabla f| \circ \mathcal{U}$ is right lower semicontinuous. Then for every t in $I \setminus \{\inf I\}$ (in I if I has minimum and $f \circ \mathcal{U}(\min I) < +\infty$) we have:

$$(3.7) \qquad \begin{cases} |\delta^+ \mathcal{U}(t)| \le \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} |\nabla f| \circ \mathcal{U}(r)\, dr = |\nabla f| \circ \mathcal{U}(t) \quad \text{and} \\ |\mathcal{U}'_+(t)| = |\nabla f| \circ \mathcal{U}(t), \quad \text{if } |\nabla f| \circ \mathcal{U}(t) < +\infty; \\ (f \circ \mathcal{U})'_+(t) = -\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \left(|\nabla f| \circ \mathcal{U}(r)\right)^2 dr = -\left(|\nabla f| \circ \mathcal{U}(t)\right)^2. \end{cases}$$

(3.8) Proof of (3.5)

By the hypotheses and by b) of proposition (2.3), applied with $Y = \mathcal{U}(I)$ we have that $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous on every t in I such that $\mathcal{U}(t) \in D(f)$. Then the assumptions of (3.6) hold and this imply (3.2).

(3.9) Lemma

If $\mathcal{U} : I \to X$ is a curve of maximal slope almost everywhere for f, then:

a) $|\nabla f| \circ \mathcal{U} \in L^2(J)$ *for any interval J contained in I such that $f \circ \mathcal{U} \in L^\infty(J)$;*

b) $\displaystyle\lim_{t \to t_0} \frac{1}{t - t_0} \int_{t_0}^{t} |\nabla f| \circ \mathcal{U}(r)\, d(\mathcal{U}(t_0), \mathcal{U}(r))\, dr = 0$ *for all t_0 in $I \setminus \{\inf I\}$ (and also for $t_0 = \min I$, if I has minimum and $f \circ \mathcal{U}(t_0) < +\infty$);*

c) if $\mathcal{U}$ is a curve of maximal slope for f, then for any t in $\mathring{I}$ (in $I \setminus \{\sup I\}$ if I has minimum and $f \circ \mathcal{U}(\min I) < +\infty$) we have:

$$|\delta^+ \mathcal{U}(t)| \le \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} |\nabla f| \circ \mathcal{U}(r)\, dr \le |\nabla f| \circ \mathcal{U}(t),$$

$$-\limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \left(|\nabla f| \circ \mathcal{U}(r)\right)^2 dr \ge D_+(f \circ \mathcal{U})(t) \ge -\left(|\nabla f| \circ \mathcal{U}(t)\right)^2.$$

Proof

a) is a trivial consequence of definition (1.2).

To prove b), set $h(t) = \int_{t_0}^{t} |\nabla f| \circ \mathcal{U}(r)\, dr$. For $t > t_0$ we have:

$$\int_{t_0}^{t} |\nabla f| \circ \mathcal{U}(r)\, d(\mathcal{U}(r), \mathcal{U}(t_0))\, dr \le \int_{t_0}^{t} h'(r) h(r)\, dr =$$

$$\frac{1}{2} h^2(t) \le \frac{1}{2}(t - t_0) \int_{t_0}^{t} \left(|\nabla f| \circ \mathcal{U}(r)\right)^2 dr.$$

For $t < t_0$ we have the opposite inequality. Then the conclusion follows from a).

To prove c), we remark that:

$$\limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \left(|\nabla f| \circ \mathcal{U}(r)\right)^2 dr \le -D_+(f \circ \mathcal{U})(t) \le$$

$$|\nabla f| \circ \mathcal{U}(t) \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} |\nabla f| \circ \mathcal{U}(r)\, dr \le$$

$$|\nabla f| \circ \mathcal{U}(t) \left(\limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \left(|\nabla f| \circ \mathcal{U}(r)\right)^2 dr\right)^{1/2}.$$

Now the conclusion follows easily.

(3.10) Lemma

Let $\mathcal{U} : I \to X$ be a curve of maximal slope almost everywhere for f such that $f \circ \mathcal{U}$ is lower semicontinuous. Let E be a negligible subset of I and suppose that the following property holds:

(3.11)
$$\begin{cases} \forall t_0 \text{ in } I, \forall (t_k)_k \text{ in } I \setminus E \text{ such that } \lim_{k \to \infty} t_k = t_0 \text{ and} \\ \lim_{k \to \infty} |\nabla f| \circ \mathcal{U}(t_k)\, d(\mathcal{U}(t_k), \mathcal{U}(t_0)) = 0 \\ \text{then} \quad \limsup_{k \to \infty} f \circ \mathcal{U}(t_k) \le f \circ \mathcal{U}(t_0) \end{cases}$$

Then $\mathcal{U}$ is a curve of maximal slope for f and $f \circ \mathcal{U}$ is continuous.

Proof

We can suppose (see definition (1.2)) that $f \circ \mathcal{U}$ is monotone on $I \setminus E$. Since $I \setminus E$ is a dense subset of I it suffices to prove that $\forall t_0$ in I:

$$\lim_{\substack{t \to t_0 \\ t \notin E}} f \circ \mathcal{U}(t) = f \circ \mathcal{U}(t_0)$$

If t_0 in I, since $f \circ \mathcal{U}$ is lower semicontinuous, it suffices to prove that

$$\limsup_{\substack{t \to t_0 \\ t \notin E}} f \circ \mathcal{U}(t) \le f \circ \mathcal{U}(t_0).$$

Now if $t_0 > \inf I$, we show that:

$$\lim_{\substack{t \to t_0^- \\ t \notin E}} f \circ \mathcal{U}(t) \le f \circ \mathcal{U}(t_0).$$

Arguing by contraddiction, if

$$\lim_{\substack{t \to t_0^- \\ t \notin E}} f \circ \mathcal{U}(t) > f \circ \mathcal{U}(t_0)$$

Then, by the hypotheses, we should have:

$$\liminf_{\substack{t \to t_0^- \\ t \notin E}} |\nabla f| \circ \mathcal{U}(t)\, d(\mathcal{U}(t), \mathcal{U}(t_0)) > 0,$$

and this contraddicts b) of (3.9).

We show now that:

$$\lim_{\substack{t \to t_0^+ \\ t \notin E}} f \circ \mathcal{U}(t) \le f \circ \mathcal{U}(t_0).$$

If $t_0 > \inf I$, this is a consequence of the monotonicity of $f \circ \mathcal{U}$ on $I \setminus E$, and of what we have proved just now; if $t_0 = \min I$ this follows from b) of definition (1.2).

(3.12) Lemma

Let $\mathcal{U} : I \to X$ be a curve of maximal slope for f such $f \circ \mathcal{U}$ is continuous, $|\nabla f| \circ \mathcal{U}$ is right lower semicontinuous and $|\nabla f| \circ \mathcal{U}(t) < +\infty$ $\forall t$ in $I \setminus \{\inf I\}$. Then:

$$f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) = -\int_{t_1}^{t_2} \left(|\nabla f| \circ \mathcal{U}(t)\right)^2 dt \qquad \forall t_1, t_2 \text{ in } I.$$

Proof

By (3.7) $(f \circ \mathcal{U})'_+(t) = -\left(|\nabla f| \circ \mathcal{U}(t)\right)^2 > -\infty$ $\forall t$ in $I \setminus \{\inf I\}$. Since $f \circ \mathcal{U}$ is continuous we get (see 10a of [14] at page 186) that:

$$f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) \ge \int_{t_1}^{t_2} (f \circ \mathcal{U})'_+(t)\, dt \qquad \forall t_1, t_2 \text{ in } I \text{ with } t_1 \le t_2.$$

which implies the result.

(3.13) Lemma

Let $\mathcal{U} : I \to X$ be a curve of maximal slope for f. Suppose that $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous and that:

$$
(3.14) \quad \begin{cases} f(v) \geq f(u) - |\nabla f|(u)\, d(u,v) \left[1 + \gamma(|\nabla f|(u)\, d(u,v))\right] - d(u,v)\omega(d(u,v)) \\ \forall u, v \text{ in } \mathcal{U}(I) \text{ with } |\nabla f|(u) < +\infty. \end{cases}
$$

where $\gamma, \omega : \mathbb{R}^+ \to \mathbb{R}^+$ are two continuous, non decreasing functions such that

$$\sigma \mapsto \frac{\gamma(\sigma)}{\sigma} \quad , \quad \sigma \mapsto \frac{\omega(\sigma)}{\sigma} \quad \text{are integrable in a right neighborhood of } 0.$$

Then the following facts hold:

a) $|\nabla f| \circ \mathcal{U}$ is right continuous on I and bounded on any compact subset of $I \setminus \{\inf I\}$ (of I if I has minimum, $f \circ \mathcal{U}(\min I) < +\infty$ and $|\nabla f| \circ \mathcal{U}(\min I) < +\infty$). Furthermore the following inequalities hold:

$$
(3.15) \quad |\nabla f| \circ \mathcal{U}(t) \leq \begin{cases} \left[\frac{f\circ\mathcal{U}(t_o) - f\circ\mathcal{U}(t)}{\ell(t_o,t)} + \int_0^{\ell(t_o,t)} \frac{\omega(\sigma)}{\sigma}\, d\sigma\right] \exp\left(\int_0^{f\circ\mathcal{U}(t_o) - f\circ\mathcal{U}(t)} \frac{\gamma(\sigma)}{\sigma}\, d\sigma\right), & \text{if } \ell(t_o,t) > 0 \\ 0, & \text{if } \ell(t_o,t) = 0 \end{cases}
$$

$\forall t_o, t$ in I with $t_o < t$ where $\ell(t_o,t) = \int_{t_o}^t |\nabla f| \circ \mathcal{U}(\tau)\, d\tau \left(\leq (t-t_o)^{1/2}(f\circ\mathcal{U}(t_o) - f\circ\mathcal{U}(t))^{1/2}\right)$,

$$
(3.16) \quad \begin{aligned} |\nabla f| \circ \mathcal{U}(t) \leq &\left[|\nabla f| \circ \mathcal{U}(t_o)\left(1 + \gamma(|\nabla f| \circ \mathcal{U}(t_o)\ell(t_o,t))\right) + \omega(\ell(t_o,t)) + \int_0^{\ell(t_o,t)} \frac{\omega(\sigma)}{\sigma}\, d\sigma\right] \times \\ &\exp\left(\int_0^{f\circ\mathcal{U}(t_o) - f\circ\mathcal{U}(t)} \frac{\omega(\sigma)}{\sigma}\, d\sigma\right) \end{aligned}
$$

$\forall t_o, t$ in $I \setminus \{\inf I\}$ with $t_o \leq t$ (and in $t_o = \min I$, if I has minimum and $f \circ \mathcal{U}(t_o) < +\infty$, $|\nabla f| \circ \mathcal{U}(t_o) < +\infty$).

b) If, in particular, $\gamma = \omega = 0$, then $|\nabla f| \circ \mathcal{U}$ is non increasing and therefore $f \circ \mathcal{U}$ is convex.

Proof

It suffices to prove a). Let t_o in I with $f \circ \mathcal{U}(t_o) < +\infty$, and set $I_o = \{t \in I \mid t \geq t_o\}$. It is easy to verify that we may assume $\ell(t_o,t) > 0\ \forall t$ in I with $t > t_o$.

By a) of theorem (3.5) of [18] we have that there exists an interval J in $\mathbb{R}$ and a strictly increasing, right continuous function $\varphi : J \to I_o$ such that, if we set $\mathcal{V} = \mathcal{U} \circ \varphi$, we get $\mathcal{V}(J) = \mathcal{U}(I_o)$ and:

$$d(\mathcal{V}(s_2), \mathcal{V}(s_1)) \leq s_2 - s_1 \quad , \quad f\circ\mathcal{V}(s_2) - f\circ\mathcal{V}(s_1) \leq -\int_{s_1}^{s_2} |\nabla f| \circ \mathcal{V}(s)\, ds$$

$\forall s_1, s_2$ in J with $s_1 \leq s_2$

($\mathcal{V}$ is a curve of maximal slope for f of unit speed, according to the definition (3.1) of [18]). Furthermore we have that $0 = \min J$ and $\mathcal{V}(\ell(t_o,t)) = \mathcal{U}(t)\ \forall t$ in I_o.

If we set $p(s) = |\nabla f| \circ \mathcal{V}(s)$ and $h(s) = \int_0^s p(\sigma)\, d\sigma$, we get, by (3.14):

$$
(3.17) \quad \begin{aligned} &h(s_2) - h(s_1) \leq p(s_1)(s_2 - s_1)(1 + \gamma(p(s_1)(s_2 - s_1))) + (s_2 - s_1)\omega(s_2 - s_1) \\ &\forall s_1, s_2 \text{ in } J \text{ with } s_1 \leq s_2, p(s_1) < +\infty. \end{aligned}
$$

Fix s' in J, then for almost every s in J we have:

$$h(s') - h(s) \leq h'(s)(s'-s)(1 + \gamma(h(s') - h(s))) + (s'-s)\omega(s'-s)$$

(in fact, if $h'(s)(s'-s) \leq h(s') - h(s)$, we use the monotonicity of γ otherwise the inequality is trivial, since γ and ω are positive functions).

Therefore for almost every $s < s'$:

$$\left(\frac{h(s') - h(s)}{s'-s}\right)' \leq \frac{\omega(s'-s)}{s'-s} + \frac{h(s') - h(s)}{s'-s}\left[h'(s)\frac{\gamma(h(s') - h(s))}{h(s') - h(s)}\right].$$

Then, if we set $\Gamma(s) = \int_0^s \frac{\gamma(\sigma)}{\sigma}\, d\sigma$, we have:

$$\left(\frac{h(s') - h(s)}{s'-s}\right)' + \left(\Gamma(h(s') - h(s))\right)' \frac{h(s') - h(s)}{s'-s} \leq \frac{\omega(s'-s)}{s'-s}.$$

By integrating between 0 and s, (using the integrating factor $\exp\left(\Gamma(h(s') - h(s))\right)$) we get that for all s in J with $s < s'$:

$$\left(\frac{h(s') - h(s)}{s'-s}\right) \leq \left[\frac{h(s')}{s'} + \int_{s'-s}^{s'} \frac{\omega(\sigma)}{\sigma}\, d\sigma\right] \exp\left(\int_{h(s')-h(s)}^{h(s')} \frac{\gamma(\sigma)}{\sigma}\, d\sigma\right).$$

Going to the limit, as $s' \to s$ we have:

$$(3.18) \quad p(s) \leq D_+ h(s) \leq \left[\frac{h(s)}{s} + \int_0^s \frac{\omega(\sigma)}{\sigma}\, d\sigma\right] \exp\left(\int_0^{h(s)} \frac{\gamma(\sigma)}{\sigma}\, d\sigma\right).$$

(the first inequality holds because p is lower semicontinuous). Using (3.17) we get:

$$(3.19) \quad p(s) \leq \left[p(0)(1 + \gamma(p(0)s)) + \omega(s) + \int_0^s \frac{\omega(\sigma)}{\sigma}\, d\sigma\right] \exp\left(\int_0^{h(s)} \frac{\gamma(\sigma)}{\sigma}\, d\sigma\right).$$

By (3.18) and (3.19) we obtain (3.15) and (3.16), by setting: $s = \ell(t_o,t)$, since:

$$p(\ell(t_o,t)) = |\nabla f| \circ \mathcal{V}(\ell(t_o,t)) = |\nabla f| \circ \mathcal{U}(t)$$

$$h(\ell(t_o,t)) \leq f\circ\mathcal{V}(0) - f\circ\mathcal{V}(\ell(t_o,t)) = f\circ\mathcal{U}(t_o) - f\circ\mathcal{U}(t)$$

(3.20) **Proof of (3.1)**

a) Since $f \in K(X; r, s)$ with $r \leq s$, the hypothesis (3.11) of lemma (3.10) is verified, then a1) and a2) hold, therefore $\mathcal{U}$ is a curve of maximal slope for f.
Since $f \in K(X; \infty, 1)$ and f is locally bounded from below on X, we obtain a3), by proposition (3.5).

b) Clearly we may suppose $r = s > 1$. Let t and T in I with $t < T$ and $f \circ \mathcal{U}(t) < +\infty$, $|\nabla f| \circ \mathcal{U}(t) < +\infty$. Since $\mathcal{U}([t, T])$ is compact and $f \circ \mathcal{U}$ is bounded on $[t, T]$ (by part a1)), then there exists $C > 0$ such that:

$$f(v) - \left[f(u) - |\nabla f|(u) d(u, v)\right] \geq -C(1 + (|\nabla f|(u))^s)(d(u, v))^s$$
$$\forall u, v \text{ in } \mathcal{U}([t, T]) \text{ with } |\nabla f|(u) < +\infty$$

Then, setting $\gamma(\sigma) = \omega(\sigma) = C\sigma^{s-1}$ $(s > 1)$, the hypothesis (3.14) holds. Moreover $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous (by a3)) and $\mathcal{U}$ is a curve of maximal slope for f (by a1)). Then the hypotheses of lemma (3.13) are verified, therefore $|\nabla f| \circ \mathcal{U}$ is right continuous and bounded on $[t, T]$. Then b2) is proved, since $|\nabla f| \circ \mathcal{U}(t) < +\infty$ for almost every t in I (if $t = \min I$, $f \circ \mathcal{U}(t) < +\infty$ and $|\nabla f| \circ \mathcal{U}(t) = +\infty$ we use the lower semicontinuity of $|\nabla f| \circ \mathcal{U}$).
b1) follows immediately by lemma (3.12).
Since $|\nabla f| \circ \mathcal{U}$ is bounded on $[t, T]$, by the previous result, it follows that $\mathcal{U}$ and $f \circ \mathcal{U}$ are Lipschitz continuous on $[t, T]$ and therefore b3) is completely proved.

We need the following lemma for the proof of theorem (3.3).

(3.21) **Lemma**

Let $\mathcal{U} : I \to X$ be a curve of maximal slope almost everywhere for f. Suppose that there exists $\eta : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ with the following properties:

$p \mapsto \eta(p, \sigma)$ *is non decreasing for every σ;*

$\sigma \mapsto \eta(p, \sigma)$ *is continuous for every p;*

$\sigma \mapsto \dfrac{\eta(p, \sigma)}{\sigma}$ *is integrable in a right neighborhood of 0 for every p*

(3.22)
$$\begin{cases} f(v) \geq f(u) - |\nabla f|(u)\, d(u, v) - \eta(|\nabla f|(u), d(u, v))\, d(u, v) \\ \quad \forall u, v \textit{ in } \mathcal{U}(I \setminus F) \textit{ with } f(u) < +\infty, |\nabla f|(u) < +\infty \\ \textit{where } F \textit{ is a negligible subset of } I. \end{cases}$$

Let $t_0 \in I$ with the properties:

$$f \circ \mathcal{U}(t_0) < +\infty \quad \textit{and} \quad |\nabla f| \circ \mathcal{U}(t_0) < +\infty$$

$$f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(t_0) \qquad \textit{for almost every } t \textit{ in } I \textit{ with } t \geq t_0.$$

Then there exists $\delta > 0$ such that:

$$|\nabla f| \circ \mathcal{U} \in L^\infty([t_0, t_0 + \delta]) \quad , \quad \limsup_{\substack{t \to t_0^+ \\ t \in I'}} |\nabla f| \circ \mathcal{U}(t) \leq |\nabla f| \circ \mathcal{U}(t_0)$$

where I' is a subset of I such that $I \setminus I'$ is negligible.

Proof
Set $I_0 = \{t \in I \mid t_0 \leq t < \sup I\}$ and, for all t in I_0, $\ell(t_0, t) = \int_{t_0}^t |\nabla f| \circ \mathcal{U}(\tau)\, d\tau$. By a) of (3.9) we have that: $\ell(t_0, t) < +\infty$ $\forall t$ in I_0, since $f \circ \mathcal{U} \in L^\infty([t_0, t])$ $\forall t$ in I_0 (in fact $f \circ \mathcal{U}(\tau) \leq f \circ \mathcal{U}(t_0) < +\infty$ almost everywhere on I_0, $f \circ \mathcal{U}$ is non increasing on I_0 with the exception of a negligible subset and $t < \sup I$).
The function $t \mapsto \ell(t_0, t)$ is absolutely continuous and non decreasing on I_0. Set $J = \{\ell(t_0, t) \mid t \in I_0\}$, we have $0 = \min J$. Put:

$$\varphi(s) = \begin{cases} \max\{t \in I_0 \mid \ell(t_0, t) \leq s\}, & \text{if } s \in J \setminus \{\sup J\} \\ \sup\{\varphi(s) \mid s \in J \setminus \{\sup J\}\}, & \text{if } s = \sup J \in J \end{cases}$$

If we define $\mathcal{V} = \mathcal{U} \circ \varphi$, it is easy to see that:

$$\ell(t_0, \varphi(s)) = s \text{ in } J \quad \text{and} \quad \mathcal{V}(\ell(t_0, t)) = \mathcal{U}(t) \text{ in } I_0$$

Furthermore (use the change of variable $s = \ell(t_0, t)$) we have:

$$d(\mathcal{V}(s_2), \mathcal{V}(s_1)) \leq s_2 - s_1 \qquad \forall s_1, s_2 \text{ in } J \text{ with } s_1 \leq s_2,$$

and, if E_0 denotes a negligible subset of I_0 such that $f \circ \mathcal{U}$ is monotone on $I_0 \setminus E_0$ (we can suppose $t \notin E_0$), if $E_1 = \{\ell(t_0, t) \mid t \in E_0\}$, we get that E_1 is negligible and:

$$f \circ \mathcal{V}(s) \leq f \circ \mathcal{V}(0) \qquad \forall s \text{ in } J \setminus E_1$$

$$f \circ \mathcal{V}(s_2) - f \circ \mathcal{V}(s_1) \leq -\int_{s_1}^{s_2} |\nabla f| \circ \mathcal{V}(s)\, ds \qquad \forall s_1, s_2 \text{ in } J \setminus E_1 \text{ with } s_1 \leq s_2$$

Set $F_1 = \{\ell(t_0, t) \mid t \in F \cap I_0\}$ and take s and s' in $J \setminus F_1$ with $0 \leq s < s'$. By (3.22), applied with $u = \mathcal{V}(s)$, $v = \mathcal{V}(s')$, it follows that:

$$f \circ \mathcal{V}(s') - f \circ \mathcal{V}(s) \geq -\left[p(s) + \eta(p(s), d(\mathcal{V}(s'), \mathcal{V}(s)))\right] d(\mathcal{V}(s'), \mathcal{V}(s))$$

where $p = |\nabla f| \circ \mathcal{V}$. If $h(s) = \int_0^s p(\sigma)\, d\sigma$, we deduce that:

$h(s) < +\infty$ $\quad \forall s$ in J, (since $f \circ \mathcal{V} \in L^\infty(0, s)$ $\forall s$ in J);

h is absolutely continuous on J;

$$\frac{h(s') - h(s)}{s' - s} \leq p(s) + \eta(p(s), s' - s) \qquad \forall s, s' \text{ in } J \setminus (F_1 \cup E_1) \text{ with } 0 \leq s < s'$$

Then, since $h' = p$ almost everywhere, and η is a non decreasing function with respect to the variable p, we have:

$$\frac{h(s') - h(s)}{s' - s} \leq h'(s) + \eta\left(\frac{h(s') - h(s)}{s' - s}, s' - s\right)$$

for almost every s', s in J with $s < s'$ (if $h'(s) \leq \dfrac{h(s') - h(s)}{s' - s}$, we use the monotonicity of η, otherwise the inequality is trivial, since $\eta \geq 0$). For any given s' in J we set:

$$k_{s'}(s) = \frac{h(s') - h(s)}{s' - s} \qquad \forall s \text{ in } I \text{ with } 0 \leq s < s'.$$

Clearly $k_{s'}(s) \geq 0$ and:

$$\begin{cases} (k_{s'})'(s) \leq \dfrac{\eta(k_{s'}(s), s'-s)}{s'-s} & \text{for almost all } s \text{ with } 0 \leq s < s' \\ k_{s'}(0) = \dfrac{h(s')}{s'}. \end{cases}$$

Now we remark that $\limsup_{s' \to 0^+} \dfrac{h(s')}{s'} \leq p(0)$, since, for every s' in $J \setminus E$, it is:

$$\frac{1}{s'}\int_0^{s'} |\nabla f| \circ \mathcal{V}(\sigma)\, d\sigma \leq \frac{f \circ \mathcal{V}(0) - f \circ \mathcal{V}(s')}{s'} \leq |\nabla f| \circ \mathcal{V}(0)\, \frac{d(\mathcal{V}(0), \mathcal{V}(s'))}{s'} +$$
$$+ \frac{o(d(\mathcal{V}(0), \mathcal{V}(s')))}{s'} \leq |\nabla f| \circ \mathcal{V}(0) + \frac{o(s')}{s'},$$

where $\lim_{\sigma \to 0} \frac{o(\sigma)}{\sigma} = 0$. Then for every $\epsilon > 0$ there exists $\bar{s}$ such that:

$$k_{s'}(0) \leq p(0) + \epsilon \qquad \forall s' \text{ with } 0 < s' \leq \bar{s}.$$

This implies

$$k_{s'}(s) \leq p(0) + \epsilon + \int_0^s \frac{\eta(k_{s'}(\sigma), s'-\sigma)}{s'-\sigma}\, d\sigma \qquad \forall s \text{ with } 0 \leq s < s' \leq \bar{s}.$$

Now if $\bar{s} > 0$ verifies also the property:

$$\int_0^{\bar{s}} \frac{\eta(p(0)+2\epsilon, \tau)}{\tau}\, d\tau < \epsilon,$$

then we have:

$$\int_0^{s'} \frac{\eta(p(0)+2\epsilon, s'-\sigma)}{s'-\sigma}\, d\sigma < \epsilon \qquad \forall s' \text{ with } 0 < s' \leq \bar{s}.$$

We claim that:

$$k_{s'}(s) \leq p(0) + 2\epsilon \qquad \forall s', s \text{ with } 0 \leq s < s' \leq \bar{s}.$$

In fact, if for some s' in $]0, \bar{s}]$ it were:

$$\lambda_{s'} = \sup\{s \mid 0 \leq s < s', k_{s'}(\sigma) \leq p(0) + 2\epsilon\ \forall \sigma \text{ in } [0, s]\} < s',$$

we should have:

$$k_{s'}(\lambda_{s'}) \leq p(0) + \epsilon + \int_0^{\lambda_{s'}} \frac{\eta(p(0)+2\epsilon, s'-\sigma)}{s'-\sigma}\, d\sigma \leq p(0) + \epsilon +$$
$$+ \int_0^{s'} \frac{\eta(p(0)+2\epsilon, s'-\sigma)}{s'-\sigma}\, d\sigma < p(0) + 2\epsilon,$$

which contraddicts the definition of $\lambda_{s'}$, since $k_{s'}$ is continuous.

Therefore, $\forall \epsilon > 0 \quad \exists \bar{s}$ in J with $\bar{s} > 0$ such that:

$$\frac{h(s') - h(s)}{s'-s} \leq p(0) + 2\epsilon \qquad \forall s, s' \text{ in } J \text{ with } 0 \leq s < s' \leq \bar{s}.$$

It follows that, for almost every s in $[0, \bar{s}]$ $p(s) = h'(s) \leq p(0) + 2\epsilon$.

Now set $t_o + \delta = \varphi(\bar{s})$ and:

$$I' = \{t \in I \mid p(\ell(t_o, t)) = 0 \text{ or } h'(\ell(t_o, t)) = p(\ell(t_o, t))\}.$$

We claim that $I \setminus I'$ is a negligible set. In fact, since $p(\ell(t_o, t))\ \left(= |\nabla f| \circ \mathcal{U}(t)\right) = \frac{d}{dt}\ell(t_o, t)$ for almost every t, it suffices to show that the set

$$A = \{t \in I \mid \frac{d}{dt}\ell(t_o, t) > 0, h'(\ell(t_o, t)) \neq p(\ell(t_o, t))\}$$

is negligible: the set $A_1 = \{s \in J \mid h'(s) \neq p(s)\}$ is negligible and the following relation holds.

$$\int_{A_1} ds = \int_A \frac{d}{dt}\ell(t_o, t)\, dt.$$

Finally we have seen that $\forall \epsilon > 0 \exists \delta > 0$ such that $|\nabla f| \circ \mathcal{U}(t) \leq p(0) + \epsilon\ \forall t \in I' \cap [t_o, t_o + \delta]$, which proves the lemma.

(3.23) Proof of (3.3)

Let $t_o \in I \setminus \{\sup I\}$ with $f \circ \mathcal{U}(t_o) < +\infty, |\nabla f| \circ \mathcal{U}(t_o) < +\infty$ and $T \in I$ with $T > t_o$. Since $f \in \mathcal{K}(X; \infty, s)$ and $s > 1$, then (3.22) holds, wih $F = \emptyset$, setting:

$$\eta(p, \sigma) = \sigma^{s-1} \sup_{t_1, t_2 \in [t_o, T]} \{\Phi\left(\mathcal{U}(t_1), \mathcal{U}(t_2), |f \circ \mathcal{U}(t_1)|, |f \circ \mathcal{U}(t_2)|, p\right)\}$$

where Φ is the function given by c) of definition (2.1).

Furthrmore $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous, by b) of proposition (2.3). Then, by lemma (3.21), there exists $\delta > 0$ such that $|\nabla f| \circ \mathcal{U}$ is bounded on $[t_o, t_o + \delta]$, and is right continuous at t_o.

Moreover $f \circ \mathcal{U}$ is continuous on $[t_o, t_o + \delta]$ because it is lower semicontinuous and also upper semicontinuous, by a) of (2.3): we have proved just now that $|\nabla f| \circ \mathcal{U}$ is bounded on $[t_o, t_o + \delta]$. Then $\mathcal{U}$ is a curve of maximal slope for f, namely a) holds.

In particular we get that, for any $\bar{t}_o$ in $]t_o, t_o + \delta[$, it is:

$$f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(\bar{t}_o)\ \forall t \text{ in } [\bar{t}_o, t_o + \delta].$$

Since $|\nabla f| \circ \mathcal{U}(\bar{t}_o) < +\infty$, we obtain, as before, that $|\nabla f| \circ \mathcal{U}$ is right continuous at $\bar{t}_o$. Furthermore, by proposition (3.6), (3.4) holds and c) is completely proved.

Finally b) follows directly by lemma (3.12).

(3.24) Counterexample to theorem (3.1)

We show now that, if $f \in \mathcal{K}(X; r, s)$ with $r > s$, it is possible that there exists a curve $\mathcal{U}$ of maximal slope for f such that $f \circ \mathcal{U}$ is not continuous.

For every r, s with $1 < s < r$ and $s \leq 2$ take the function $f : [0, 1] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} x^s, & \text{if } x > 0 \\ -1, & \text{if } x = 0 \end{cases}$$

where $\epsilon = 1 - s/r$. Then $f \in K([0,1]; r, s)$, since, for a suitable constant $C > 0$ the following inequality holds:

$$f(y) \geq f(x) - |\nabla f|(x)|y - x| - C\left(|\nabla f|(x)\right)^r |y - x|^s \qquad \forall x, y \text{ in } [0,1].$$

On the other hand, if $x_0 \in [0,1]$, the curve $\mathcal{U} : [0, +\infty[\to [0,1]$ defined by:

$$\mathcal{U}(t) = \begin{cases} \left(x_0^{2-\epsilon} - \epsilon(2-\epsilon)t\right)^{\frac{1}{2-\epsilon}}, & \text{if } 0 \leq t \leq t_0 = \frac{x_0^{2-\epsilon}}{\epsilon(2-\epsilon)} \\ 0, & \text{if } t \geq t_0 \end{cases}$$

is a curve of maximal slope for f. Nevertheless $f \circ \mathcal{U}$ is not continuous, if $x_0 > 0$.

(3.25) Counterexample to b) of theorem (3.1) and to lemma (3.13)

We show that, if $f \in K(X; 0, 1)$, it may happen that there exists a curve $\mathcal{U} : I \to X$ of maximal slope for f such that $|\nabla f| \circ \mathcal{U}$ is unbounded on the compact subsets of $\mathring{I}$; more precisely we show that such a $\mathcal{U}$ may exist, also if f verifies the inequality (3.14), with $\gamma = 0$, and with an ω such that $\lim_{\sigma \to 0} \omega(\sigma) = 0$ but $\frac{\omega(\sigma)}{\sigma}$ is not integrable on any right neighborhood of 0.

Let $X = \left[-\frac{1}{2}, \frac{1}{2}\right]$ and define $f : X \to \mathbb{R}$ by:

$$f(x) = \begin{cases} -x \ln|\ln|x||, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

It is easy to see that:

$$f(y) \geq f(x) - |\nabla f|(x)|y - x| - \omega(|y - x|)|y - x| \qquad \forall x, y \text{ in } X,$$

where:

$$\omega(\sigma) = \begin{cases} -\frac{2}{\ln \sigma}, & \text{if } \sigma > 0 \\ 0, & \text{if } \sigma = 0. \end{cases}$$

We remark that $\lim_{\sigma \to 0^+} \omega(\sigma) = 0$, but $\sigma \mapsto \frac{\omega(\sigma)}{\sigma}$ is not integrable.

On the other hand it is clear that there exists a curve $\mathcal{U} : [0, T] \to X$ of maximal slope for f such that $\mathcal{U}(0) < 0$ and $\mathcal{U}(T) > 0$. For such a $\mathcal{U}$ $|\nabla f| \circ \mathcal{U}$ is not bounded

§4. A constructive procedure and some existence theorems.

In this section we consider a very simple procedure, which allows to construct a curve of maximal slope. In such a procedure we use in an essential way the variational character of the evolution problem we are dealing with. We deduce the existence theorem (4.10), where we point out the minimal hypotheses needed for the existence. We deduce also theorem (4.2), which, using the class $K(X; \infty, 1)$ has the advantage to have more sinthetic assumptions.

Let, as usual, X be a metric space, with metric d and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function.

(4.1) Definition

Let Y be a subset of X. We say that f is "coercive on Y", if for any C in $\mathbb{R}$ the set $\{u \mid f(u) \leq C\} \cap Y$ is compact.

Let u_0 in $D(f)$. We say that f is "coercive at u_0", if there exists $R > 0$ such that f is coercive on:

$$\{u \mid d(u, u_0) \leq R, f(u) \leq f(u_0)\}.$$

(4.2) Theorem

Suppose $f \in K(X; \infty, 1)$ (see d) of definition (2.1)) and let u_0 in $D(f)$ be such that f is coercive at u_0.

Then there exist $T > 0$ and an absolutely continuous curve $\mathcal{U} : [0, T] \to X$, such that $\mathcal{U}$ is a curve of maximal slope almost everywhere for f with:

$$\mathcal{U}(0) = u_0 \quad , \quad f \circ \mathcal{U}(t) \leq f(u_0) \quad \forall t \text{ in } [0, T],$$
$$f \circ \mathcal{U}, |\nabla f| \circ \mathcal{U} \text{ are lower semicontinuous on } [0, T].$$

In particular we recall that (see(1.4)):

$$\text{(4.3)} \quad \begin{cases} |\mathcal{U}'(t)| = |\nabla f| \circ \mathcal{U}(t) & \text{almost everywhere on } [0, T], \\ g'(t) = -\left(|\nabla f| \circ \mathcal{U}(t)\right)^2 & \text{almost everywhere on } [0, T], \\ \text{where } g : [0, T] \to \mathbb{R} \cup \{+\infty\} \text{ is a suitable non increasing function} \\ \text{such that } g(t) = f \circ \mathcal{U}(t) \text{ almost everywhere on } [0, T]. \end{cases}$$

The proof is carried out in (4.11).

Problem

Does a curve $\mathcal{U}$ of maximal slope for f exist, with $\mathcal{U}(0) = u_0$, under the assumptions of (4.2) ?

(4.4) A constructive procedure

If $u \in X, \rho > 0$, we recall that $\overline{B(u, \rho)}$ is the set $\{v \in X \mid d(u, v) \leq \rho\}$. Assign u_0 in $D(f)$ and $R > 0$. We say that the k-uple $P = (\theta_0, \ldots, \theta_k)$ (k in $\mathbb{N}$) is a partition of $[0, R]$, if $0 = \theta_0 < \theta_1 < \cdots < \theta_k = R$. The number $\delta(P) = \max_{i=1..k} \{\theta_i - \theta_{i-1}\}$ will be called the amplitude of P.

(4.5) **First step**

Suppose that there exists a sequence $(P_h)_h$ of partitions of $[0, R]$ with $\lim_{h\to+\infty} \delta(P_h) = 0$ and such that, for any $P_h = (\theta_o, \dots, \theta_{k_h})$ there exist the minimum points $\mathcal{V}_h(s)$, for $0 \le s \le R$, with the properties:

$$\mathcal{V}_h(0) = u_o \qquad \textit{and } \forall i = 1, \dots, k_h, \forall s \textit{ in }]\theta_{i-1}, \theta_i]$$

$$\mathcal{V}_h(s) \textit{ is a minimum point for } f \textit{ in } \overline{B(\mathcal{V}_h(\theta_{i-1}), s - \theta_{i-1})}.$$

In such a way we have defined, for all h in $\mathbf{N}$, a curve $\mathcal{V}_h : [0, R] \to \mathrm{X}$ with the properties:

a) $$d(\mathcal{V}_h(s_2), \mathcal{V}_h(s_1)) \le |s_2 - s_1| + 2\delta(P_h) \qquad \forall s_1, s_2 \textit{ in } [0, R]$$

b) $$f \circ \mathcal{V}_h(s_2) - f \circ \mathcal{V}_h(s_1) \le -\overset{\bullet}{\int_{s_1}^{s_2}} \left(|\nabla f| \circ \mathcal{V}_h(s)\right)^2 \, ds \qquad \forall s_1, s_2 \textit{ in } [0, R] \textit{ with } s_1 \le s_2.$$

Proof

The first inequality is clear.

To prove the second one we recall the definition:

$$\chi_u(s) = \inf\{f(v) \mid d(u, v) \le s\} \qquad \forall u \text{ in } \mathrm{X}, \forall s \ge 0$$

and remark that, if u_s is a minimum point for f in $\overline{B(u, s)}$, then it is easy to see that:

$$D_+\chi_u(s) \le D_+\chi_{u_s}(0) = -|\nabla f|(u_s).$$

Since χ_u is non increasing, we have that, if $0 \le s_1 \le s_2 \le R$:

$$\chi_u(s_2) - \chi_u(s_1) \le \int_{s_1}^{s_2} \chi_u'(s) \, ds \le -\overset{\bullet}{\int_{s_1}^{s_2}} |\nabla f|(u_s) \, ds.$$

This implies b).

(4.6) **Second step**

Let the hypotheses of (4.5) be verified and suppose that, for any s in $[0, R]$, the set $\{\mathcal{V}_h(s) \mid h \text{ in } \mathbf{N}\}$ is compact.

Then it is easy to verify that there exists $\mathcal{V} : [0, R] \to \mathrm{X}$ and a sequence $(h_l)_l$ such that $\mathcal{V}(0) = u_o$ and:

a) $$(\mathcal{V}_{h_l})_l \textit{ converges to } \mathcal{V} \textit{ uniformly on } [0, R];$$

b) $$d(\mathcal{V}(s_2), \mathcal{V}(s_1)) \le |s_2 - s_1| \qquad \forall s_1, s_2 \textit{ in } [0, R].$$

(4.7) **Third step**

Let the hypotheses of (4.6) be verified. Suppose that:

$$B_o = \{v \textit{ in } \mathrm{X} \mid d(v, u_o) \le R, f(v) \le f(u_o)\} \textit{ is closed;}$$

$$f \textit{ is lower semicontinuous and bounded from below on } B_o;$$

$$\limsup_{\substack{v \to u \\ v \in B_o, |\nabla f|(v) \le c}} f(v) \le f(u) \qquad \forall u \textit{ in } B_o, \forall c \textit{ in } \mathbf{R}.$$

Then there exists a negligible subset F of $[0, R]$ such that:

a) $$f \circ \mathcal{V}(s) \le f(u_o) \qquad \forall s \textit{ in } [0, R]$$

b) $$f \circ \mathcal{V}(s_2) - f \circ \mathcal{V}(s_1) \le -\overset{\bullet}{\int_{s_1}^{s_2}} \liminf_{l\to+\infty} |\nabla f| \circ \mathcal{V}_{h_l}(s) \, ds \qquad \forall s_1, s_2 \textit{ in } [0, R] \setminus F \textit{ with } s_1 \le s_2.$$

Proof

By the lower semicontinuity of f in B_o a) follows. To get b), we remark that, by Fatou's lemma, we have, for $s_1 \le s_2$ in $[0, R]$:

$$\overset{\bullet}{\int_{s_1}^{s_2}} \liminf_{l\to+\infty} |\nabla f| \circ \mathcal{V}_{h_l}(s) \, ds \le \liminf_{l\to+\infty} \overset{\bullet}{\int_{s_1}^{s_2}} |\nabla f| \circ \mathcal{V}_{h_l}(s) \, ds \le$$
$$\liminf_{l\to+\infty} \left(f \circ \mathcal{V}_{h_l}(s_1) - f \circ \mathcal{V}_{h_l}(s_2)\right) \le f(u_o) - \inf_{v \in B_o} f(v).$$

Therefore there exists a negligible set F contained in $[0, R]$ such that:

$$\liminf_{l\to+\infty} |\nabla f| \circ \mathcal{V}_{h_l}(s) < +\infty \qquad \forall s \text{ in } [0, R].$$

Now, if $s_1 \notin F$, there exist a sequence $(l_i)_i$ and a constant C such that:

$$f \circ \mathcal{V}_{h_{l_i}}(s_1) \le f(u_o) \quad , \quad |\nabla f| \circ \mathcal{V}_{h_{l_i}}(s_1) \le C \qquad \forall i \text{ in } \mathbf{N}.$$

Then, by hypotheses:

$$\lim_{i\to+\infty} f \circ \mathcal{V}_{h_{l_i}}(s_1) = f \circ \mathcal{V}(s_1).$$

Since $\liminf_{i\to+\infty} f \circ \mathcal{V}_{h_{l_i}}(s_2) \ge f \circ \mathcal{V}(s_2)$, for all s_2 in $[0, R]$, then b) holds, by b) of (4.5).

(4.8) **Fourth step**

Let the hypotheses of (4.7) be verified. Then there exist $T > 0$ an a curve $\mathcal{U} : [0, T] \to \mathrm{X}$, which is absolutely continuous and such that $\mathcal{U}(0) = u_o$, $f \circ \mathcal{U}(t) \le f(u_o)$ $\forall t$ in $[0, T]$ and:

$$(4.9) \quad \begin{cases} d(\mathcal{U}(t_2), \mathcal{U}(t_1)) \le \int_{t_1}^{t_2} \overline{|\nabla f|} \circ \mathcal{U}(t) \, dt & \forall t_1, t_2 \textit{ in } [0, T] \textit{ with } t_1 \le t_2, \\ f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) \le -\int_{t_1}^{t_2} \left(\overline{|\nabla f|} \circ \mathcal{U}(t)\right)^2 \, dt & \textit{for almost all } t_1, t_2 \textit{ in } [0, T] \textit{ with } t_1 \le t_2, \end{cases}$$

where:

$$\overline{|\nabla f|}(u) = \liminf_{\substack{v \to u \\ v \in B_o}} |\nabla f|(v).$$

Furthermore $f \circ \mathcal{U}$ is lower semicontinuous in B_*.

Proof

We remark that $\overline{|\nabla f|}$ is lower semicontinuous on B_*. If $\overline{|\nabla f|}(u_*) = 0$, then the conclusion is trivial. If $\overline{|\nabla f|}(u_*) > 0$, we can suppose that $\overline{|\nabla f|} \circ \mathcal{V}(s) \geq \epsilon > 0$ on $[0, R]$. Take the function $\psi(s) = \int_0^s \left(\overline{|\nabla f|} \circ \mathcal{V}(\sigma)\right)^{-1} d\sigma$. ψ is an absolutely continuous and strictly increasing function on $[0, R]$ with $\psi(0) = 0$. Let $T = \psi(R)$, $\varphi = \psi^{-1} : [0, T] \to [0, R]$. φ is a continuous and strictly increasing function. Set $\mathcal{U} = \mathcal{V} \circ \varphi$. To prove (4.9) it suffices to change variable, setting $t = \psi(s)$, in the integrals:

$$\int_{t_1}^{t_2} \overline{|\nabla f|} \circ \mathcal{U}(t)\, dt \quad , \quad \int_{t_1}^{t_2} \left(\overline{|\nabla f|} \circ \mathcal{U}(t)\right)^2 dt$$

and to remark that ψ maps negligible sets into negligible sets, being an absolutely continuous function.

From the previous procedure we deduce, in particular, the following statement.

(4.10) Theorem

Let u_* in $D(f)$ and suppose that there exists $R > 0$ such that:

a) f *is coercive on* $B_* = \{v \in X \mid d(v, u_*) \leq R, f(v) \leq f(u_*)\}$;

b) $\displaystyle\limsup_{\substack{v \to u \\ v \in B_*, \overline{|\nabla f|}(v) \leq C}} f(v) \leq f(u) \qquad \forall u \text{ in } B_*, \forall C \text{ in } \mathbb{R}.$

Then there exist $T > 0$ and an absolutely continuous curve $\mathcal{U} : [0, T] \to X$ such that $\mathcal{U}(0) = u_*$, $f \circ \mathcal{U}(t) \leq f(u_*)$ $\forall t$ in $[0, T]$, (4.9) hold and $f \circ \mathcal{U}$ is lower semicontinuous.
Furthermore, if in addition:

c) $$\liminf_{\substack{v \to u \\ v \in B_*}} |\nabla f|(v) \geq |\nabla f|(u) \qquad \forall u \text{ in } B_*$$

then $\mathcal{U}$ is a curve of maximal slope almost everywhere for f, $|\nabla f| \circ \mathcal{U}$ is lower semicontinuous and (4.3) hold.

Proof

The conclusion follows clearly from (4.8) by remarking that $\overline{|\nabla f|}(u) = |\nabla f|(u)$ for every u in B_*, if c) holds,and by using proposition (1.4).

(4.11) Proof of (4.2)

Since f is coercive at u_*, there exists $R > 0$ such that f is coercive on $B_* = \{v \mid d(v, u_*) \leq R, f(v) \leq f(u_*)\}$, in particular f_{B_*} is lower semicontinuous and bounded from below. Then f is also locally bounded from below at any u in $\{v \mid d(v, u_*) < R, f(v) \leq f(u_*)\}$. Therefore, decreasing R if necessary, we can suppose that f is locally bounded from below at any u in B_*. Since $f \in K(X; \infty, 1)$ by b) of proposition (2.3), applied with $Y = B_*$, then c) of theorem (4.10) holds. Finally, by a) of (2.3), b)of (4.10) is verified too. Then the thesis follows by theorem (4.10).

We conclude this section by a statement concerning the maximal interval of existence of a curve $\mathcal{U}$ of maximal slope for f. This will point out an important link between $\mathcal{U}$ and f.

(4.12) Theorem

Suppose that X is a subspace of a complete metric space X_1, f is lower semicontinuous and for every u in $D(f)$ there exist $T > 0$ and a curve $\mathcal{U} : [0, T) \to X$ of maximal slope (almost everywhere) for f such that $\mathcal{U}(0) = u$
Then for any u_* in $D(f)$ there exist $\bar{T} > 0$ and $\bar{\mathcal{U}} : [0, \bar{T}[\to X$ such that $\mathcal{U}$ is a curve of maximal slope (almost everywhere) for f with $\bar{\mathcal{U}}(0) = u_*$ and at least one of the following properties holds:

$$\bar{T} = +\infty \quad , \quad \operatorname*{ess\,inf}_{t \in [0, \bar{T}[} \{f \circ \bar{\mathcal{U}}(t)\} = -\infty \quad , \quad \lim_{t \to \bar{T}^-} \bar{\mathcal{U}}(t) = \bar{u} \notin X.$$

Proof

Clearly, if $\mathcal{U} : [0, T[\to X$ is a curve of maximal slope almost everywhere for f, then:

$$d(\mathcal{U}(t_2), \mathcal{U}(t_1)) \leq (t_2 - t_1)^{1/2} \left(f(u_*) - \operatorname{ess\,inf}\{f \circ \bar{\mathcal{U}}(t) \mid t \in [0, \bar{T}[\}\right)^{1/2}$$
$$\forall t_1, t_2 \text{ in } [0, \bar{T}[\text{ with } t_1 \leq t_2.$$

Applying this property the conclusion follows easily.

§5. Some classes of functions defined in Hilbert spaces.

To study the strong evolution curves associated with functions defined on a Hilbert space H (see definition (1.8)), we introduce now some classes of functions, analogous to those considered, in metric spaces, in §2. The goal is always that of considering evolution problems also when non convex contraints, of the type described in §7, are involved. In this section we deal with a Hilbert space H ,a subset W of H an a function $f : W \to \mathbb{R} \cup \{+\infty\}$. We recall that $D(f) = \{v \in W \mid f(v) < +\infty\}$. We shall use the concepts of subdifferential and subgradient introduced in (1.6).

(5.1) Definition

If $u, v \in D(f)$ with $\partial^- f(u) \neq \emptyset$, we set:

$$R_f(u, v) = f(v) - \left(f(u) + (\operatorname{grad}^- f(u), v - u)\right).$$

Let r and s be two numbers such that:

$$0 \leq r \leq +\infty \quad , \quad 1 \leq s < +\infty.$$

We define the class $\mathcal{H}(W; r, s)$ in the following way:

a) if $0 \leq r < +\infty$, $1 < s$, we say that $f \in \mathcal{H}(W; r, s)$, if the following inequality holds:

$$R_f(u, v) \geq -\Psi(u, v, |f(u)|, |f(v)|)(1 + \|\operatorname{grad}^- f(u)\|^r)\|v - u\|^s \qquad \forall u, v \text{ in } D(f) \text{ with } \partial^- f(u) \neq \emptyset,$$

where $\Psi : D(f)^2 \times (\mathbb{R}^+)^2 \to \mathbb{R}^+$ is a function which is non decreasing in its real arguments and such that $(u, v) \mapsto \Psi(u, v, C_1, C_2)$ is continuous on $\{w \in W \mid |f(w)| \leq C\}^2$ for any C_1, C_2, C in $\mathbb{R}^+$;

b) if $0 \leq r < +\infty$, $1 = s$, we say that $f \in \mathcal{H}(W; r, 1)$, if the inequality of case a) holds with $s = 1$, and

Ψ has the additional property:

$$\Psi(u,u,C_1,C_2) = 0 \qquad \forall u \text{ in } D(f), \forall C_1, C_2 \text{ in } \mathbb{R}^+;$$

c) if $r = +\infty$, $1 < s$, we say that $f \in \mathcal{N}(W;\infty,s)$, if the following inequality holds:

$$R_f(u,v) \geq -\Phi(u,v,|f(u)|,|f(v)|,\|\mathrm{grad}^- f(u)\|)\|v-u\|^s \qquad \forall u, v \text{ in } D(f) \text{ with } \partial^- f(u) \neq \emptyset,$$

where $\Phi : D(f)^2 \times (\mathbb{R}^+)^3 \to \mathbb{R}^+$ is a function which is non decreasing in its real arguments and such that $(u,v) \mapsto \Phi(u,v,C_1,C_2,p)$ is continuous $\{w \in W \mid |f(w)| \leq C\}^2$ for any C_1, C_2, C, p in $\mathbb{R}^+$;

d) if $r = +\infty$, $1 = s$, we say that $f \in \mathcal{N}(W;\infty,1)$, if the inequality of case c) holds with $s = 1$ and Φ has the additional property:

$$\Phi(u,u,C_1,C_2,p) = 0 \qquad \forall u \text{ in } D(f), \forall C_1, C_2, p \text{ in } \mathbb{R}^+.$$

(5.2) Remark

Suppose that f is lower semicontinuous.It is clear that:

a) if f is convex, then $R_f \geq 0$;

b) if $f = f_o + f_1$, where f_o is convex and $f_1 \in C^{1,\epsilon}_{loc}$ with $\epsilon > 0$ (or C^1), then $f \in \mathcal{N}(H;0,1+\epsilon)$ ($f \in \mathcal{N}(H;0,1)$);

c) if f is (p,q)-convex (see definition (1.1) and theorem (2.5) of [7] and see [9], [10]), or if $f \in C(p,q)$ (see definition (1.6) of [3]), then $f \in \mathcal{N}(H;1,2)$;

d) if f is ϕ-convex of order r (see definition (4.1) of [15]), then $f \in \mathcal{N}(H;r,2)$;

e) if f is ϕ-convex (see definition (1.16) of [11], or also [5] and [15]), then $f \in \mathcal{N}(H;\infty,2)$.

We shall prove, in this section, that, under suitable compactness assumptions, if f belongs to one of the classes introduced above, then the following property holds:

$$\text{(5.3)} \qquad \forall u \text{ in } D(f): \quad |\nabla f|(u) < +\infty \Rightarrow \partial^- f(u) \neq \emptyset \text{ and } |\nabla f|(u) = \|\mathrm{grad}^- f(u)\|,$$

whose importance has been already pointed out at the end of §1. On the other hand it is clear that, if f verifies (5.3), then (see definition (2.1)):

$$f \in \mathcal{N}(W;r,s) \Rightarrow f \in K(W;r,s) \qquad \forall r \text{ in } [0,+\infty], \forall s \text{ in } [1,+\infty[$$

These are facts of basic importance to obtain existence theorems for strong evolution curves for functions in the classes introduced before, by using the theorems proved in §4.

(5.4) Theorem

a) Suppose that $f \in \mathcal{N}(W;\infty,1)$ and f is coercive at a point u in $D(f)$ (see definition (4.1)). Then, if $|\nabla f|(u) < +\infty$, we have that:

$$\partial^- f(u) \neq \emptyset \qquad and \qquad |\nabla f|(u) = \|\mathrm{grad}^- f(u)\|.$$

b) If $f \in \mathcal{N}(W;r,s)$ with $r \in [0,+\infty]$, $s \in [1,+\infty[$ and if f is coercive at every u in $D(f)$, then (5.3) holds and $f \in K(W;r,s)$.

The proof is carried out in (5.9).

Let us remark, first of all, that for the classes $\mathcal{N}(W;r,s)$ we can easily prove properties like those stated in proposition (2.3) for the corresponding classes $K(X;r,s)$: it suffices to replace, in those statements $|\nabla f|(u)$ by $\|\mathrm{grad}^- f(u)\|$.

We point out now some important facts.

(5.5) Lemma

Let $u \in D(f)$.

a) If $f \in \mathcal{N}(W;\infty,1)$, f is locally bounded from below at u, then:

$$\text{(5.6)} \quad \begin{cases} \text{for every sequence } (u_h)_h \text{ in } D(f), \text{ for every } \alpha \text{ in } H \text{ such that:} \\ \lim_{h\to\infty} u_h = u \ , \ \sup_{h\in\mathbb{N}}\{f(u_h)\} < +\infty \ , \ \liminf_{h\to\infty} f(u_h) \geq f(u) \text{ and} \\ \quad \partial^- f(u_h) \neq \emptyset \ , \ (\mathrm{grad}^- f(u_h))_h \text{ converges weakly to } \alpha \text{ then:} \\ \qquad \alpha \in \partial^- f(u) \ , \ \lim_{h\to\infty} f(u_h) = f(u). \end{cases}$$

b) If (5.6) holds and f is lower semicontinuous, then:

$$\text{(5.7)} \qquad \liminf_{\substack{v\to u \\ f(v)\leq C}} \|\mathrm{grad}^- f(v)\| \geq \|\mathrm{grad}^- f(u)\| \qquad \forall C \text{ in } \mathbb{R},$$

with the convention that, if $w \in D(f)$, $\partial^- f(w) = \emptyset$, then $\|\mathrm{grad}^- f(w)\| = +\infty$.

c) If (5.7)holds, at least for $C = f(u)$, and if the following property is verified:

$$\text{(5.8)} \quad \begin{cases} \text{for any } \epsilon > 0 \text{ there exist } \rho > 0, \mu_o > 0 \text{ such that the function:} \\ \qquad v \mapsto f(v) + \mu\|v-u\|^{1+\epsilon} \\ \text{has minimum on } \overline{B(u,\rho)} \text{ for any } \mu \geq \mu_o \end{cases}$$

(this is the case if, for instance, f is coercive at u), then:

$$|\nabla f|(u) < +\infty \Rightarrow \partial^- f(u) \neq \emptyset \text{ and } |\nabla f|(u) = \|\mathrm{grad}^- f(u)\|.$$

Proof

a) The thesis follows clearly by the inequality:

$$f(v) \geq f(u_h) + (\mathrm{grad}^- f(u_h), v - u_h) - \Phi(u_h, v, \sup_h\{|f(u_h)|\}, |f(v)|, \|\mathrm{grad}^- f(u_h)\|)\|v - u_h\|$$

$\forall v$ in $D(f)$, $\forall h$ in $\mathbb{N}$, where Φ is given by d) of definition (5.1).

b) The thesis follows immediately from (5.6), by the lower semicontinuity of the norm, with respect to weak convergence.

c) If $\epsilon > 0$ is given, there exist $\rho > 0, \mu_\circ \geq 0$ such that f is locally bounded from below on $\overline{B(u,\rho)}$ and such that there exists the minimum point u_μ of the function $v \mapsto f(v) + \mu\|v-u\|^{1+\epsilon}$, for every $\mu \geq \mu_\circ$. Then we have:

$$f(u_\mu) + \mu\|u_\mu - u\|^{1+\epsilon} \leq f(u) \qquad \forall \mu \geq \mu_\circ.$$

Therefore $\lim_{\mu\to+\infty} u_\mu = u$. Since $\|\cdot\|$ is differentiable, we have clearly:

$$0 \in \partial^- f(u_\mu) + \mu\alpha_\mu \quad \text{namely} \quad -\mu\alpha_\mu \in \partial^- f(u_\mu)$$

where

$$\alpha_\mu = (1+\epsilon)\|u_\mu - u\|^\epsilon \frac{u_\mu - u}{\|u_\mu - u\|}, \quad \text{if } u_\mu \neq u \quad , \quad \alpha_\mu = 0, \qquad \text{if } u_\mu = u.$$

Therefore:

$$f(u) \geq f(u_\mu) + \mu\|u_\mu - u\|^{1+\epsilon} \geq f(u_\mu) + \frac{\|\mu\alpha_\mu\|}{1+\epsilon}\|u_\mu - u\|,$$

which implies that:

$$\limsup_{\mu\to+\infty} \|\mu\alpha_\mu\| \leq (1+\epsilon)|\nabla f|(u).$$

By (5.7), since $f(u_\mu) \leq f(u)$ for every $\mu \geq \mu_\circ$, we have that, if $|\nabla f|(u) < +\infty$, then $\partial^- f(u) \neq \emptyset$ and $\|\mathrm{grad}^- f(u)\| \leq (1+\epsilon)|\nabla f|(u)$. Since ϵ is arbitrary, and since $|\nabla f|(u) \leq \|\mathrm{grad}^- f(u)\|$ (see (1.7)), then we concude that $|\nabla f|(u) = \|\mathrm{grad}^- f(u)\|$.

(5.9) **Proof of theorem (5.4)**

a) Since $f \in \mathcal{N}(W;\infty,1)$, and f is locally bounded from below at u (being coercive at u), then (5.6) holds. Furthermore, by the coerciveness of f at u, we have that (5.8) holds and f is lower semicontinuous at u, then (5.7) holds too. Now c) of lemma (5.5) gives the result.

b) It is an immediate consequence of a).

§6. Existence and regularity theorems in Hilbert spaces.

The existence and regularity theorems stated in this section are proved by going back to the analogous theorems for the metric case.

As in §5, W denotes a subset of a Hilbert space H and $f : W \to \mathbb{R} \cup \{+\infty\}$ is a function.

We shall prove the following theorems.

(6.1) **Theorem (existence)**

Suppose that $f \in \mathcal{N}(W;\infty,1)$ (see d) of definition (5.1)), $u_\circ \in \mathcal{D}(f)$ and f is coercive at $u_\circ$ (see definition (4.1))). Then there exist $T > 0$ and an absolutely continuous curve $\mathcal{U} : [0,T] \to W$ such that $\mathcal{U}$

is a strong evolution curve almost everywhere for f (see definition (1.8)) with $\mathcal{U}(0) = u_\circ$ and $f \circ \mathcal{U}(t) \leq f(u_\circ)$ $\forall t$ in $[0,T]$. Then (by (1.9)) we have that: $\partial^- f(\mathcal{U}(t)) \neq \emptyset$ almost everywhere on $[0,T]$ and:

$$\mathcal{U}'(t) = -\mathrm{grad}^- f(\mathcal{U}(t)) \qquad \textit{almost everywhere on } [0,T],$$
$$g'(t) = -\|\mathrm{grad}^- f(\mathcal{U}(t))\|^2 \qquad \textit{almost everywhere on } [0,T],$$

where $g : [0,T] \to \mathbb{R} \cup \{+\infty\}$ is a non increasing function such that $g(t) = f \circ \mathcal{U}(t)$ almost everywhere on $[0,T]$. Moreover $f \circ \mathcal{U}$ and $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ are lower semicontinuous on $[0,T]$ (with the convention that, if $w \in \mathcal{D}(f)$ and $\partial^- f(w) = \emptyset$, then we set $\|\mathrm{grad}^- f(w)\| = +\infty$).

The proof is in (6.5).

(6.2) **Theorem (regularity)**

Let $\mathcal{U} : I \to X$ be a strong evolution curve almost everywhere for f such that $f \circ \mathcal{U}$ is lower semicontinuous. Suppose that f is locally bounded from below on W. Then the following facts hold.

a) Suppose that $f \in \mathcal{N}(W;r,s)$ with $r \leq s$ (see a) and b) of definition (5.1)).
Then $f \circ \mathcal{U}$ is continuous, therefore $\mathcal{U}$ is a strong evolution curve for f (see definition (1.8)) and $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ is lower semicontinuous (with the convention that if $w \in \mathcal{D}(f)$ and $\partial^- f(w) = \emptyset$, then we set $\|\mathrm{grad}^- f(w)\| = +\infty$).
Moreover, for any t with $\partial^- f(\mathcal{U}(t)) \neq \emptyset$ (therefore almost everywhere on I) we have:

$$(6.3) \qquad \begin{cases} \mathcal{U}'_+(t) = -\mathrm{grad}^- f(\mathcal{U}(t)) \\ (f \circ \mathcal{U})'_+(t) = -\|\mathrm{grad}^- f(\mathcal{U}(t))\|^2 \end{cases}$$

and $(f \circ \mathcal{U})'_+(t) = -\infty$, if $\partial^- f(\mathcal{U}(t)) = \emptyset$.
Moreover we have that

$$\forall t \textit{ in } I: \qquad |\nabla f| \circ \mathcal{U}(t) < +\infty \Rightarrow \partial^- f(\mathcal{U}(t)) \neq \emptyset \quad \textit{and} \quad \|\mathrm{grad}^- f(\mathcal{U}(t))\| = |\nabla f| \circ \mathcal{U}(t).$$

b) Suppose that $f \in \mathcal{N}(W;r,s)$ with $r \leq s$ and $s > 1$.
Then, in addition to the properties stated in a), the following ones hold:

$$(6.4) \quad f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) = -\int_{t_1}^{t_2} \|\mathrm{grad}^- f(\mathcal{U}(t))\|^2 \, dt \qquad \forall t_1, t_2 \textit{ in } I;$$

$\partial^- f(\mathcal{U}(t)) \neq \emptyset \quad \forall t$ *in* $I \setminus \{\inf I\}$, *which implies that (6.3) hold* $\forall t$ *in* $I \setminus \{\inf I\}$
(and also for $t = \min I$ if I has minimum $f \circ \mathcal{U}(t) < +\infty, \partial^- f(\mathcal{U}(t)) \neq \emptyset$);

$\mathrm{grad}^- f(\mathcal{U}(\cdot))$ *is right continuous at t $\forall t$ in $I \setminus \{\inf I\}$ and bounded on $[t,T]$ $\forall T > t$*
therefore $\mathcal{U}$ and $f \circ \mathcal{U}$ are Lipschitz-continuous on $[t,T]$
(and also for $t = \min I$ if I has minimum $f \circ \mathcal{U}(t) < +\infty, \partial^- f(\mathcal{U}(t)) \neq \emptyset$).

c) Suppose that $f \in \mathcal{N}(W;\infty,s)$ with $s > 1$.
Then for every $t_\circ$ in $I \setminus \{\sup I\}$ such that $f \circ \mathcal{U}(t_\circ) < +\infty, \partial^- f(\mathcal{U}(t_\circ)) \neq \emptyset$ and $f \circ \mathcal{U}(t_\circ) \geq f \circ \mathcal{U}(t)$ for almost every $t \geq t_\circ$, there exists $\delta > 0$ such that the following properties hold on $[t_\circ, t_\circ + \delta]$:

$\mathcal{U}$ is a strong evolution curve for f;

$\partial^- f(\mathcal{U}(t)) \neq \emptyset \quad \forall t \quad$ *and (6.3), (6.4) hold;*

$\mathrm{grad}^- f(\mathcal{U}(\cdot))$ *is bounded and right continuous, therefore $\mathcal{U}$ and $f \circ \mathcal{U}$ are Lipschitz-continuous;*

$\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ *is lower semicontinuous.*

The proof is carried out in (6.8).

(6.5) Proof of (6.1)

Since f is coercive at u_0, there is $R > 0$ such that f is coercive at any u of $W_0 = \{u \mid \|u - u_0\| < R, f(u) \le f(u_0)\}$. Since $f \in \mathcal{K}(W; \infty, 1)$, we get, by a) of (5.4), that for every u in W_0 with $|\nabla f|(u) < +\infty$ it is $\partial^- f(u) \neq \emptyset$ and $\|\mathrm{grad}^- f(u)\| = |\nabla f|(u)$.

On the other hand it is clear that $|\nabla f_{W_0}|(u) = |\nabla f|(u)$ for any u in W_0, if we take in W_0 the metric induced by H and $f_{W_0} : W_0 \to \mathbb{R} \cup \{+\infty\}$ is the function defined by $f_{W_0}(v) = f(v)$. It follows that $f_{W_0} \in K(W_0; \infty, 1)$ and it is, of course, coercive at u_0.

By theorem (4.2) (applied with $X = W_0$) we have that there exist $T > 0$ and an absolutely continuous curve $\mathcal{U} : [0, T] \to W_0$, such that $\mathcal{U}$ is a strong evolution curve almost everywhere for f with $\mathcal{U}(0) = u_0$, $f \circ \mathcal{U}(t) \le f(u_0)$ $\forall t$ in $[0, T]$, $f \circ \mathcal{U}$ and $|\nabla f| \circ \mathcal{U}$ are lower semicontinuous.

On the other hand it is clear that, for what we have seen before, $|\nabla f_{W_0}|(u) = \|\mathrm{grad}^- f(\mathcal{U}(t))\|$ (with the usual convention), and then, for almost every t (precisely for all t's such that $|\nabla f| \circ \mathcal{U}(t) < +\infty$) we have that $\partial^- f(\mathcal{U}(t)) \neq \emptyset$.

Consequentely, by theorem (1.11), $\mathcal{U}$ is a strong evolution curve almost everywhere for f. a) of (1.9) completes the proof.

We need the following lemma to prove theorem (6.2).

(6.6) Lemma

Let $\mathcal{U} : I \to W$ be a strong evolution curve for f such that:

(6.7) $$\liminf_{t \to t_0} \|\mathrm{grad}^- f(\mathcal{U}(t))\| \ge \|\mathrm{grad}^- f(\mathcal{U}(t_0))\| \qquad \forall t_0 \text{ in } I$$

(with the convention that, if $w \in D(f)$ and $\partial^- f(w) = \emptyset$, then we set $\|\mathrm{grad}^- f(w)\| = +\infty$).
Then $|\nabla f| \circ \mathcal{U}(t) = \|\mathrm{grad}^- f(\mathcal{U}(t))\|$ for every t in I.

Proof

Let $t \in I$. By (6.7), since $|\nabla f| \circ \mathcal{U}(t) = \|\mathrm{grad}^- f(\mathcal{U}(t))\|$ for almost any t in I (see (1.9)), we get

$$\|\mathrm{grad}^- f(\mathcal{U}(t))\| \le \liminf_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|\mathrm{grad}^- f(\mathcal{U}(\tau))\| \, d\tau \le$$
$$\le \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} |\nabla f| \circ \mathcal{U}(\tau) \, d\tau \le |\nabla f| \circ \mathcal{U}(t),$$

where the last inequality is a consequence of lemma (3.9) part c). On the other hand $|\nabla f| \circ \mathcal{U}(t) \le \|\mathrm{grad}^- f(\mathcal{U}(t))\|$ (see (1.7)).

(6.8) Proof of theorem (6.2)

a) By the hypotheses, and (1.9), $\mathcal{U}$ is a curve of maximal slope almost everywhere for f, and there exists a negligible subset E of I such that:

$$\partial^- f(\mathcal{U}(t)) \neq \emptyset \text{ and } |\nabla f| \circ \mathcal{U}(t) = \|\mathrm{grad}^- f(\mathcal{U}(t))\| \qquad \forall t \text{ in } I \setminus E.$$

On the other hand, since $f \in \mathcal{K}(W; r, s)$, with $r \le s$, the following property is true:

$$\forall t_0 \text{ in } I, \forall (t_k)_k \text{ in } I \text{ such that } \partial^- f(\mathcal{U}(t_k)) \neq \emptyset \; \forall k \text{ in } \mathbb{N} \text{ and}$$
$$\lim_{k \to \infty} t_k = t_0, \lim_{k \to \infty} \|\mathrm{grad}^- f(\mathcal{U}(t_k))\| \|\mathcal{U}(t_k) - \mathcal{U}(t_0)\| = 0 \text{ then:}$$
$$\limsup_{k \to \infty} f \circ \mathcal{U}(t_k) \le f \circ \mathcal{U}(t_0).$$

hence (3.11) is true where E is the set introduced just now. By lemma (3.10) $f \circ \mathcal{U}$ is continuous and then $\mathcal{U}$ is a strong evolution curve for f.

Since $f \in \mathcal{K}(W; r, s) \subset \mathcal{K}(W; \infty, 1)$, $f \circ \mathcal{U}$ is lower semicontinuous and locally bounded from below on W, we have, by a) and b) of lemma (5.5), that (6.7) of lemma (6.6) holds. Therefore:

$$|\nabla f| \circ \mathcal{U}(t) = \|\mathrm{grad}^- f(\mathcal{U}(t))\| \text{ and } \liminf_{s \to t} |\nabla f| \circ \mathcal{U}(s) \ge |\nabla f| \circ \mathcal{U}(t) \qquad \forall t \text{ in } I.$$

Then (3.7) are verified, which imply, by lemma (1.12), that the equations (6.3) hold and $(f \circ \mathcal{U})'_+(t) = -\infty$, if $\partial^- f(\mathcal{U}(t)) = \emptyset$, since in such a case $|\nabla f| \circ \mathcal{U}(t) = +\infty$.

b) Since $s > 1$, we have that (see a) of definition (5.1)):

$$f(v) \ge f(u) + \langle \mathrm{grad}^- f(u), v - u \rangle - \Psi(u, v, |f(u)|, |f(v)|)(1 + \|\mathrm{grad}^- f(u)\|^r)\|v - u\|^s$$
$$\forall u, v \text{ in } D(f) \text{ with } \partial^- f(u) \neq \emptyset,$$

and we have seen in a) that $|\nabla f|(u) = \|\mathrm{grad}^- f(u)\|$ for every u in $\mathcal{U}(I)$. Therefore the assumptions of (3.13) hold, on any given interval $[t, T]$ contained in I, with $\omega(\sigma) = \gamma(\sigma) = C\sigma^{s-1}$, where C is a suitable constant (clearly we can suppose $r = s$).

It follows that $|\nabla f| \circ \mathcal{U}$ is right continuous and bounded on $[t, T]$, if $|\nabla f| \circ \mathcal{U}(t) < +\infty$. Since we have, for almost every t, that $|\nabla f| \circ \mathcal{U}(t) < +\infty$ and by step a), we get that $\partial^- f(\mathcal{U}(t)) \neq \emptyset$ for every t in $I \setminus \{\inf I\}$, and furthermore $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ is right-continuous at every t in $I \setminus \{\inf I\}$ with $\partial^- f(\mathcal{U}(t)) \neq \emptyset$.

On the other hand, by lemma (5.5), for any given t with $\partial^- f(\mathcal{U}(t)) \neq \emptyset$, we have that for every sequence $(t_k)_k$ converging to t from the right, and such that $(\mathrm{grad}^- f(\mathcal{U}(t_k)))_k$ converges weakly to an element α in H, it turns out that either $\alpha = \mathrm{grad}^- f(\mathcal{U}(t))$ or $\|\alpha\| > \|\mathrm{grad}^- f(\mathcal{U}(t))\|$. By the right continuity of $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ it follows that $\alpha = \mathrm{grad}^- f(\mathcal{U}(t))$, and then $\mathrm{grad}^- f(\mathcal{U}(\cdot))$ is right continuous.

Now (6.4) follows immediately from (3.12).

c) Since $f \in \mathcal{K}(W; \infty, s)$ with $s > 1$, we have that (see c) of (5.1)):

$$f(v) \ge f(u) + \langle \mathrm{grad}^- f(u), v - u \rangle - \Phi(u, v, |f(u)|, |f(v)|, \|\mathrm{grad}^- f(u)\|)\|v - u\|^s$$
$$\forall u, v \text{ in } D(f) \text{ with } \partial^- f(u) \neq \emptyset.$$

Since $\mathcal{U}$ is a strong evolution curve almost everywhere for f, we can find a negligible subset F of I such that:

$$\partial^- f(\mathcal{U}(t)) \neq \emptyset \text{ and } |\nabla f| \circ \mathcal{U}(t) = \|\mathrm{grad}^- f(\mathcal{U}(t))\| \qquad \forall t \text{ in } I \setminus F.$$

Then, if t_0 verifies the given hypotheses and $T \in I$ with $T > t_0$, then the assumptions of lemma (3.21) hold on $[t_0, T]$. It follows that $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\| \in L^\infty(t_0, t_0 + \delta)$ for a suitable $\delta > 0$ and there exists a subset I' of I such that $I \setminus I'$ is negligible and

$$\limsup_{\substack{t \to t_0^+ \\ t \in I'}} \|\mathrm{grad}^- f(\mathcal{U}(t))\| \le |\nabla f| \circ \mathcal{U}(t_0).$$

Since $f \in \mathcal{K}(W;\infty,1)$, $f \circ \mathcal{U}$ is lower semicontinuous and locally bounded from below on W, we get, by a) of lemma (5.6), that $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ is lower semicontinuous on I, therefore $\partial^- f(\mathcal{U}(t)) \neq \emptyset$ for every t in $[t_o, t_o+\delta]$ and $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ is bounded on $[t_o, t_o+\delta]$. Moreover:

$$\|\mathrm{grad}^- f(\mathcal{U}(t_o))\| \le \liminf_{t \to t_o^+} \|\mathrm{grad}^- f(\mathcal{U}(t))\| \le \limsup_{t \to t_o^+} \|\mathrm{grad}^- f(\mathcal{U}(t))\| \le$$

$$\limsup_{\substack{t \to t_o^+ \\ t \in J'}} \|\mathrm{grad}^- f(\mathcal{U}(t))\| \le |\nabla f| \circ \mathcal{U}(t_o) \le \|\mathrm{grad}^- f(\mathcal{U}(t_o))\|.$$

Then $\|\mathrm{grad}^- f(\mathcal{U}(t_o))\| = |\nabla f| \circ \mathcal{U}(t_o)$ and $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ is right continuous at t_o.

Finally $f \circ \mathcal{U}$ is continuous on $[t_o, t_o+\delta]$ because it is upper semicontinuous on $[t_o, t_o+\delta]$, since $\|\mathrm{grad}^- f(\mathcal{U}(\cdot))\|$ is bounded on $[t_o, t_o+\delta]$ and $f \in \mathcal{K}(W;\infty,1)$. It follows that $\mathcal{U}$ is a strong evolution curve for f on $[t_o, t_o+\delta]$, therefore $f \circ \mathcal{U}$ is non increasing on $[t_o, t_o+\delta]$. This implies that we can repeat the previous reasoning, made at the point t_o, for any other t of $[t_o, t_o+\delta]$. Then:

$$|\nabla f| \circ \mathcal{U}(t) = \|\mathrm{grad}^- f(\mathcal{U}(t))\| \text{ and } \lim_{s \to t^+} \|\mathrm{grad}^- f(\mathcal{U}(s))\| = \|\mathrm{grad}^- f(\mathcal{U}(t))\| \qquad \forall t \text{ in } [t_o, t_o+\delta].$$

It follows that $|\nabla f| \circ \mathcal{U}$ is right continuous and bounded on $[t_o, t_o+\delta]$. Then, as usual, we get (6.3) thanks to proposition (3.6) and lemma (1.12).

(6.4) follows by lemma (3.12).

To prove the right continuity of $\mathrm{grad}^- f(\mathcal{U}(\cdot))$ we reason as in b).

It is easy to prove the following result, analogous to (4.12).

(6.9) Theorem

Suppose that f is lower semicontinuous and that for every u in $D(f)$ there exist $T > 0$ and $\mathcal{U} : [0,T] \to W$, which is a strong evolution curve (almost everywhere) for f such that $\mathcal{U}(0) = u$.
Then for every u_o in $D(f)$ there exist $\bar{T} > 0$ and $\bar{\mathcal{U}} : [0,\bar{T}[\to W$, such that $\bar{\mathcal{U}}$ is a strong evolution curve (almost everywhere) for f with $\bar{\mathcal{U}}(0) = u_o$ and at least one of the following properties holds:

$$\bar{T} = +\infty \quad , \quad \operatorname*{ess\,inf}_{t \in [0,\bar{T}[} \{ f \circ \bar{\mathcal{U}}(t) \} = -\infty \quad , \quad \lim_{t \to \bar{T}^-} \bar{\mathcal{U}}(t) = \bar{u} \notin W.$$

§7. Some applications.

We illustrate here some problems which can be studied using the theory developed so far.

The problem of "geodesics with respect to an obstacle", trated below in (7.1), has been studied in [16] making use of precisely the results stated in [6], for the curves of maximal slope in metric spaces, whose proofs are given in this paper.

The problem treated in (7.2), concerning the "eigenvalues of the Laplace operator with respect to an obstacle", has been studied in [17] and [3], using the theory developed in [11], which takes into account cases with lack of coerciveness conditions, but requires stronger estimates for the function. As we shall see, such problem can be as well treated with the theory developed in this paper.

The problem treated in (7.3) concerns the "heat equation", perturbed by a merely continuous term, on a C^1 non convex constraint. Owing to the lack of regularity of both the perturbation and the constraint, the theory developed in [11] does not apply, nevertheless one can use the theorems proved in this paper.

We remark that, if the constraint were more regular ($C^{1,1}$ for instance), then such a problem could be also studied by the results of the paper [20].

(7.1) Geodesics with respect to an obstacle (see [2], [16], [23], [24])

Let K be a smooth compact submanifold of $\mathbb{R}^n$, of dimension n ($\mathring{K} \neq \emptyset$, ∂K is an hypersurface).
We say that a curve $\gamma : [0,T] \to \mathbb{R}^n \setminus \mathring{K}$ is a "geodesic with respect to the obstacle K", if

γ and $\dot\gamma$ are absolutely continuous,

there exists $\lambda : [0,1] \to [0,+\infty[$ such that, denoting by $\nu(x)$ the exterior normal to K at x in ∂K :

$\ddot\gamma(s) = 0$ for almost every s with $\gamma(s) \notin K$

$\ddot\gamma = \lambda(s)\nu(\gamma(s))$ for almost every s with $\gamma(s) \in \partial K$.

In [16] it is proved that:

if A and B are "outside of K" (that is if are in the unbounded connected component of $\mathbb{R}^n \setminus \mathring{K}$), then there exist infinitely many geodesics with respect to K joining A and B.

For this goal one considers the Hilbert space $H = L^2(0,1;\mathbb{R}^n)$ with the usual inner product:

$$(\gamma,\delta) = \int_0^1 (\gamma(s),\delta(s))\, ds \qquad \forall \gamma,\delta \text{ in } H$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^n$ and ,if $A, B \in \mathbb{R}^n \setminus \mathring{K}$ are given , the function $f : H \to \mathbb{R} \cup \{+\infty\}$ defined by:

$$f(\gamma) = \begin{cases} \frac{1}{2}\int_0^1 |\dot\gamma(s)|^2\, ds, & \text{if } \gamma \in D(f) \\ +\infty, & \text{otherwise} \end{cases}$$

where:

$$D(f) = \{\gamma \in H^{1,2}(0,1;\mathbb{R}^n) \mid \gamma(0) = A, \gamma(1) = B, \gamma(s) \notin \mathring{K}\ \forall s \text{ in } [0,1]\}$$

The proof, given in [16], is carried out through three steps:

1) the geodesics with respect to K joining A an B are "critical points from below"for f;

2) $\forall\gamma_0$ in $D(f)$ there exist a strong evolution curve $\mathcal{U} : [0,+\infty[\to H$ for f such that $\mathcal{U}(0) = \gamma_0$; for this goal an existence theorem is stated (see (2.3) of [16]) with no proof: such theorem is a particular case of (6.1)-(6.2); furthermore , $\forall C$ in $\mathbb{R}$, $\mathcal{U}$ depends continuously on γ_0, as γ_0 varies in $\{\gamma \mid f(\gamma) \le C\}$;

3) by means of the flow of the strong evolution curves for f, one gets the result, adjusting in a suitable way Ljusternik Schnirelmann's techniques to a class of lower semicontinuous functions.

We illustrate now in a more detailed fashion how step 2) is carried out.

a) Let $\gamma \in D(f)$. From theorems (1.6) and (2.4) step a) of [16] it follows

$$\partial^- f(\gamma) \ne \emptyset \Leftrightarrow \gamma \in \mathrm{H}^{2,2}(0,1;\mathbb{R}^n);$$
$$\partial^- f(\gamma) \ne \emptyset \Rightarrow \mathrm{grad}^- f(\gamma) = -\left[\ddot\gamma - (\ddot\gamma, \nu\circ\gamma)^+ I_{C(\gamma)}(\nu\circ\gamma)\right]$$

where $I_{C(\gamma)} : [0,1] \to \mathbb{R}$ has value 1 on the set $C(\gamma) = \{s \in [0,1] \mid \gamma(s) \in \partial K\}$ and value 0 elsewhere; if $a \in \mathbb{R}$, then a^+ denotes the positive part of a.
In particular $0 \in \partial^- f(\gamma)$ if and only if γ is a geodesic with respect to K joining A and B.

b) Let $\gamma \in D(f)$ and $\partial^- f(\gamma) \ne \emptyset$. From theorem (2.1) of [16] it follows:

$$f(\gamma+\delta) \ge f(\gamma) + \int_0^1 \left(\mathrm{grad}^- f(\gamma)(s), \delta(s)\right)\, ds +$$
$$- C(f(\gamma))^2 \int_0^1 (\delta(s))^2\, ds \qquad \forall\delta \text{ in } H.$$

for a suitable constant C. Then $f \in \mathcal{N}(H;0,2)$.

c) It follows, by theorems (6.1), (6.2) and (6.9) of this paper, that, for every γ_0 in $D(f)$ there exists an absolutely continuous strong evolution curve for f, $\mathcal{U} : [0,+\infty[\to D(f)$ with $\mathcal{U}(0) = \gamma_0$, such that for every $t > 0$ there exists $\mathcal{U}'_+(t)$, $\mathcal{U}(t) \in \mathrm{H}^{2,2}(0,1;\mathbb{R}^n)$ and:

$$\mathcal{U}'_+(t)(s) = \frac{d^2}{ds^2}\mathcal{U}(t)(s) \qquad \text{for almost any } s \text{ with } \mathcal{U}(t)(s) \notin K,$$
$$\mathcal{U}'_+(t)(s) = \frac{d^2}{ds^2}\mathcal{U}(t)(s) - \left(\frac{d^2}{ds^2}\mathcal{U}(t)(s), \nu(\mathcal{U}(t)(s))\right)^+ \nu(\mathcal{U}(t)(s))$$
$$\text{for almost any } s \text{ with } \mathcal{U}(t)(s) \text{ in } \partial K$$

$$f\circ\mathcal{U}(t_2) - f\circ\mathcal{U}(t_1) = -\int_{t_1}^{t_2}\int_0^1 |\mathcal{U}'(t)(s)|^2\, ds\, dt \qquad \forall t_1, t_2 \text{ in } [0,+\infty[.$$

Furthermore all the properties listed in (6.2) hold. In [16] it is also proved that $\mathcal{U}$ is unique and depends continuously on $(\gamma_0, f(\gamma_0))$, using in a standard way the inequality:

$$\langle \mathrm{grad}^- f(\gamma_2) - \mathrm{grad}^- f(\gamma_1), \gamma_2 - \gamma_1 \rangle \ge -C\left((f(\gamma_1))^2 + (f(\gamma_2))^2\right)\|\gamma_2 - \gamma_1\|^2$$

which follows immediately from the one written in b).

A sligthly different problem arises from the study of the geodesics with respect to K, submitted to the condition that the end points are forced to lie in a given submanifold M of $\mathbb{R}^n \setminus \mathring{K}$; in [23] it is shown that, if $D(f)$ is replaced by:

$$D(f) = \{\gamma \in \mathrm{H}^{1,2}(0,1;\mathbb{R}^n) \mid \gamma(0) \in M, \gamma(1) \in M, \gamma(s) \notin \mathring{K}\ \forall s \text{ in } [0,1]\},$$

then the function f belongs to $\mathcal{N}(H;2,2)$. Also in this case, multiplicity results are proved.
Finally the problem of closed geodesics with respect to K is faced in [24].

(7.2) Eigenvalues of the Laplace operator with respect to an obstacle (see [3], [4] , [17])

Let Ω be a bounded open subset of $\mathbb{R}^n$. Suppose that $g : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Caratheodory function, namely $g(x,\cdot)$ is continuous for almost every x and $g(\cdot,s)$ is measurable for every s, $\varphi_1, \varphi_2 : \Omega \to \mathbb{R}$ are measurable functions with $\varphi_1 \le \varphi_2$ almost everywhere in Ω. Let $\rho > 0$.
We make the following hypotheses:

(g.1) $$G(x,s) = \int_0^s g(x,\sigma)\, d\sigma \ge -a(x) - bs^2 \qquad \forall x \text{ in } \Omega, \forall s \text{ in } \mathbb{R},$$
for suitable a in $\mathrm{L}^1(\Omega)$, b in $\mathbb{R}$;

(g.2) $$\frac{g(x,s_2) - g(x,s_1)}{s_2 - s_1} \ge -C \qquad \forall x \text{ in } \Omega, \forall s_1, s_2 \text{ in } \mathbb{R};$$
for a suitable C in $\mathbb{R}$

(g.3) $$G(\cdot,s) \in \mathrm{L}^1(\Omega) \qquad \forall s \text{ in } \mathbb{R}.$$

Set $H = \mathrm{L}^2(\Omega)$ with the usual inner product, and:

$$K = \{u \in H \mid \varphi_1 \le u \le \varphi_2 \text{ almost everywhere on } \Omega\} \quad , \quad S_\rho = \left\{u \in H \mid \int_\Omega u^2\, dx = \rho\right\},$$
$$K_g = \{u \in \mathrm{H}^1_0(\Omega) \cap K \mid G(\cdot,u) \in \mathrm{L}^1(\Omega)\}.$$

Let us consider the function $f : H \to \mathbb{R}\cup\{+\infty\}$, defined by:

$$f(u) = \begin{cases} \frac{1}{2}\int_\Omega |Du|^2\, dx + \int_\Omega G(x,u)\, dx, & \text{if } u \in K_g \cap S_\rho \\ +\infty, & \text{otherwise.} \end{cases}$$

We remark that the "constraint"$K_g$is neither convex nor regular. It turns out that:

a) $\mathcal{D}(f) = K_g \cap S_\rho$, f is lower semicontinuous and the sets $\{u \mid f(u) \leq C\}$ are compact for any C;
b) for every u in $\mathcal{D}(f)$ such that:

$$\mathrm{meas}(\{x \in \Omega \mid \varphi_1(x) < u(x) < 0\} \cup \{x \in \Omega \mid \varphi_2(x) > u(x) > 0\}) > 0$$

(see the following point d)) we have that:

b1) if $\alpha \in H$, then:

$$\alpha \in \partial^- f(u) \Leftrightarrow \begin{cases} g(\cdot, u) \in \mathrm{L}^1(\Omega) \\ \exists\lambda \text{ in } \mathbb{R} \text{ such that:} \\ \int_\Omega \mathrm{D}u\mathrm{D}(v-u)\,\mathrm{d}x + \int_\Omega g(x,u)(v-u)\,\mathrm{d}x + \\ +\lambda\int_\Omega u(v-u)\,\mathrm{d}x \geq \int_\Omega \alpha(v-u)\,\mathrm{d}x \qquad \forall v \text{ in } K_g \end{cases}$$

(see (3.13) of [3]); if $0 \in \partial^- f(u)$, we say that u is an eigenfunction of the operator $v \mapsto \Delta v - g(\cdot, v)$ with respect to φ_1 and φ_2, with eigenvalue λ;

b2) there exists a neigborhood W of u such that $f \in \mathcal{N}(W; 1, 2)$ (see a1) of (3.13) and definition (1.6) of [3]);

c) by theorems (6.1) and (6.2) it follows that, for any u_o in $\mathcal{D}(f)$ such that:

$$\mathrm{meas}(\{x \in \Omega \mid \varphi_1(x) < u_o(x) < 0\} \cup \{x \in \Omega \mid \varphi_2(x) > u_o(x) > 0\}) > 0$$

there exist $T > 0$, $\mathcal{U} : [0,T] \to \mathrm{L}^2(\Omega)$, with $\mathcal{U}$ is absolutely continuous, $\mathcal{U}(0) = u_o$ and $\Lambda : [0,T] \to \mathbb{R}$, such that $\mathcal{U}(t) \in \mathcal{D}(f) \quad \forall t$ in $[0,T]$ and for almost every t in $[0,T]$:

$$g(\cdot, \mathcal{U}(t)) \in \mathrm{L}^1(\Omega),$$

$$\int_\Omega \mathcal{U}'(t)(v - \mathcal{U}(t))\,\mathrm{d}x + \int_\Omega \mathrm{D}\mathcal{U}(t)\mathrm{D}(v - \mathcal{U}(t))\,\mathrm{d}x +$$

$$+ \int_\Omega g(x, \mathcal{U}(t))(v - \mathcal{U}(t))\,\mathrm{d}x + \Lambda(t)\int_\Omega \mathcal{U}(t)(v - \mathcal{U}(t))\,\mathrm{d}x \geq 0 \quad \forall v \text{ in } K_g;$$

(from the variational inequality above, by usual techniques, it is possible to deduce the unicity of $\mathcal{U}$ and its continuous dependence on $(u_o, f(u_o))$);

d) the hypothesis made on u, in b), implies that K and S_ρ "are not tangent at u", in a suitable sense (see (3.12) of [3]) and this fact ensures, by theorem (3.13) of [3], that b2) holds. In [4] some assumptions on φ_1, φ_2 and ρ are considered, which ensure that K and S_ρ are not tangent at any u in $\mathcal{D}(f)$.

Under some additional symmetry assumptions a multiplicity results for eigenfunctions of $v \mapsto \Delta v - g(\cdot, v)$ with respect to φ_1 and φ_2 is proved in [4].

(7.3) HEAT EQUATION WITH C^1 NON CONVEX CONSTRAINTS (see [21])

Let Ω be a bounded open set of $\mathbb{R}^n$ with $n \geq 3$. Let $g, h : \Omega \times \mathbb{R} \to \mathbb{R}$ be two Caratheodory functions and $\rho > 0$.

We make the following hypotheses.

(g.1) there exist a_o in $\mathrm{L}^1(\Omega)$, b_o in $\mathbb{R}$, $p_o \leq 2 + 4/n$ such that:

$$G(x,s) = \int_0^s g(x,\sigma)\,\mathrm{d}\sigma \geq -a_o(x) - b_o|s|^{p_o} \qquad \forall x \text{ in } \Omega, \forall s \text{ in } \mathbb{R};$$

(g.2) there exist a_1 in $\mathrm{L}^2(\Omega)$, b_1 in $\mathbb{R}$, $p_1 \leq 2^* = 2 + 4/(n-2)$ such that:

$$g(x,s_2) - g(x,s_1) \geq -a_1(x) - b_1(|s_1| + |s_2|)^{p_1/2}$$
$$\forall x \text{ in } \Omega, \forall s_1, s_2 \text{ in } \mathbb{R} \text{ with } s_1 \leq s_2;$$

(g.3) $G(\cdot, s)$ is integrable on Ω for every s in $\mathbb{R}$;

(h) there exist c in $\mathrm{L}^2(\Omega)$, d in $\mathbb{R}$ such that:

$$|h(x,s)| \leq c(x) + d|s| \qquad \forall x \text{ in } \Omega, \forall s \text{ in } \mathbb{R}.$$

Let $H = \mathrm{L}^2(\Omega)$, with the usual inner product, and consider the constraint V_ρ defined by:

$$V_\rho = \left\{ v \in H \mid \int_\Omega \left(\int_0^{v(x)} h(x,\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}x = \rho \right\}.$$

Let $f_1, f : H \to \mathbb{R} \cup \{+\infty\}$ be the functions defined by:

$$f_1(u) = \begin{cases} \int_\Omega |\mathrm{D}u|^2\,\mathrm{d}x + \int_\Omega G(x,u)\,\mathrm{d}x, & \text{if } u \in \mathrm{H}^1_o(\Omega) \\ +\infty, & \text{otherwise} \end{cases}, \quad f(u) = \begin{cases} f_1(u), & \text{if } u \in V_\rho \\ +\infty, & \text{if } u \notin V_\rho \end{cases}$$

The following facts hold:

a) $\mathcal{D}(f_1) = \{u \in \mathrm{H}^1_o(\Omega) \mid G(\cdot, u) \in \mathrm{L}^1(\Omega)\}$, $\mathcal{D}(f) = \mathcal{D}(f_1) \cap V_\rho$.

b) f_1 and f lower semicontinuous and the sets;

$$\{v \mid f_1(v) \leq C\} \quad , \quad \{v \mid f(v) \leq C\}$$

are compact in H for every C in $\mathbb{R}$.

c) $f_1 \in \mathcal{N}(H; 0, 1)$ (see definition (5.1)) and, if $u \in \mathcal{D}(f_1)$, $\alpha \in H$:

$$\alpha \in \partial^- f_1(u) \Leftrightarrow \begin{cases} g(\cdot, u) \in \mathrm{L}^1(\Omega) \\ \Delta u - g(\cdot, u) = \alpha \quad \text{(in the distributional sense).} \end{cases}$$

d) For every u_o in $\mathcal{D}(f)$ such that $h(\cdot, u_o) \not\equiv 0$ there exists a neighborhood W_o of u_o such that $f \in \mathcal{N}(W_o; 1, 1)$ and if $u \in \mathcal{D}(f) \cap W_o$, $\alpha \in H$, we have that:

$$\alpha \in \partial^- f(u) \Leftrightarrow \exists\lambda \text{ in } \mathbb{R}, \exists\alpha_1 \text{ in } \partial^- f_1(u) \text{ such that } \alpha = \alpha_1 - \lambda h(\cdot, u),$$

in particular, if $\partial^- f(u) \neq \emptyset$, we have that:

$$g(\cdot, u) \in \mathrm{L}^1(\Omega) \quad , \quad \Delta u - g(\cdot, u) \in \mathrm{L}^2(\Omega)$$
$$\mathrm{grad}^- f(u) = -\Delta u + g(\cdot, u) - \lambda_o h(\cdot, u),$$

where

$$\lambda_* = \frac{\int_\Omega(-\Delta u + g(x,u))h(x,u)\,dx}{\int_\Omega(h(x,u))^2\,dx}.$$

Using the theory developed in this paper we obtain the following result:

e) If assumptions (g.1), (g.2), (g.3) and (h) hold, if $\rho \in \mathbb{R}$, $u_* \in D(f) = \{v \in H^1_0(\Omega) \mid G(\cdot,u) \in L^1(\Omega), \int_\Omega \left(\int_0^u h(x,\sigma)\,d\sigma\right)\,dx = \rho\}$, $h(\cdot,u_*) \neq 0$, then there exist $T > 0$, $U : [0,T] \to H$, $\Lambda : [0,T] \to \mathbb{R}$, such that U is absolutely continuous $U(0) = u_*$, $U(t) \in D(f)$ $\forall t$ in $[0,T]$ and for almost every t in $[0,T]$ we have:

$$\begin{cases} g(\cdot,u) \in L^1(\Omega) \\ U'(t) - \Delta U(t) + g(\cdot,U(t)) + \Lambda(t)h(\cdot,U(t)) = 0 \\ \quad\text{(in the distributional sense.)} \end{cases}$$

Furthermore:

$$\Lambda(t) = \frac{\int_\Omega(-\Delta U(t) + g(x,U(t)))h(x,U(t))\,dx}{\int_\Omega(h(x,U(t)))^2\,dx},,$$

the functions:

$$t \mapsto \int_\Omega |DU(t)|^2\,dx \quad , \quad t \mapsto \int_\Omega G(x,U(t))\,dx$$

are continuous and their sum is non increasing.

REFERENCES

[1] H.BREZIS , *Opérateurs maximaux monotones et semigroupes de contraction dans les espaces de Hilbert* ,North-Holland Mathematics Studies , n. 5 , Notas de Matematica (50) , Amsterdam-London , 1973.

[2] A.CANINO, *On p-convex sets and geodesics*, to appear

[3] G.CHOBANOV , A.MARINO ,D.SCOLOZZI , *Evolution equations for the eigenvalue problem for the Laplace operator with respect to an obstacle*, submitted to Ann. Scuola Normale Sup.Pisa.

[4] G.CHOBANOV , A.MARINO ,D.SCOLOZZI , to appear.

[5] E.DE GIORGI , M.DEGIOVANNI , A.MARINO , M.TOSQUES , *Evolution equations for a class of non linear operators* , Atti Accadem. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 75 (1983) , 1-8.

[6] E.DE GIORGI , A.MARINO , M.TOSQUES , *Problemi di evoluzione in spazi metrici e curve di massima pendenza* , Atti Accadem. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 68 (1980) , 180-187.

[7] E.DE GIORGI A.MARINO , M.TOSQUES , *Funzioni (p,q)-convesse* , Atti Accadem. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 73 (1982) , 6-14.

[8] M.DEGIOVANNI , *Parabolic equations with time-dependent boundary conditions* , Ann. Mat. Pura Appl. , (IV), Vol. CXLI, 223-264.

[9] M.DEGIOVANNI , A.MARINO , M.TOSQUES , *General properties of (p,q)-convex functions and (p,q)-monotone operators* , Ricerche Mat. 32 (1983) , 285-319.

[10] M.DEGIOVANNI , A.MARINO , M.TOSQUES , *Evolution equations associated with (p,q)-convex functions and (p,q)-monotone operators* , Ricerche Mat. 33 (1984) , 81-112.

[11] M.DEGIOVANNI , A.MARINO , M.TOSQUES , *Evolution equations with lack of convexity* , Nonlinear Anal. The. Meth. and Appl., Vol. 9, 12 (1985), 1401-1443.

[12] M.DEGIOVANNI , M.TOSQUES ,*Evolution equations for (φ,f)-monotone operators* , Boll. Un. Mat.Ital. (6) 6-B, 537-568.

[13] D.KINDERLEHRER , G.STAMPACCHIA , *An introduction to variational inequalities and their applications* , Pure and applied Mathematics , 88 ,Academic Press , New York , London, Toronto , Ont. 1980.

[14] G.LETTA , *Teoria elementare dell'integrazione* , Boringhieri 1976.

[15] A.MARINO , *Evolution equation and multiplicity of critical points with respect to an obstacle* ,Contribution to Modern Calculus of Variations , Cesari ed. Res.Notes in Math. , Pitman (to appear).

[16] A.MARINO , D. SCOLOZZI , *Geodetiche con ostacolo* , Boll. Un. Mat. Ital. B (6) 2 (1983) , 1-31.

[17] A.MARINO , D.SCOLOZZI , *Autovalori dell'operatore di Laplace ed equazioni evoluzione in presenza di ostacolo* , Problemi differenziali e teoria dei punti critici (Bari , 1984) , pp. 137-155 , Pitagora , Bologna , 1984.

[18] A.MARINO , M.TOSQUES , *Curves of maximal slope for a certain class of non regular functions* , Boll.Un.Mat.Ital.B (6) 1 (1982) , 143-170.

[19] E.MITIDIERI, M.TOSQUES ,*Volterra integral equations associated with a class of nonlinear operators in Hilbert spaces* , Ann. Fac. Sci. Tolouse Math., (5), VIII, n. 2 (1986-87).

[20] E.MITIDIERI, M.TOSQUES, *Nonlinear integrodifferential equation in Hilbert spaces: the variational case*, Preoceedings of the congress "Volterra integral equations in Banach spaces and applications" (Trento, Feb. 1987), to appear.

[21] C.SACCON , *Perturbazioni continue della equazione del calore in presenza di vincoli non convessi* , to appear

[22] C.SACCON , *On an evolution problem with free boundary* , to appear in Houston J. of Math.

[23] D.SCOLOZZI, *Esistenza e molteplicita' di geodetiche con ostacolo con estremi variabili*, Ricerche Mat. 33 (1984), 171-201

[24] D.SCOLOZZI, *Un teorema di esistenza di una geodetica chiusa su varieta' con bordo*, Boll. U.M.I. (6) 4-A (1985), 451-457

[25] M.TOSQUES , *Quasi autonomous evolution equations associated with* (ϕ,f)*-monotone operators* , to appear