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ELLIPTIC EQUATIONS WITH CRITICAL GROWTH

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Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$. We shall be concerned with the problem of existence of solutions for the nonlinear elliptic equation

$$(1.1) \quad \begin{cases} -\Delta u - a(x)\, u = |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $a(x)$ is a given smooth function and $2^* = \frac{2N}{N-2}$ is the limit exponent in the Sobolev inequality, that is the exponent such that the embedding

$$j : H^1_0(\Omega) \hookrightarrow L^{p}(\Omega)$$

is continuous but not compact. So, as we shall see better, in this problem a lack of compactness occurs. Before starting to study problem (1.1) in detail, let me recall that during the past few years there has been a considerable amount of research on this problem. The motivation for investigating (1.1) comes from the fact that it is a simplified model of some variational problems in Geometry and in Physics that also exhibit a loss of compactness. The most notorious example is Yamabe's problem: find a function u satisfying

$$\begin{cases} -4\,\frac{N-1}{N-2}\,\Delta u = R'\, u^{(N+2)/(N-2)} - R(x)\, u & \text{on } M \\ u > 0 & \text{on } M \end{cases}$$

for some constant R'.

Here M is an n-dimensional Riemannian manifold, Δ is the Laplace-Beltrami operator, $R(x)$ is the scalar currature of M.

But there are many others:

a) Existence of extremal functions for isoperimetric inequalities, Hardy-Littlewood-Sobolev inequalities (see Jacobs [J], Lieb [L])

b) Existence of non-minimal solutions for Yang-Mills functionals (see Taubes [T])

etc.

It is known that to study the existence of solutions for (1.1) is equivalent to finding critical points of the free energy functional defined on $H_0^1(\Omega)$ by

$$E(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - a(x)\,u^2)\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx$$

or critical points of

$$\hat{E}(u) = \int_\Omega (|\nabla u|^2 - a(x)\,u^2)\,dx$$

constrained on the manifold

$$V = \left\{ u \in H_0^1(\Omega) \;:\; \int_\Omega |u|^{2^*} = 1 \right\}$$

Technically the lack of compactness of the embedding $j : H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ implies that the manifold V, the unit sphere in L^{2^*}, is not closed for the weak $H_0^1(\Omega)$ topology, and, more, the functionals E and $\hat{E}_{|V}$ do not satisfy in all the energy range the following Palais-Smale-c condition. Let $f \in \mathcal{C}^1(B,\mathbb{R})$, B real Hilbert space. f sastifies the P-S condition at the level c if any sequence $\{u_m\} \subset B$ such that $f(u_m) \xrightarrow[m\to+\infty]{} c$ and $f'(u_m) \xrightarrow[m\to+\infty]{} 0$ possesses a convergent subsequence.

The difficulties one meets in trying to find critical points by usual variational methods can be at this point, easily understood. The following example is useful to explain them, by difference.

Let's suppose $a(x) = \lambda$, $\lambda \in (-\infty, \lambda_1)$, $2 < p < 2^*$, and look for positive solutions of

$$(1.2) \qquad \begin{cases} -\Delta u - \lambda u = |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

A natural approach to this problem is to try to solve the minimization problem

$$\inf \left\{ \int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx, \quad u \in H_0^1(\Omega), \quad \int_\Omega |u|^p = 1 \right\}$$

So call S_λ this infimum and consider a minimizing sequence $\{u_m\} \subset H_0^1(\Omega)$:

$$(1.3) \qquad \begin{aligned} &\int_\Omega (|\nabla u_m|^2 - \lambda u_m^2)\, dx = S_\lambda + o(1) \\ &\int_\Omega |u_m|^p\, dx = 1 \end{aligned}$$

Therefore u_m is bounded in $H_0^1(\Omega)$ and up to a subsequence still denoted by u_m

$$u_m \longrightarrow \bar{u} \qquad \text{weakly in } H_0^1(\Omega)$$

and by the compactness of the embedding $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$

$$u_m \longrightarrow \bar{u} \qquad \text{strongly in } L^p(\Omega)$$

Hence passing to the limit as $m \to +\infty$ in (1.3) we obtain

$$\int_\Omega |\nabla \bar{u}|^2 - \lambda \bar{u}^2 \le S_\lambda \qquad |\bar{u}|_{L^p} = 1$$

so $\bar{u}$ is a minimizer, and, taking account that $\lambda < \lambda_1$, $S_\lambda > 0$. Moreover we can assume $\bar{u} \ge 0$ (otherwise we replace $\bar{u}$ by $|\bar{u}|$). Also, since $\bar{u}$ is a solution of

$$-\Delta u - \lambda u = S_\lambda u^{p-1} \qquad \text{in } \Omega$$

and $S_\lambda > 0$, $v = S_\lambda^{1/p-2}\bar{u}$ solves (1.2), and, by the strong maximum principle $v > 0$.

It is clear where the above argument fails if we try to use it in the same equation (1.2) with $p = 2^*$. Indeed for the limit of a minimizing sequence $\{u_m\}$ it is possible in this case only to say that

$$\int_\Omega |\nabla \bar{u}|^2 - \lambda \bar{u}^2 \le S_\lambda \qquad |\bar{u}|_{L^{2^*}(\Omega)} \le 1$$

In fact $\{u_m\}$ need not converge strongly in $L^{2^*}(\Omega)$, so its limit may not be a minimizer.

The first result about problem (1.1) was given in 1965 by Pohozaev who proved the following nonexistence theorem

<u>Theorem 1.1</u> [Pohozaev [P]] Let suppose Ω starshaped and $a(x) = \lambda$, $\lambda \in (-\infty, 0)$. Then there is no solu-

tion of (1.1).

Pohozaev theorem is a consequence of Pohozaev identity: if u is a solution of

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and g is a continuous function then

$$\left(1 - \frac{N}{2}\right) \int_\Omega g(u)\, u + N \int_\Omega G(u) = \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial u}{\partial \nu}\right)^2$$

where $G(t) = \int_0^t g(s)\, ds$ and ν is the outward normal. In the special case

$$g(s) = \lambda s + |s|^{2^*-2} s$$

Pohozaev identity reduces to

$$(1.4) \qquad \lambda \int_\Omega u^2 = \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial u}{\partial \nu}\right)^2$$

from which if Ω is starshaped (i.e. $x \cdot \nu > 0$ on $\partial\Omega$ a.e.) it follows $\lambda \geq 0$. If $\lambda = 0$ from (1.4) we deduce $\frac{\partial u}{\partial \nu} = 0$, so we can exclude at once, using the strong maximum principle, that $-\Delta u = |u|^{2^*-2} u$ has a positive solution. Moreover neither solutions that change sign can exist. In this case we would have that u solves

$$(1.5) \qquad \begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \Omega \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

Then we could extend smoothly u, for example, in a ball $B \supset \Omega$ putting $u \equiv 0$ on $B \setminus \Omega$. Thus u would be a regular solution of (1.5) in B, so should enjoy of the unique continuation property, hence u must be zero.

Pohozaev's interesting negative result had a strong impact in the mathematical community. It combined with the objective difficulties due to the lack of compactness, led many people into the opinion that it was better avoid problems with limiting nonlinearities.

On the other hand some hope to be able to manage problem (1.1) arises from these observations:

a) Assume Ω is any bounded domain and $\bar{\lambda}$ due to a simple eigenvalue of $-\Delta$ with zero Dirichlet boundary data, then for λ belonging to a suitably small neighborhood of $\bar{\lambda}$, (1.2) has a solution whatever $p > 2$ is. This follows from the bifurcation theory (see [R]).

b) Assume Ω is an annulus i.e. $\Omega = \{x \in \mathbb{R}^N : 0 < r_1 < |x - x_0| < r_2\}$ and a is a radial function $a(x) = a(|x|)$. Then it is easy to see (as pointed out by Kazdan-Warner [K-W]) that (1.1) has a radial solution u.

The first successful investigation of problem (1.1) was given by Brezis and Nirenberg in a celebrated paper [B.N], inspired by a previous research of Aubin on the Yamabe problem. In that article the authors, studying the existence of positive solutions for (1.1) without any geometrical restriction on Ω, showed how to overcome the difficulty coming from the lack of compactness and proved the following results:

Theorem 1.2 Assume $N \geq 4$, $(-\Delta - a)$ coercive. Let denote

$$(1.6) \qquad S_a = \inf \{ \hat{E}(u),\ u \in V \}$$

$$(1.7) \qquad S = \inf \left\{ \int_\Omega |\nabla u|^2 dx,\ u \in V \right\}.$$

Then the following conditions are equivalent

i) $a(x) > 0$ somewhere in Ω

ii) $S_a < S$

iii) S_a is achieved.

Theorem 1.3 Assume $N = 3$, $a(x) = \lambda$, then there exists a $\lambda^* \in [0, \lambda_1)$ such that for any $\lambda \in (\lambda^*, \lambda_1)$ problem (1.1) admits at least a positive solution. Moreover if Ω is a ball $\lambda^* = \lambda_1/4$ and there is no solution for $\lambda \in [0, \lambda_1/4]$

Remark 4. If we look for positive solutions of (1.1) a necessary condition for the existence of a solution is that $Lu = (-\Delta - a(x))u$ should be coercive that is

$$(1.8) \quad \exists\, \delta > 0 \text{ such that } \int_\Omega (|\nabla \varphi|^2 - a(x)\varphi^2)\, dx \geq \delta \|\varphi\|^2_{H_0^1} \quad \forall \varphi \in H_0^1(\Omega)$$

In fact let $\mu_1 = \min\limits_{\varphi \in H_0^1} \left(\dfrac{\int |\nabla \varphi|^2 - a\varphi^2}{\|\varphi\|^2} \right)$ denote the first eigenvalue of L and let $\varphi_1 > 0$ be the corresponding eigenfunction so that

$$-\Delta \varphi_1 - a\varphi_1 = \mu_1 \varphi_1$$

If a positive solution $\bar{u}$ of (1.1) exists, multiplying by φ_1 and integrating by parts, we obtain

$$\int_\Omega (-\Delta \bar{u} - a(x)\bar{u})\varphi_1 = \mu_1 \int_\Omega \bar{u}\varphi_1 = \int_\Omega \bar{u}^{2^*-1} \varphi_1$$

then μ_1 has to be positive, so L is coercive.
Note that the coercivity in the case $a(x) = \lambda$ means $\lambda < \lambda_1$.
This condition is not sufficient as the Pohozaev's theorem shows: for $\lambda \leq 0$ there are no solutions for (1.1) if $a(x)=\lambda$.
Taking account of these remarks Theorem 2 gives, if $a(x) = \lambda$, that, for $N \geq 4$, there is a positive solution for (1.1) if $\lambda \in (0, \lambda_1)$. ∎

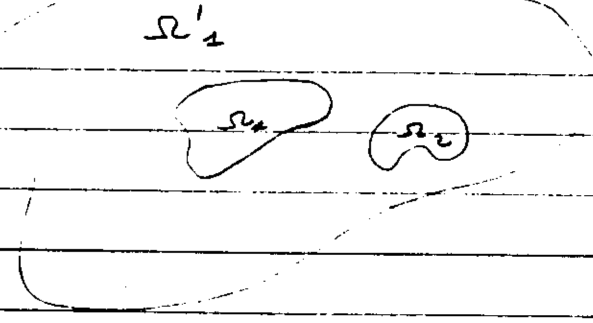
Remark 5 iii) implies that (1.1) has a positive solution. Let us assume that the infimum (1.6) is achieved by a function u (note that $S_a > 0$ if 1.8 holds), then as before we can assume $u \geq 0$.
The Euler equation related to (1.6) is

$$-\Delta u - a(x)\, u = S_a\, u^{2^*-1}$$

So $v = S_a^{(N-2)/4} u$ solves (1.1) and, as before, using the maximum principle we conclude $v > 0$. ∎

The Brezis - Nirenberg method was based on the observation that the compactness is preserved in some energy range and that the best constant S for the embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$ (defined by (1.7)) plays a very important role.
Before proving Theorem 2 I want to recall some properties of the best Sobolev constant S.

a) S is independent of Ω and depends only on N.
This follows from the fact that the ratio $\frac{\|u\|_{H_0^1}}{|u|_{2^*}}$ is invariant under scaling, in other words, the ratio $\|u_k(x)\|/|u_k|_{2^*}$ is independent of k, where $u_k(x) = u(kx)$.
If Ω_1 and Ω_2 are two domains call S_{Ω_1} and S_{Ω_2} the best constant related to Ω_1 and Ω_2



We can expand in a linear way Ω_1 to Ω'_1 so that $\Omega'_1 \supset \Omega_2$ then we will have

$$S_{\Omega_1} = \inf_{u \in H^1_0(\Omega_1)} \frac{\|u\|_{H^1_0}}{|u|_{L^{2^*}}} = \inf_{u \in H^1_0(\Omega'_1)} \frac{\|u\|_{H^1_0}}{|u|_{L^{2^*}}} \leq \inf_{u \in H^1_0(\Omega_2)} \frac{\|u\|_{H^1_0}}{|u|_{L^{2^*}}} = S_{\Omega_2}$$

Expanding Ω_2 to an Ω'_2 such that $\Omega'_2 \supset \Omega_1$ we can obtain the opposite relation

$$S_{\Omega_2} \leq S_{\Omega_1}$$ ∎

b) the infimum in (1.7) is never achieved when Ω is a bounded domain. Indeed, suppose that S were attained by some function $u \in H^1_0(\Omega)$. We may assume $u \geq 0$ on Ω. Fix a ball $\tilde{\Omega} \supset \Omega$ and set

$$\tilde{u} = \begin{cases} u & \text{on } \Omega \\ 0 & \text{on } \tilde{\Omega} \setminus \Omega \end{cases}$$

Thus S is also achieved on $\tilde{\Omega}$ by $\tilde{u}$ and $\tilde{u}$, satisfies $-\Delta \tilde{u} = \mu \tilde{u}^{2^*-1}$ for some constant $\mu > 0$, contradicting Pohozaev's result.

c) when $\Omega = \mathbb{R}^N$, the infimum in (1.7) is achieved by the function

$$\Psi(x) = \frac{C}{(1 + |x|^2)^{\frac{N-2}{2}}} \qquad C = \text{normalization constant}$$

or by any function $\Psi_{\mu,x_0}(x)$ obtained from it by scaling and translation

$$(1.8)' \qquad \Psi_{\mu,x_0}(x) = \frac{\bar{C} \, \mu^{\frac{N-2}{4}}}{(\mu + |x - x_0|^2)^{\frac{N-2}{2}}} \qquad \mu > 0, \; x_0 \in \mathbb{R}^N, \quad \bar{C} = \frac{1}{S^{\frac{N}{4}}} [N(N-2)]^{\frac{N-2}{4}} \text{ norm const.}$$

(see [A], [Ta])

Proof of Theorem 1.1 We shall prove $1 \Rightarrow 2$, $2 \Rightarrow 3$, $3 \Rightarrow 1$

We start proving $2 \Rightarrow 3$ namely the following compactness result : If $S_a < S$ then any minimizing sequence for S_a is relatively compact.

Let u_j be a minimizing sequence for S_a, that is

$$(1.9) \qquad \int_\Omega |\nabla u_j|^2 - a(x)\, u_j^2 = S_a + o(1)$$

$$(1.10) \qquad \int_\Omega |u_j|^{2^*} = 1$$

Since u_j is bounded in $H_0^1(\Omega)$, we may say that, up to a subsequence, still denoted by u_j

$$u_j \longrightarrow \bar{u} \quad \text{weakly in } H_0^1(\Omega)$$
$$u_j \longrightarrow \bar{u} \quad \text{strongly in } L^2(\Omega)$$
$$u_j(x) \longrightarrow \bar{u}(x) \quad \text{a.e. in } \Omega$$

so

$$|u_j|^{2^*-2} u_j \longrightarrow |\bar{u}|^{2^*-2}\bar{u} \quad \text{weakly in } L^{(2^*)'}$$

Now put

$$u_j - \bar{u} = v_j$$

hence $u_j = \bar{u} + v_j$, we must prove that v_j, that goes weakly to 0 in $H_0^1(\Omega)$, goes strongly to 0 in $H_0^1(\Omega)$.

From (1.9) we obtain

$$(1.11) \qquad \int_\Omega |\nabla \bar{u}|^2 + \int |\nabla v_j|^2 - \int a(x)\, \bar{u}^2 = S_a + o(1)$$

On the other hand we may use the following

Lemma (Brezis-Lieb [B.L].) Suppose $u \in L^p$, v_j is bounded in

L^p and $v_j \to 0$ a.e. Then

$$\int_\Omega |u+v_j|^p = \int_\Omega |u|^p + \int_\Omega |v_j|^p + o(1)$$

So

$$1 = \int_\Omega |\bar{u}+v_j|^{2^*} = \int_\Omega |\bar{u}|^{2^*} + \int_\Omega |v_j|^{2^*} + o(1)$$

and by convexity

$$1 = \left[\int_\Omega |\bar{u}|^{2^*} + \int_\Omega |v_j|^{2^*}\right]^{2/2^*} + o(1) \leq |\bar{u}|^2_{L^{2^*}} + |v_j|^2_{L^{2^*}} + o(1).$$

Using this in (1.11) we deduce

$$(1.12)\quad \int_\Omega |\nabla \bar{u}|^2 + \int_\Omega |\nabla v_j|^2 - \int_\Omega a(x)\,\bar{u}^2 \leq S_a\left[|\bar{u}|^2_{L^{2^*}} + |v_j|^2_{L^{2^*}}\right] + o(1).$$

Since by definition of S_a we have

$$\int_\Omega |\nabla \bar{u}|^2 - a(x)\,\bar{u}^2 \geq S_a\,|\bar{u}|^2_{L^{2^*}}$$

we conclude that

$$\int_\Omega |\nabla v_j|^2 \leq S_a\,|v_j|^2_{L^{2^*}} + o(1).$$

By definition of S it is

$$|v_j|^2_{L^{2^*}} \leq \frac{1}{S}\int_\Omega |\nabla v_j|^2\,dx.$$

and thus we are led to

$$\left(1 - \frac{S_a}{S}\right)\int_\Omega |\nabla v_j|^2\,dx = o(1)$$

and since $S_a < S$ we can conclude $v_j \to 0$ strongly in $H^1_0(\Omega)$.

The most technical part is $1 \Rightarrow 2$ which is achieved by constructing explicitly a function φ such that

$$Q(\varphi) = \frac{\int_\Omega (|\nabla \varphi|^2 - a(x)\varphi^2)\,dx}{|\varphi|^2_{L^{2^*}}} < S$$

The idea of the choice of φ can be explained easily if we refer to the case $a(x) = \lambda > 0$. In this case we have to compare

$$\inf_{\varphi \in H_0^1(\Omega)} \frac{\int |\nabla \varphi|^2 - \lambda \varphi^2}{|\varphi|^2_{L^{2^*}}} \quad \text{with} \quad \inf_{\varphi \in H_0^1(\Omega)} \frac{\int |\nabla \varphi|^2}{|\varphi|^2_{L^{2^*}}}$$

If S were achieved in Ω by some φ, φ could be used as testing function to obtain the wanted relation. Since, unfortunately S is never achieved when Ω is bounded, but S in $\mathbb{R}^N$ is attained by the functions $\Psi_{\mu x_0}$, the idea is try to use as test function one of such functions suitably truncated.
Without any loss of generality we may assume that $0 \in \Omega$ and that $a(0) > 0$. So in a suitably small ball $B_{\rho,0} \subset \Omega$ we will have $a(0) \geq a(0)/2 > 0$. Fix a cut off function $\zeta \in C_0^\infty(\Omega)$ such that $\zeta(x) = 1$ in $B_{\rho/2,0}$ $\zeta(x) = 0$ in $\Omega \setminus B_{\rho,0}$ and set

$$\varphi_{\mu,0}(x) = \zeta(x)\,\Psi_{\mu,0}$$

A careful expansion as $\mu \to 0$ leads to

$$Q(\varphi_{\mu_0}(x)) \leq \begin{cases} S - \frac{a(0)}{2} k\mu + O(\mu^{\frac{N-2}{2}}) & \text{if } N \geq 5 \\[2ex] S - \frac{a(0)}{2} k\mu\,|\log \mu| + O(\mu) & \text{if } N = 4 \end{cases}$$

where k depends only on N. The conclusion follows choosing $\mu > 0$ small enough.
Finally, to prove $3 \Rightarrow 1$, suppose $a(x) \le 0$ everywhere on Ω. Let φ be a function such that $|\varphi|_{L^{2^*}} = 1$ & $\int_\Omega |\nabla\varphi|^2 - a\varphi^2 = S_a$. Therefore $\int_\Omega |\nabla\varphi|^2 < S_a$. On the other hand from the above expansion if we consider the functions $\varphi_{1/m,0}(x)$, $m \to +\infty$, we see that $S_a \le S$. Thus φ would be a minimizer for the Sobolev inequality, a contradiction. Q. E. D. ∎

Remark 6. $S_a < S$ is a sufficient but not necessary condition for the existence of a solution of (1.1). In fact if Ω is an annulus and $a(x)$ is radial function with $a(x) \le 0$ (1.1) has a radial positive solution, but S_a is not achieved. Clearly this solution correspond to critical points of $\hat{E}$ on V at "high energy" levels.

Remark 7. An approach alternative to the minimization method would be to consider the "free" functional E and to use the Mountain-Pass Lemma.
First of all, we note that using an argument analogous to that used before it is possible to give for E the compactness result:
Lemma. If $c < \frac{1}{N} S^{N/2}$ and u_m is a sequence along which E satisfies the Palais-Smale assumptions in c, then u_m is relatively compact. Then, by the growth of E, it is easy to verify in the standard way that 0 is a local strict minimum for E and that $\exists\, w$ s.t. $E(tw) \xrightarrow[t \to +\infty]{} -\infty$ hence for a $\bar{t}$ big enough $E(\bar{t}w) \le 0$ [any $w \ge 0$, $w \ne 0$ is O.k.]. So the level

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in (0,1)} E(\gamma(t))$$

$$\Gamma = \{ \gamma \in \mathcal{C}[0,1],\ \gamma(0) = 0,\ \gamma(1) = \bar{t}w \}$$

is such that $c > 0$. Now in order to reach the conclusion of the Ambrosetti - Rabinowitz theorem we need the P-S condition. Namely we need $c < \frac{1}{N} S^{N/2}$

Hence we had to choose w in an appropriate way, taking account that it suffices to find w so that

$$w \in H_0^1(\Omega), \quad w \geq 0, \quad w \not\equiv 0, \quad \sup_{t>0} E(tw) < \frac{1}{N} S^{N/2}$$

As before for the computation of S_a, the right function for this purpose is a function

$$\varphi_{\mu,x_0} = \int(x)\, \Psi_{\mu,x_0}$$

where $x_0 \in \Omega$, $\mu > 0$ and $\int$ is a cut off function. Indeed a careful expansion shows (for $N \geq 4$)

$$\sup_{t\geq 0} E(t\,\varphi_{\mu,x_0}) \leq \begin{cases} \frac{1}{N} S^{N/2} + O\left(\mu^{(N-2)/2}\right) - k\,\mu & \text{if } N \geq 5 \\ 1/N\, S^{N/2} - k\,\mu \log\mu + O\,\mu & \text{if } N = 4 \end{cases}$$

with $k > 0$ constant depending only on $a(0)$ and N.

<u>Remark 8</u> It is not difficult to show that the $(P\text{-}S)_c$ condition does not hold at the level $c = \frac{1}{N} S^{N/2}$. It suffices to consider the sequence

$$\Psi_{1/m,x_0}(x) = \frac{\left[N(N-2)(1/m)\right]^{\frac{N-2}{4}}}{\left[1/m + |x-x_0|^2\right]^{\frac{N-2}{2}}}$$

and then $\quad u_m(x) = \int(x)\, \Psi_{1/m,x_0}(x)$

Since $\Psi_{1/m,x_0}(x)$ solves

$$-\Delta u = u^{2^*-1} \quad \text{in } \mathbb{R}^N$$

$$E(u_m) \to \frac{1}{N} S^{N/2} \quad \text{and} \quad E'(u_m) \longrightarrow 0 \quad \text{in } H^{-1}(\Omega)$$

moreover

$$u_m \rightharpoonup 0 \quad \text{weakly in } H^1_0(\Omega)$$

thus u_m cannot be relatively compact in $H^1_0(\Omega)$.
Analogously the sequence

$$\hat{u}_m = \frac{u_m(x)}{|u_m|_{L^{2^*}}}$$

is such that $|\hat{u}_m|_{L^{2^*}} = 1$, $\hat{E}(\hat{u}_m) \to S$ and $\hat{E}'_{|V}(\hat{u}_m) \to 0$ but is not relatively compact. So if $S_a = S$ we have found a minimizing sequence that does not contains a strongly convergent subsequence. ■

We turn now to the problem (1.1) when $N=3$ and $a(x)=\lambda$. In the case $N=3$ the question is more delicate and a complete solution has been found only when Ω is a ball. I shall sketch out, before, the proof of theorem 3 in this case and I'll assume for simplicity

$$\Omega = \{ x \in \mathbb{R}^3 : |x| = 1 \} \quad \text{so that} \quad \lambda_1 = \pi^2, \quad \varphi_1 = |x|^{-1} \sin(\pi |x|)$$

Since we are looking for positive solutions, we have already pointed out that we must consider $\lambda \in (0, \lambda_1)$. Moreover we remark the compactness result for the minimizing sequences does not depend on the dimension of the space. Hence, to prove the sufficient condition for the existence of a solution, we can show that

$$S_\lambda < S \qquad \forall \quad \lambda > \lambda_1/4 .$$

Choose then
$$u_\mu(x) = u_\mu(|x|) = \frac{\varphi(|x|)}{(\mu + |x|^2)^{1/2}}$$

$$\varphi(0) = 1, \quad \varphi'(0) = 0, \quad \varphi(1) = 0 \qquad \varphi \text{ smooth}$$

A careful estimate gives

$$Q_\lambda(u_\mu) = \frac{\|u_\mu\|^2 - \lambda |u_\mu|^2_{L^2}}{|u_\mu|^2_{L^{2*}}} = S + \left(\frac{1}{4}\pi^2 - \lambda\right) k \mu^{1/2} + O(\mu)$$

k positive constant, and the first part of the theorem follows choosing $\mu > 0$ small enough.
The proof of the necessary condition is given by contradiction. We suppose in fact that a positive solution u of

$$-\Delta u - \lambda u = u^5 \qquad \text{in } \{x \in \mathbb{R}^3 : |x| = 1\}$$

for $0 < \lambda < \frac{\lambda_1}{4}$ exists, by the Gidas-Ni-Nirenberg result [ ] it must be spherically symmetric. We can, hence, write $u(x) = u(r)$ where $r = |x|$, thus u verifies

$$\begin{cases} -u'' - \frac{2}{r} u' = u^5 + \lambda u & \text{on } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

The crucial step, then, is to prove the identity

$$\int_0^1 u^2 \left(\lambda \psi' + \frac{1}{4}\psi'''\right) r^2 \, dr = \frac{2}{3} \int_0^1 u^6 (r\psi - r^2 \psi') \, dr + \frac{1}{2} |u'(1)|^2 \psi(1)$$

$\forall \psi$ smooth s.t. $\psi(0) = 0$
that generalizes the Pohozaev identity (that corresponds to the case $\psi(r) = r$). This done we choose

$$\psi(r) = \sin(\sqrt{4\lambda}\, r) \qquad \text{so that}$$

$$\psi(1) \geq 0 \quad , \quad \lambda \psi' + \frac{1}{4}\psi''' = 0$$

and

$$r\psi - r^2\psi' = r \sin(\sqrt{4\lambda}\, r) - r^2 \cdot \sqrt{4\lambda} \cos \sqrt{4\lambda}\, r > 0 \quad \text{on } (0,1)$$

since $\sin\vartheta - \vartheta\cos\vartheta > 0$ for $\vartheta \in (0, \pi]$. Therefore we obtain a contradiction.

The last part of Teorem 3 can be obtained from the following inequality which sharps the Sobolev inequality:

Assume $\Omega \subset \mathbb{R}^3$ is a bounded domain. Then there exists a constant λ_*, depending on Ω, $0 < \lambda_* < \lambda_1$ such that

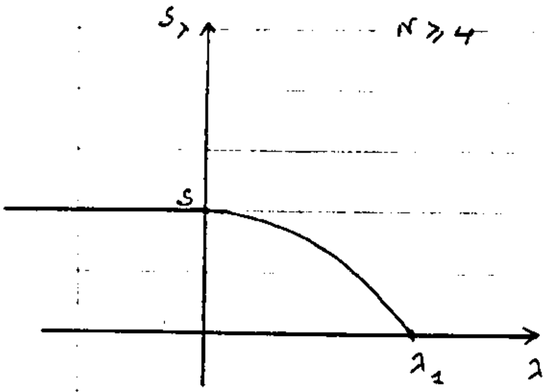
$$(1.12) \qquad \|u\|^2_{H^1_0} \geq S\, |u|^2_6 + \lambda_* |u|^2_2 \qquad \forall\, u \in H^1_0(\Omega)$$

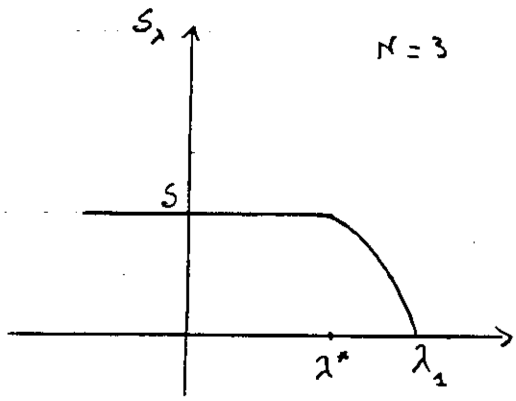
From this it follows $S_{\lambda_*} \geq S$ that combined with the obvious $S_\lambda \leq S$ $\forall\, \lambda \in \mathbb{R}$ gives $S_{\lambda_*} = S$. Therefore, since if $\lambda < \lambda_*$ $S_\lambda \geq S_{\lambda_*}$, we conclude that given $\Omega \subset \mathbb{R}^3$ there is a number λ^* attached to Ω, with $0 < \lambda^* < \lambda_1$ such that

$$S_\lambda < S \qquad \text{for} \quad \lambda > \lambda^*$$
$$S_\lambda = S \qquad \text{for} \quad 0 \leq \lambda \leq \lambda^*$$

In other words the graph of the function

$\lambda \to S_\lambda$ looks in the cases $N \geq 4$ and $N = 3$ in two different ways





To prove (1.12) let Ω^* a ball such that meas $\Omega^*=$ meas Ω. Let u^* denote the symmetric rearrangement of u. It is known (see [ ], [ ]) that if $u \in H_0^1(\Omega)$, then $u^* \in H_0^1(\Omega^*)$ and

$$\| u^* \|_{H_0^1(\Omega^*)} \leq \| u \|_{H_0^1(\Omega)}$$

and

$$| u^* |_{L^p(\Omega^*)} = | u |_{L^p(\Omega)}$$

On the other hand, if $\lambda = \frac{1}{4}\lambda_1(\Omega^*)$ $S_\lambda \geq S$, otherwise, in fact, the strict inequality $S < S_\lambda$ would imply the existence of a solution in Ω^* with $\lambda = \frac{\lambda_1(\Omega^*)}{4}$, then

$$\| u \|^2_{H_0^1(\Omega)} \geq \| u^* \|_{H_0^1(\Omega^*)} \geq S\, | u |^2_{L^6(\Omega^*)} + \frac{1}{4}\lambda_1(\Omega^*)\, | u |^2_{L^2(\Omega^*)} =$$

$$= S\, | u |^2_{L^6(\Omega)} + \lambda_*\, | u |^2_{L^2(\Omega)}$$

<u>Remark 3</u> If $\lambda < \lambda^*$ then the infimum for S_λ is not achieved. Suppose, indeed, by contradiction that it is achieved by some u_0. Let μ be such that $\lambda < \mu < \lambda^*$. Then

$$S_\mu < S_\lambda = S$$

because

$$S_\mu \leq \frac{\int_\Omega |\nabla u_0|^2 - \mu \int u_0^2}{|u_0|^2_{L^{2^*}}} < \frac{\int |\nabla u_0|^2 - \lambda \int u_0^2}{u_0} = S_\lambda = S$$

This is absurd since $S_\mu = S$ for $\mu \leq \lambda^*$.

Thus if there are solutions of (1.1) when $\lambda < \lambda^*$, they must correspond to critical points of the functional $\hat{E}(u)$ on the sphere V but they are not minima.

2.

From now on we shall consider questions related to problem 1.1 when $a(x) = \lambda = \text{const.}$

We shall be concerned with the problem

$$(2.1)\begin{cases} -\Delta u - \lambda u = |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

A first observation that one can do is that, even if the condition $\lambda < \lambda_1$ is necessary for the existence of positive solutions of (2.1), however one can expect the existence of nontrivial solutions for $\lambda > \lambda_1$. Clearly these solutions, if exist, will not be minima for

$$\hat{E}(u) = \int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx \qquad \text{on}$$

$$V = \left\{ u \in H_0^1 : \int_\Omega |u|^{2^*} = 1 \right\}$$

In fact if $\lambda > \lambda_1$, $\hat{E}(u)$ is no more positive definite.

A first partial answer to this question was given in [CFS] by the following theorem which extends well known bifurcation results by Marino [M] and Böhme [Bo]

<u>Theorem 2.1</u> [Cerami - Fortunato - Struwe [CFS]] Let $N \geq 3$, $\lambda > 0$ and put

$$\lambda_+ = \min \{ \lambda_j : \lambda < \lambda_j \}$$

$$m = \text{multiplicity of } \lambda_+$$

$$\lambda_+ - \lambda < S\, |\text{meas } \Omega|^{-2/N}$$

Then problem 2.1 admits at least m pairs of non trivial solutions

$$\{ u_K(\lambda), \; -u_K(\lambda) \} \qquad K = 1, 2 \dots m$$

such that

$$\| u_K(\lambda) \|_{H_0^1(\Omega)} \longrightarrow 0 \qquad \text{as} \quad \lambda \to \lambda_+$$

■

Subsequently the question was completely settled by Capozzi-Fortunato-Palmieri [C.F.P] who proved the following

<u>Theorem 2.2</u> Let $N \geq 4$. For any $\lambda > 0$, problem (2.1) has at least a (pair of) nontrivial solution. ■

Both the above stated theorems were proved looking for critical points of the free functional on $H_0^1(\Omega)$

$$E(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda u^2) \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx$$

Namely the idea was to construct, using a variant of the saddle point theorem ([R2], [BBF]), a min-max level $c < \frac{1}{N} S^{N/2}$ "candidate to be critical" for the functional E.

In particular let me recall the following version of the saddle point theorem due to Bartolo-Benci-Fortunato

<u>Proposition 2.3</u> Let H be a real Hilbert space with norm $\| \cdot \|$ and suppose $I \in \mathcal{C}^1(H, \mathbb{R})$ is a functional on H satisfying the following conditions

$$h_1) \quad I(u) = I(-u) \qquad I(0) = 0$$

h_2) There exists a constant $\beta > 0$ s.t. (P-S) condition holds in $]0, \beta[$

h_3) There exists two closed subspaces $V, W \subset H$ and positive constants ρ, δ, β' with $\delta < \beta' < \beta$ such that

i) $I(u) \le \beta'$ $\qquad \forall u \in W$

ii) $I(u) \ge \delta$ $\qquad \forall u \in V, \ \|u\| = \rho$

iii) $\operatorname{codim} V < +\infty$, $\quad \dim W \ge \operatorname{codim} V$

Then there exists at least

$$\dim W - \operatorname{codim} V$$

pairs of critical points of I with critical values belonging to the interval $[\delta, \beta']$. ∎

The proof of the Cerami - Fortunato - Palmieri result was made using this theorem and choosing in a clever way the subspaces V and W.
Indeed called $\lambda^+ = \min\{\lambda_j : \lambda < \lambda_j\}$, put

$$H_1 = \overline{\bigoplus_{\lambda_j \ge \lambda^+} M_{\lambda_j}} \quad \text{and} \quad H_2 = \bigoplus_{\lambda_j < \lambda^+} M_{\lambda_j}$$

where M_{λ_j} is the eigenspace corresponding to λ_j, and the closure is taken in $H_0^1(\Omega)$.
Choose then $V = H_1$ and set

$$\bar W_\mu = \{u \in H_0^1(\Omega) : u = u^- + t\,\Psi_{\mu,0}, \ u^- \in H_2\}$$

where $\Psi_{\mu,0}$ is a function defined in $(1.8)'$.
Thus a careful extimation gives if μ is suitebly small that (i) of the above theorem is verified for $W = \bar W_\mu$ with $\beta = \frac{1}{N} S^{N/2}$ moreover it is easy to verify

$$f_\lambda(u) \ge \frac{1}{2}\left(\lambda - \frac{\lambda}{\lambda^+}\right)\|u\|^2_{H_0^1} - \text{const}\,\|u\|^{2^*}_{H_0^1} \ge \delta > 0$$

if $\|u\| = \rho$, ρ suitably small. So the conclusion follows applying Proposition 2.3.

Alternative proofs of Theorems 2.1 and 2.2 were given there in [A.S-] and [A.S] using a dual approach which is similar to the dual formulation of Clarke-Ekeland [C-E] for Hamiltonian systems.
But, given some answers, a lot of other stimulating problems occurred to the researchers. I'll discuss the results related to the following two:

I) Suppose $\lambda > 0$, then problem (2.1) has at least one solution. Is it possible to show that (2.1) has more than one (pair of) solution? In particular infinitely many?

II) Suppose $\lambda = 0$. Is it possible to find geometrical conditions which insure the existence of solutions for (2.1)?

Question (I) arises naturally from the observation that problem (2.1) has a $\mathbb{Z}^2$-symmetry, and that the presence of the symmetry induces in the subcritical case the existence of infinitely many solutions.
Question (II), although the Pohozaev's theorem is conclusive for starshaped domains, is supported by the existence results for ring-shaped domains.
However, it is easy to realize that the above mentioned approach can be not useful to give an answer to these questions.
Actually, if we are looking for multiple (pairs of) solutions of (2.1), in particular for infinitely many ones, we expect to find critical points at larger and larger positive levels.
Likewise when $\lambda = 0$ the infimum of $\hat{E}$ on V is exactly

S, and the minimization problem has no solution because S is never achieved in a bounded domain. Analogously, there is no hope to construct min-max levels for $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*}$ <u>below</u> $(1/N)\, S^{N/2}$.

A proof, in the spirit of the Brezis - Nirenberg method, of the fact that (2.1) has infinitely many solutions, if $\lambda > 0$ and Ω has a suitable symmetry, has been given by Fortunato - Jannelli [F.J]. By working on suitable subspaces of $H^1_0(\Omega)$, made up by "anti-symmetric" functions with many sign changes, they could increase the bound from above to the compactness of E, replacing $(1/N)\, S^{N/2}$ with larger and larger values. A rough explanation of the underlying idea, in the case Ω is a ball, is the following: for any integer K, large enough, split Ω in $2K$ equal sectors, put in one of them the positive solution found by theorem 1.2 and replace it in the other sectors by odd specular reproduction.

In order to state precisely the Fortunato - Jannelli result we need to give a definition of "rotational symmetry" in $\mathbb{R}^N$. Let V be a 2-dimensional subspace of $\mathbb{R}^N$ and $V^\perp$ be its orthogonal complement. Let Σ_V be the group of the rotations of V. Any $\sigma \in \Sigma_V$ induces in a natural way a map $\gamma_\sigma : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\gamma_\sigma(y_1, y_2) = (\sigma(y_1), y_2) \qquad y_1 \in V,\ y_2 \in V^\perp$$

<u>Definition</u> We say that an open subset $\Omega \subset \mathbb{R}^N$ has a rotational symmetry if there exists a 2-dimensional subspace V of $\mathbb{R}^N$ such that $\gamma_\sigma(\Omega) = \Omega \quad \forall\, \sigma \in \Sigma_V$

Theorem 2.4. Let $N \geq 4$, $\lambda > 0$ and suppose that $\Omega \subset \mathbb{R}^N$ bounded and smooth has a rotational symmetry. Then, for any $M > 0$ there exists a solution u_M of (2.1) such that $E(u_M) > M$.

Obviously this technique does not work if Ω has no symmetry, nor if one looks for solutions which do not allow that use of symmetry, for instance radial solutions in a ball.

It is clear then that a better knowledge of the obstacles to the global compactness, in the usual sense, is necessary. We already know that the Palais-Smale condition fails at the level $1/N \, S^{N/2}$ for the functional E and at the level S for $\hat{E}$ on V. Furthermore, arguing as in Remark 8, it is not difficult to construct sequences of functions showing that P-S condition fails for E at levels $\frac{k}{N} S^{N/2}$ $(k \in \mathbb{N})$ [It suffices to consider $f_1(x)\Psi_{\sigma^1_m, x_1}(x) + f_2(x)\Psi_{\sigma^2_m, x_2}(x) + \dots + f_k(x)\Psi_{\sigma^k_m, x_k}(x)$, where $\sigma^i_m \xrightarrow[m \to +\infty]{} 0$ and $x_j \neq x_i$ $j \neq i$] or, analogously at the levels $k^{2/N} S$ for $\hat{E}$ on V. Nevertheless we need to know more: all the obstacles.

A good picture of the situation is given by the following theorem stated in [S] (see also [PLL])

Theorem 2.5 Let $N \geq 3$, $\lambda \in \mathbb{R}$. Suppose $\{u_m\} \subset H^1_0(\Omega)$ satisfies

$$E(u_m) \to c$$

$$E'(u_m) \to 0 \qquad \text{strongly in } H^{-1}(\Omega) = (H^1_0(\Omega))'$$

Then there exist a number $k \in \mathbb{N}_0$, k sequences of points

$\{x_m^j\}_m \subset \mathbb{R}^N$, $1 \le j \le K$, K sequences of positive numbers $\{\sigma_m^j\}_m \subset \mathbb{R}^+$, $\sigma_m^j \xrightarrow[m \to +\infty]{} 0$, $1 \le j \le K$ and $(K+1)$ sequences of functions $\{u_m^j\}_m$, $0 \le j \le K$ such that for some subsequence still denoted by u_m

$$u_m(x) = u_m^0(x) + \sum_{j=1}^{K} \frac{1}{(\sigma_m^j)^{\frac{N-2}{2}}} u_m^j\left(\frac{x - x_m^j}{\sigma_m^j}\right)$$

and

$$u_m^0(x) \xrightarrow[m \to +\infty]{} u^0(x) \qquad \text{strongly in } H_0^1(\Omega)$$

$$u_m^j(x) \longrightarrow u^j(x) \qquad \text{strongly in } \mathcal{D}^{1,2}\,\mathbb{R}^N \ (*)$$

$$1 \le j \le K$$

where u^0 is a solution of (2.1) and u^j are solutions of

$$(2.2) \quad \begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \mathbb{R}^N \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \end{cases}$$

Moreover as $m \to +\infty$

$$(2.3) \quad \|u_m\|^2_{H_0^1(\Omega)} \longrightarrow \|u^0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{K} \|u^j\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$$

$$(2.4) \quad E(u_m) \longrightarrow E(u^0) + \sum_{j=1}^{K} E_\infty(u^j)$$

where

$$E_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx$$

(*) $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the Sobolev space obtained as closure of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{1/2}$$

and

$$(2.5) \qquad \frac{1}{\sigma_m^j}\, \operatorname{dist}(x_m^j, \partial\Omega) \xrightarrow[m\to+\infty]{} +\infty \qquad j = 1, 2, \dots$$

$$(2.6)\ \forall j \neq i \qquad \max\left(\frac{\sigma_m^j}{\sigma_m^i}, \frac{\sigma_m^i}{\sigma_m^j}, \frac{|x_m^i - x_m^j|}{\sigma_m^i + \sigma_m^j}\right) \xrightarrow[m\to+\infty]{} +\infty$$

The above result indicates that the only obstructions to the compactness are the solutions of the problem (2.2). In fact the behaviour of a Palais-Smale sequence is analyzed and the conclusion is that such a sequence either goes strongly to its weak limit or differs from it by one or more sequences that, after suitable translations and rescalings, converge to a solution of (2.2). Roughly speaking when a loss of compactness occurs we have something which concentrates. Suppose, for example, $k = 1$ and $u_m^0(x) \to 0$, we may assume (modulo a subsequence) that $x_m^1 \to \bar{x}$; if $\bar{x} \in \Omega$, $u_m(x)$ is roughly equivalent to $\frac{1}{(\sigma_m^1)^{\frac{n-2}{2}}} u^1\left(\frac{x-\bar{x}}{\sigma_m^1}\right)$ u^1 being a solution of (2.2) and $\bar{x}$ the concentration point. When $\bar{x} \in \partial\Omega$ property (2.5) says that the "boundary effect" is negligible compared to concentration effect. If $k > 1$, called $\bar{x}^j$ the concentration points, [i.e. $x_m^j \xrightarrow[m\to+\infty]{} \bar{x}^j$ (modulo subsequences)] if all the points $\bar{x}^j$ are distinct and belong to Ω we can think that $u_m(x) = u_m^0(x) + \sum \frac{1}{(\sigma_m^j)^{\frac{n-2}{2}}} u^j\left(\frac{x-\bar{x}^j}{\sigma_m^j}\right)$. However, if some of the points $\bar{x}^j$ coincide: $\bar{x}^j = \bar{x}^i$, $i \neq j$, property (2.6) says that the speed at which the singularities coalesce is much slower than the speed of concentration, so that one sees two distinct waves, as in the case the points are distinct.

The energy levels of E where the Palais-Smale condition can fail (and actually fails) are, by (2.4), characterized by the critical levels of E_∞.
So we need to know the critical points of E_∞.
We already know the family (1.8)' of minimizers of (1.7) when $\Omega = \mathbb{R}^N$, to this family corresponds by the relation $v = S^{\frac{N-2}{4}} \Psi_{\mu, x_0}$ a family of ground state solutions of 2.2. whose energy is exactly $\frac{1}{N} S^{N/2}$. Hence $\frac{K}{N} S^{N/2}$, $K = 1, 2 \ldots n$ are energy levels "bad" for the Palais-Smale condition. We wonder if they are the only levels we have to worry about. Unfortunately, the existence of infinitely many solutions of (2.2) of changing sign and not identificable each other by scaling and translation has been proved by Ding [D]. Thus there are infinitely many energy levels different from $\frac{K}{N} S^{N/2}$ where P-S can fail! Moreover if $c_0 \neq 0$ is a critical level for E, any level obtained adding to c_0, $\frac{K}{N} S^{N/2}$ or the energy of a solution of (2.2) found by Ding, is again a bad level.
Nevertheless some observation can help us to work.
1) By the Gidas-Ni-Nirenberg [G.N.N] theorem a positive solution of (2.2) must be radial, so using the Gidas[G] analysis of the spherically symmetric solutions of (2.2) it is possible to conclude that every positive regular solution of (2.2) must belong to the above family of ground state solutions.
2) It is not difficult to verify that for the energy of any changing sign solution u of (2.2) the extimation

$$E_\infty(u) > \frac{2}{N} S^{N/2}$$

holds.
Hence if we restrict ourselves to the positive cone of

$H^1_0(\Omega)$ or to the subspace made up by radially symmetric functions if Ω is a ball the energy levels of E_∞ that we have to worry about are precisely $k/N\, S^{N/2}$ $k \in \mathbb{N}$. Moreover, whatever Ω is, below $2/N\, S^{N/2}$ there are only two bad levels for Palais-Smale condition : $1/N\, S^{N/2}$ and $c_0 + 1/N\, S^{N/2}$ where c_0 is the energy level of the solution whose existence we already know.

Collecting these informations the following progresses have been achieved for the question I:

Theorem 2.5 ([ESS], [Z]) Assume Ω is any bounded domain in $\mathbb{R}^N$, with $N \geq 5$. Then for every $\lambda \in (0, \lambda_1)$ problem (2.1) has at least two pairs of solutions.

Theorem 2.7 ([ESS], [CWZ], [S]) Assume Ω is a ball in $\mathbb{R}^N$, $N \geq 7$. Then for every $\lambda > 0$ problem (2.1) has infinitely many radial solutions.

The idea of the proof is to use min-max arguments [like saddle point theorem] with additional devices [like to consider classes of functions with prescribed number of nodal lines] to construct "candidate critical levels" in the energy intervals where the Palais-Smale condition holds.

Question II has been studied by Bahri-Coron in a very remarkable paper where they exhibit a pure topological condition to the existence of a solution of (2.1) when $\lambda = 0$

Theorem [B-C] Suppose $N \geq 3$, $\lambda = 0$, $\Omega \subset \mathbb{R}^N$ topologically nontrivial i.e.

$\exists\ k \in \mathbb{Z}$, $k \geq 1$ such that either

$$H_{2k-1}(\Omega, \mathbb{Q}) \neq 0 \quad \text{or} \quad H_k(\Omega, \mathbb{Z}/2\mathbb{Z}) \neq 0$$

then (2.1) has at least a positive solution.

Very briefly the argument of the proof can be explained in this way. Look at the critical points of the functional

$$E^+(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{2^*}\int_\Omega (u^+)^{2^*} dx$$

for this functional the compactness theorem says that the obstacles to the compactness are the critical points of

$$E^+_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2}\int_{\mathbb{R}^N} (u^+)^{2^*} dx$$

that is the positive solutions of (2.2). So the energy levels where Palais-Smale fails are only $K/N\ S^{N/2}$ if there are no nontrivial solutions of (2.1).

Now call

$$E^+_\gamma = \{ u \in H^1_0 : E^+(u) \leq \gamma \}$$

Then the following facts can be proved

1) There is some integer k_0 ($k_0(\Omega)$) such that
$$E^+_{K/N\, S^{N/2}+\varepsilon} \simeq E^+_{K/N\, S^{N/2}-\varepsilon} \quad \text{(homotopy equivalence)}$$
$\forall\ K \geq k_0$

2) If Ω has nontrivial topology, for each $K > 1$ the pair $\left(E^+_{K/N\, S^{N/2}+\varepsilon},\ E^+_{K/N\, S^{N/2}-\varepsilon}\right)$ is nontrivial

in other words $E^{+}_{\frac{k}{N}S^{N/2}+\varepsilon}$ and $E^{+}_{\frac{k}{N}S^{N/2}-\varepsilon}$ are not homotopical-ly equivalent.

To explain better the meaning of the concept of topological nontriviality first we can say that

Ω topologically nontrivial $\Rightarrow$ Ω not contractible.

If $N = 3$ the converse is also true.
If $N \geq 4$, on the contrary, there exist domains which are not contractible and such that

$$H_{2k-1}(\Omega, \mathbb{Q}) = 0 \qquad H_k(\Omega, \mathbb{Z}/2\mathbb{Z}) = 0$$

$\forall\ k \geq 1$.
In any case the assumption covers a large class of domains: for instance any domain with a hole satisfies $H_{N-1}(\Omega, \mathbb{Z}/2\mathbb{Z}) \neq 0$ and any domain topologically equivalent to a solid torus verifies $H_1(\Omega, \mathbb{Q}) \neq 0$.

3

Let's come back now to problem 1.1 when $a(x) \le 0$. In this case when we consider the related minimization problem

$$S_a = \inf \left\{ \int_\Omega (|\nabla u|^2 - a(x)\, u^2)\, dx \;, \quad u \in V \right\}$$

we cannot expect that the second order term helps to lowering S_a with respect to S. On the contrary $\int (-a(x))\, u^2 \ge 0$ so $S \le S_a$, and, by theorem 1.1 S_a cannot be achieved [if N≥4]. Nevertheless we have seen, in the case $a(x) = 0$, that, all the same, there are solutions for (1.1) if Ω is topologically nontrivial. Moreover if Ω is an annulus and $a(x)$ is a radial function : $a(x) = a(|x|)$ there exist a positive solution for (1.1). It is clear, then, that these solutions are not minima.

These examples show the fact that the infimum is not achieved does not imply problem (1.1) has no solution. It is then a challenge to find an appropriate tool to give an aswer to the following question :

Assume $a(x) \le 0$ on Ω, $a(x) \not\equiv 0$. Find conditions on $a(x)$ which guarantee that Problem (1.1) has a solution.

If Ω is bounded (even if Ω is a ball and $a(x)$ is radial) the answer is, as far as I know, unknown. If $\Omega = \mathbb{R}^N$ we have the following

<u>Theorem</u> (Benci - C. [BeC]) If $\Omega = \mathbb{R}^N$ and

(3.1) $\qquad a(x) \ge 0 \qquad \forall\, x \in \mathbb{R}^N \qquad a(x) \ge \nu > 0$ in a neighborhood of a point $\bar{x}$

(3.2) $\exists\ p_1 < N/2$ and $p_2 > N/2$ such that
$a \in L^p \quad , \forall\ p \in [p_1, p_2]$

(3.3) $|a|_{L^{N/2}} < S\ (2^{2/N} - 1)$

then the problem (1.1) has at least one positive solution.

Before giving an idea of the proof method let me do some remarks.

Remark 3.1 The assumption (3.2) cannot be removed. In fact a variant of the Pohozaev identity implies

$$\int_{\mathbb{R}^N} \left[a(x) + \frac{1}{2} \left(\nabla a(x) \mid x \right) \right] u^2 \, dx = 0 \quad .$$

So if we consider

$$a(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ |x|^{-\beta} & \text{if} \quad |x| \geq 1 \end{cases}$$

we obtain $\beta > 2$ that is equivalent to $a \in L^{p_1}(\mathbb{R}^N)$ $p_1 < N/2$,
Analogously it is possible to construct an $a(x)$ that shows p_2 must be bigger than $N/2$.

Remark 3.2 The assumption (3.3) is related to the approach used to prove the existence of the solution; namely that is a constructive method. Probably arguing by contradiction, as in Bahri - Coron paper, using Morse theory it is possible to remove this assumption.

Remark 3.3 The infimum S_a in our case is just S. In fact $S_a \geq S$ and using the functions

$$\varphi_{1/m,0} = \frac{1}{S^{\frac{N-2}{4}}} \frac{\left[N(N-2)\ 1/m\right]^{\frac{N-2}{4}}}{\left[1/m + |x|^2\right]^{\frac{N-2}{2}}}$$

we see that

$$\hat{E}(\varphi_{1/m,0}) = \int_{\mathbb{R}^N} |\nabla \varphi_{1/m,0}|^2\, dx + \int_{\mathbb{R}^N} a(x)\,(\varphi_{1/m,0})^2\, dx \leq$$

$$\leq S + |a|_{L^p}\ |\varphi_{1/m,0}|^2_{2p'}$$

with $p \in \left(\frac{N}{2}, p_2\right]$ $2p' \in \left(\frac{N}{N-2}, 2^*\right)$. Moreover it is easy to verify that

$$|\varphi_{1/m,0}|_{L^q} \xrightarrow[m \to +\infty]{} 0 \qquad \forall\, q \in \left(\frac{N}{N-2}, 2^*\right)$$

So $S_a \leq S$.

It is clear that S is not achieved (in the opposite case we would have $\exists\, v \neq 0$ such that

$$\int_{\mathbb{R}^N} (\nabla v)^2\, dx < \int_{\mathbb{R}^N} \left[|\nabla v|^2 + a(x)\, v^2\right] dx = S$$

contradicting the definition of S)

The first step in the proof is to look into the reason of the lack of compactness. When Ω is a bounded domain there can be found, roughly speaking, in the invariance of the $H^1_0(\Omega)$ and $L^{2^*}(\Omega)$ norms by the re-scaling $u \to r^{\frac{N-2}{2}}(u(r(\cdot - x_0)))$. When Ω is unbounded to this type of invariance we have ~~to~~ to add the invariance by translations. We have the following

Theorem 3.1 Let $\{u_m\}$ be a Palais - Smale sequence:

$$E(u_m) \to c \qquad E'(u_m) \to 0 \quad \text{in } (\mathcal{D}^{1,2})$$

Then there exist a number k, k sequences of points $\{y^j_m\} \subset \mathbb{R}^N$, $1 \le j \le k$, k sequences of positive numbers $\{\sigma^j_m\}$ $1 \le j \le k$, $k+1$ sequences of functions $\{u^j_m\} \subset \mathcal{D}^{1,2}(\mathbb{R}$ $0 \le j \le k$ such that for some subsequence, still denot by u_m

$$u_m = u^0_m(x) + \sum_{j=1}^{k} \frac{1}{(\sigma^j_m)^{\frac{N-2}{2}}} u^j\left(\frac{x-y^j_m}{\sigma^j_m}\right)$$

and

$$u^j_m(x) \xrightarrow[m\to+\infty]{} u^j(x) \qquad \text{strongly in } \mathcal{D}^{1,2}(\mathbb{R}^N)$$
$$0 \le j \le k$$

where u^0 is a solution of (1.1) and u^j are solutions of (2.2) and

if $\quad y^j_m \xrightarrow[m\to+\infty]{} \bar{y}^j \quad$ then either $\sigma^j_m \xrightarrow[m\to+\infty]{} +\infty$ or $\sigma^j_m \xrightarrow[m\to+\infty]{}$

if $\quad |y^j_m| \xrightarrow[m\to+\infty]{} +\infty$ then as $m\to+\infty$ each of $\begin{cases} \sigma^j_m \to +\infty \\ \sigma^j_m \to 0 \\ \sigma^j_m \to \bar{\sigma}^j \qquad 0<\bar{\sigma}^j<+\infty \end{cases}$

can occur.

Moreover

$$\|u_m\|^2 \longrightarrow \sum_{j=0}^{k} \|u^j\|^2$$

$$E(u_m) \longrightarrow E(u^0) + \sum_{j=1}^{k} E_\infty(u^j) \quad .$$

So we understand that when a loss of compactness happen one of these cases occurs : something which concentrates at a point, something which flattens, something which travel keeping the same concentration, something which travel and at the same time concentrates or becomes flat.

As before if we restrict to the positive cone the energy of any such "wave" is just $\frac{1}{N} S^{N/2}$. So the "bad" energy levels for E are $\frac{k}{N} S^{N/2}$, $k = 1, 2 \ldots$ and for $\hat{E}$ constrained on V are $k^{2/N} S$, $k = 1, 2 \ldots$

The idea is try to construct a "candidate critical level" between S and $2^{2/N} S$.

First of all we consider $\forall u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$\beta(u) = \frac{1}{S} \int_{\mathbb{R}^N} \frac{x}{|x|} \, |\nabla u|^2 \, dx$$

$$\gamma(u) = \frac{1}{S} \int_{\mathbb{R}^N} \sigma(x) \, |\nabla u|^2 \, dx \qquad \sigma(x) = \begin{cases} 0 & |x| < 1 \\ 1 & |x| \geq 1 \end{cases}$$

β is a "weighted" barycenter and γ is something that helps to consider the concentration of u.

Called

$$\mathcal{B} = \{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : (\beta(u), \gamma(u)) = (0, \tfrac{1}{2}) \}$$

we prove that

$$c_0 = \inf_{\mathcal{B}} \hat{E}(u) > S$$

Then we define an operator

$$\Phi : \mathbb{R}^N \times (0, +\infty) \longrightarrow \mathcal{D}^{1,2}(\mathbb{R}^N)$$

$$\Phi : (y, \delta) \to \varphi_{\delta, y} = \frac{[N(N-2)\delta]^{\frac{N-2}{4}}}{[\delta + |x-y|^2]^{\frac{N-2}{2}}} \cdot \frac{1}{S^{\frac{N-2}{4}}}$$

and we use the condition on a to prove that

$$\hat{E}(\Phi(y,\delta)) = \hat{E}(\varphi_{\delta,y}) < 2^{2/N} S \qquad \forall y \in \mathbb{R}^N \ \forall \delta > 0$$

Hence we construct a "box"

$$\Sigma = \{ (y, \delta) \in \mathbb{R}^{N+1} : |y| < R \quad \delta \in [\delta_1, \delta_2] \}$$

with $R > 0$ and $0 < \delta_1 < 1/2 < \delta_2$ chosen in such a way that

$$\forall (y, \delta) \in \partial\Sigma \qquad \hat{E}(\Phi(y,\delta)) < \frac{S + c_0}{2}$$

and that $\forall$ set A belonging to the family Γ obtained by continuous deformations of the set $\Phi(\Sigma)$

$$\Gamma = \{ A \in \mathcal{P} \cap V : A = h(\Phi(\Sigma)),\ h \in \mathcal{C}(\mathcal{P}\cap V, \mathcal{P}\cap V) \quad h(u) = u \quad \forall u : \hat{E}(u) < \frac{c_0 + S}{2} \}$$

we have $A \cap \mathcal{B} \neq \emptyset$.

And we put

$$c = \inf_{A \in \Gamma} \sup_{u \in A} \hat{E}(u)$$

$$c \leq \sup_{\Phi(\Sigma)} \hat{E}(u) \leq \sup_{\substack{y \in \mathbb{R}^N \\ \delta \in \mathbb{R}^+}} \hat{E}(\varphi_{\delta,y}) < 2^{2/N} S \, .$$

$$c \geq \inf_{\mathcal{B}} \hat{E}(u) > S$$

and we prove by standard deformation arguments that c is a critical level.

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