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COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS
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EQUIVARIANT MORSE THEORY

PART I

A review of equivariant Morse theory

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Lecture 1

A review of equivariant Morse theory

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1 Introduction

Whenever a mathematical problem has a symmetry group, it is recommended to use the symmetry to gain insight on the problem. Variational problems are no exception, and the aim of this first lecture is to show how one should formulate the critical point theories of Morse or Ljusternik-Schnirelman in this framework. In the next lectures we shall illustrate this by proving some new results and giving simpler proofs or improving known ones in the field of periodic solutions for Hamiltonian systems.

2 Ordinary and Equivariant Morse theory

Let X be a Hilbert manifold without boundary (in most cases we could limit ourselves to finite dimensional or at the expense of complicating the proofs extend the results to Banach manifolds) on which a compact Lie group operates by isometries.

Remark 1 : If $m(x)(\xi,\xi)$ is any metric on X, and G operates on X, then $m^*(x)(\xi,\xi) = \int_G m(gx)(g^*\xi, g^*\xi)\, dg$ is a new metric invariant by G (dg is the Haar measure on G)

Remark 2 : We are only interested in the cases $G = S^1$ or $G = \mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$

Let now f be a C^2 function on X which is G-equivariant (and satisfies the Palais Smale condition (cf infra)). Let us first see how one could work out a Morse theory for f that takes into account the symmetry group.

Ordinary Morse theory can be summarized as follows :

(this is the case $G = \{e\}$)

if f has c_k nondegenerate critical points of index k,

set $M(f;t) = \sum_{k=0}^{\infty} c_k t^k \in \mathbb{N}[[t]]$

$$P(X,t) = \sum_{k=0}^{\infty} b_k t^k \in \mathbb{N}[[t]]$$

where $b_k = \dim H^k(X, \mathbb{F})$ for some field $\mathbb{F}$.

Then the main result in Morse theory is:

<u>Theorem 1.1</u> (Morse inequalities)

There exists $Q(f,t) \in \mathbb{N}[[t]]$ such that

$$M(f,t) - P(X,t) = (1+t)\, Q(f,t) \qquad \square$$

As a result $m_k \geq b_k$, if one has $m_k = b_k$, then f is said to be perfect. Here, every critical point of f of index k contributes to the Morse polynomial by t^k and we shall say that $P(X,t)$ is the Poincare polynomial of the ordinary cohomology of X with coefficients in $\mathbb{F}$.

Remark 3 : We shall here assume that $\mathbb{F}$ is given. We will take $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}_p$ in our applications.

Now, still for $G = \{e\}$, we can extend Theorem 1.1 to the case where f has nondegenerate critical manifolds : $f'(x) = 0$ is the union of compact manifolds C_i, and the null space of $f''(x)$ for x in C_i is equal to $T_x C_i$, the tangent space to C_i at x.

Now at each x in C_i, $f''(x)$ has a negative eigenspace $\nu_i^-(x)$. The $\nu_i^-(x)$ for x in C_i can be glued together, and yield a vector bundle ν_i^- over C_i. Let θ_i be the orientation bundle of ν_i^-, θ_i can be defined as $\Lambda^{\dim \nu_i^-} \nu_i^-$

or as the determinant bundle of ν_i^-.

Then the spaces $H^k(C_i, \theta_i)$ are defined, and let $P(C_i, \theta_i; t) = P_i(t)$ be the corresponding Poincaré polynomial. If $k_i = \dim \nu_i^-$, set

$$M(f,t) = \sum t^{k_i} P_i(t)$$

where the sum is on all critical manifolds C_i.

Then with this new Morse polynomial, and keeping $P(X,t)$ as before, theorem 1.1 still holds.

Before we deal with the case $G \neq \{e\}$, we must define equivariant cohomology of X. If the action of G on X is free (i.e $gx = x \Rightarrow g = e$) then X/G is a manifold, and f induces a function f_G on X/G which is a Morse function if and only if the critical points of f are nondegenerate critical orbits. Then we can define $M_G(f;t) = M(f_G, t)$ and a critical orbit of index k contributes to $M_G(f,t)$ as t^k, and the new Poincaré polynomial is $P(X/G;t)$.

If the action of G is not free, we try to make it free as follows:

Let S be a compact manifold with a free G action, consider $X_S = X \times S$ with the diagonal free G action $g.(x,s) = (gx, gs)$, and the equivariant function $f_S(x,s) = f(x)$. Then if Gx is a critical orbit of f, $Gx \times S$ is a critical manifold of f_S. Going to the quotient as before, the critical manifolds of

$f_{S,G}$ are $Gx \times S/G \simeq S/I(x)$ ($I(x) \subset G$ is the isotropy group of x) and it is nondegenerate if and only if Gx is a nondegenerate critical orbit. Then we define θ_x as before, and $t^k P(S/I(x), \theta_x; t)$ will be the contribution of Gx to $M_{S,G}(f;t) = M(f_{S,G};t)$, the new Poincaré polynomial being $P_{S,G}(X;t) = P(X_S/G\,;t)$.

The unpleasant fact about this construction, is that the inequality we get depends on the choice of S. Also if the action is free, $M_{S,G}(f;t)$ and $P_{S,G}(X;t)$ are the product of $M(f_G;t)$ and $P(X/G;t)$ by $P(S;t)$.

This is easy to check, $X_S/G = X/G \times S$ if the G action is free on X, so $P_{S,G}(X;t) = P(X/G;t) \times P(S;t)$. For the Morse polynomial the proof is the same

<u>Remark 4</u>: If we multiply the Morse inequalities by a polynomial with nonnegative coefficients, some information is lost, since we only now that $Q(t)\,P(S;t)$ has nonnegative coefficients, a weaker statement than knowing that $Q(t)$ itself has nonnegative coefficients (unless $P(S;t) \equiv 1$)

According to the last remark, the best choice of S is given by $P(S;t) \equiv 1$, that is S is contractible.

It is a fact of Algebraic Topology, that for G a compact Lie group, there is always a contractible space, denoted by EG endowed with a free G-action.

<u>Example:</u>

Let $G = S^1$ or $\mathbb{Z}_p$, and S^∞ be the unit sphere of a numerable Hilbert space with the canonical G action, i.e.

$$e^{i\theta}(z_1, \ldots, z_n, \ldots) = (e^{i\theta}z_1, \ldots, e^{i\theta}z_n, \ldots)$$

$$e^{i\theta} \in G \subset S^1.$$

Note that EG is not a finite dimensional compact manifold, but we can define for all N a compact G space EG^N such that $P(EG^N; t) = 1 + t^N R(t)$. Then $EG = \lim EG^N$ in homotopy, and this allows us to simplify the Algebraic Topology (many results only hold for compact spaces)

Example:

For $G = S^1$ or $\mathbb{Z}_k$, we can set $EG^N = S^{2N-1}$ where S^{2N-1} is the unit sphere of $\mathbb{C}^N$ with the action

$$e^{i\theta}.(z_1, \dots, z_N) = (e^{i\theta}z_1, \dots, e^{i\theta}z_N).$$

We now set $X_G = X \times EG/G$ f_G the function induced by f on X_G, and $H^*_G(X) = H^*(X_G)$,

$$M_G(f;t) = M(f_G;t), \quad P_G(X;t) = P(X_G;t).$$

Theorem 1.2': If f has nondegenerate critical orbits, we have

$$M_G(f;t) - P_G(X;t) = (1+t)\,Q(t)$$

where $Q(t)$ is a formal power series with nonnegative coefficients. □

We shall now give without proof the contribution of the critical orbits to $M_G(f;t)$, that is $P(t) =$ $P(EG/I(x)\,;\,\theta_x\,;t)$.

Let us remark that since $I(x) \subset G$, EG is a free contractible $I(x)$ space, and one can prove that $EG/I = BI$ does depend on I only.

We are only intrested in the cases $G = S^1$ $\mathbb{F} = \mathbb{Q}$ or $G = \mathbb{Z}_k$ $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}_p$ p divides k.

In the first case $I = S^1$ or $\mathbb{Z}_k$, and

$$P(BS^1, \theta\,; t) = \sum_{k \geq 0} t^{2k} = \frac{1}{1-t^2}$$

$$P(B\mathbb{Z}_k, \theta\,; t) = \begin{cases} 1 & \text{if } \theta \text{ is orientable} \\ 0 & \text{if } \theta \text{ is non orientable} \end{cases}$$

<u>Remark:</u> One can see, using that $\pi_1(B\mathbb{Z}_k) = \mathbb{Z}_k$, that if θ is nonorientable, k is necessarily even.

In the second case $I = \mathbb{Z}_d$ d divides k. If $\mathbb{F} = \mathbb{Q}$,

or p does not divide d

$$P(B\mathbb{Z}_d, \theta; t) = \begin{cases} 1 & \text{for } \theta \text{ orientable} \\ 0 & \text{for } \theta \text{ non orientable} \end{cases}$$

If $\mathbb{F} = \mathbb{Z}_p$, p divides d, we have

$$P(B\mathbb{Z}_d, \theta; t) = \begin{cases} \sum_{k \geq 0} t^k = \frac{1}{1-t} & \text{for } \theta \text{ orientable or } p=2 \\ 0 & \text{for } \theta \text{ non orientable} \end{cases}$$

Example

$$X = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

$$G = S^1, \quad e^{i\theta}(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$$

$f(z_1, z_2) = |z_1|^2$ has for critical orbits $(0, e^{i\theta})$ and $(e^{i\theta}, 0)$

The action is free, so $P_G(X; t) = 1 + t^2$, also

$M_G(f; t) = 1 + t^2$, so here $M_G - P_G = 0$

Example 2 :

$X = S^3 \quad G = S^1$, and the action is now $e^{i\theta}(z_1, z_2) = (e^{i\theta} z_1, e^{2i\theta} z_2)$

For $f = |z_1|^2$, we have :

a minimum $(0, e^{i\theta})$ $\quad I = \mathbb{Z}_2$

a maximum $(e^{i\theta}, 0)$ $\quad I = \{e\}$

If we try to guess whether the minimum has an orientable θ or not, we see that in the first case $M(t) = 1 + t^2$, in the second $M(t) = t^2$. Since $P(t) = 1 + t^2$, the second case can not occur.