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EXISTENCE AND REGULARITY RESULTS FOR ELLIPTIC EQUATIONS AND SYSTEMS

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# Existence and Regularity Results for Elliptic Equations and Systems

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This talk is an introduction to the problems which arise in the theory of existence and regularity for variational elliptic equations and (especially) systems. The point of view we adopt is the **minimization of functionals:** thus, no Euler equation will be really written, instead we will develop our theory for integral functionals.

A general energy functional has the form

$$\int_A f\big(x, u(x), Du(x), \dots, D^k u(x)\big)\, dx;$$

for the sake of simplicity we will confine ourselves to the case $k = 1$: then the functional takes the usual form

$$F(u) = \int_A f\big(x, u(x), Du(x)\big)\, dx, \tag{1}$$

and we want to solve the problem

$$\min\{F(u) : u \in S\}, \tag{2}$$

where $S$ is the space of "admissible" functions. The starting point of the direct method of the calculus of variations is this well-known result:

*if $F : S \to \mathbf{R}$ is lower semicontinuous and coercive (i.e., $\{F \le c\}$ is compact) then problem (2) has a solution.*

Since the topology of the space $S$ is not given *a priori*, it is customary to take a fairly weak one, in order to have as many compact sets in $S$ as possible. This cannot be done easily if $S$ is a "strong" space, e.g., a space of continuous functions, thus we choose to enlarge $S$, to make the existence problem easier, but we create the problem of regularity.

Our scheme is then:

a) find $S' \supset S$ in which problem (2) has a solution $u^*$, via the result above;

b) prove that in fact $u^* \in S$.

Usually, we handle the coercivity of $F$ by the choice of $S'$, and this leaves us with the problems of semicontinuity and regularity. In order to make the integral in (2) have sense, we assume that

$$f(x,s,\xi) \text{ is measurable in } x \text{ and continuous in } (s,\xi); \tag{3}$$

$$0 \le f(x,s,\xi) \le c(a(x) + |s|^p + |\xi|^p), \tag{4}$$

with $a \in L^1$ and $p \ge 1$. In the case of equations, $u : A \subset \mathbf{R}^n \to \mathbf{R}$, we have the

**Scalar semicontinuity theorem.** *Assume $f$ satisfies* (3),(4). *Then the functional $F$ is lower semicontinuous in the weak topology of $W^{1,p}$ if and only if $f$ is convex with respect to $\xi$.*

This result dates back to Lebesgue, Tonelli, ..., and more recently (necessity of the convexity) to Marcellini - Sbordone. When we deal with systems, $u : A \subset \mathbf{R}^n \to \mathbf{R}^m$, some simple physical considerations (see a famous example by Ball) rule out convexity as a reasonable assumption for existence theorems. Instead, its place is taken by the quasiconvexity condition:

$$\int_A f(x_0, s_0, \xi + D\varphi(x))\, dx \ge \int_A f(x_0, s_0, \xi)\, dx$$

for every $x_0 \in A$, $s_0 \in \mathbf{R}^m$, $\xi \in \mathbf{R}^{nm}$ and $\varphi \in C_0^1(A; \mathbf{R}^m)$. This means, roughly speaking, that for a homogeneous portion of the body the linear deformation $\xi x$ minimizes the energy integral among all deformations which are linear at the boundary. With this definition (introduced by Morrey), we may state the

**Vector-valued semicontinuity theorem.** *Assume $f$ satisfies* (3),(4). *Then the functional $F$ is lower semicontinuous in the weak topology of $W^{1,p}$ if and only if $f$ is* **quasi***convex with respect to $\xi$.*

This result (due to Dacorogna and Acerbi - Fusco) contains the scalar case, because for $m = 1$ quasiconvexity is equivalent to convexity; it is also a strong generalization: an example of quasiconvex function is any *convex function of the subdeterminants of the matrix $\xi$*, which of course need not be convex in $\xi$.

As for regularity, even in the scalar case we have to strengthen the assumptions on $f$; moreover, for the sake of simplicity, we confine ourselves to the case independent of $(x,u)$, although the results would still hold (under less readable assumptions).

We begin with the case $p \ge 2$, and we assume that

$$|\xi|^p \le f(\xi) \le c(1 + |\xi|^p), \tag{5}$$

$$f \text{ is twice continuously differentiable, with } |f_{\xi\xi}(\eta)| \le c(1 + |\eta|^2)^{(p-2)/2}, \tag{6}$$



$$f \text{ is uniformly strictly convex, i.e., } f_{\xi\xi}(\eta)\mu\mu \ge c(1 + |\eta|^2)^{(p-2)/2}|\mu|^2. \tag{7}$$

We say that $u$ is a local minimizer of $F$ if

$$\int_B f(Du(x))\, dx \le \int_B f(Du(x) + D\varphi(x))\, dx$$

whenever $\mathrm{spt}(\varphi) \subset B$. Giaquinta, Giaquinta - Modica and others proved a partial regularity result, i.e., regularity in a subset of $A$:

**Scalar regularity theorem.** *Assume $f$ satisfies* (5),(6),(7): *then every local minimizer of $F$ is of class $C^{1,\alpha}(A')$, where $A'$ is an open subset of $A$ and meas$(A \setminus A') = 0$.*

When we pass to systems, the existence theorem above obliges us to modify assumptions (7), which implies convexity, and (6), since boundedness of the second derivatives is somewhat natural if $f$ is convex, but not if $f$ is quasiconvex. Then we assume

$$f \text{ is twice continuously differentiable,} \tag{6'}$$

$$\int_A f(\xi + D\varphi(x))\, dx \ge \int_A [f(\xi) + c(|D\varphi|^2 + |D\varphi|^p)]\, dx; \tag{7'}$$

this condition is called uniform strict quasiconvexity. Then, always for $p \ge 2$, we have:

**Vector-valued regularity theorem.** *Assume $f$ satisfies* (5),(6'),(7'): *then every local minimizer of $F$ is of class $C^{1,\alpha}(A')$, where $A'$ is an open subset of $A$ and meas$(A \setminus A') = 0$.*

This result has been proved by Evans, still with assumption (6), and by Acerbi - Fusco.

Very little is known in the case $p < 2$: K. Uhlenbeck proved the everywhere regularity of the local minimizers of $\int |Du|^p\, dx$, with $p \ge 2$, and this was later generalized to the case $p < 2$, but essentially for the same kind of functional, by Di Benedetto, Manfredi and Tolksdorff (scalar case) and by Hamburger and Acerbi - Fusco (vector-valued case).

Recently, a very interesting paper by Anzellotti - Giaquinta proved that in the scalar case, under very general assumptions on $f$ and for all $p$, a local minimizer $u$ of $F$ is regular near the points where $Du \ne 0$. This seems to have been generalized (a last control of the proof is due) by Acerbi - Fusco to the quasiconvex case, but only for $p \ge 2$: we examined the lower exponents, but with no success at the moment.

Another important direction in which a theory of this type has been developed (by Chang) is the study not of minimum points, but of some saddle points of $F$.

## Bibliography

The paper by Anzellotti - Giaquinta is as yet unpublished; all the other papers quoted may be found in the References of the following (all by E. Acerbi & N. Fusco):

*Semicontinuity problems in the calculus of variations.* Arch. Rational Mech. Anal. **86.**

*A regularity theorem for minimizers of quasiconvex integrals.* Arch. Rational Mech. Anal. **99.**

*Regularity theorems for non-quadratic functionals. The case* $1 < p < 2$. J. Math. Anal. Appl., to appear.