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PERIODIC BOUNCE TRAJECTORIES WITH A LOW NUMBER OF BOUNCE POINTS

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PRELIMINARY VERSION

PERIODIC BOUNCE TRAJECTORIES
WITH A LOW NUMBER OF BOUNCE POINTS [1]

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In this paper we study the existence of a periodic trajectory with prescribed period, which bounces against the boundary of an open subset of $\mathbb{R}^N$, in presence of a potential field. For every T>0 we found a T-periodic nonconstant solution with at most N+1 bounce points.

1. Introduction.

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with boundary $\partial\Omega$ of class C^2.

A bounce trajectory in $\bar{\Omega}$ is a piecewise linear path with corners at $\partial\Omega$, for wich the usual low of reflection is satisfied, namely the segments make equal angles with the tangent plane. A bounce point is a corner point for our path.

The main result of this paper is the following:

THEOREM (1.1). Let Ω be as above. Then there exists at least one periodic nonconstant trajectory in $\bar{\Omega}$ with at most N+1 bounce points.

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Remark (1.2). The conclusion of Theorem (1.1) is optimal in the sense that it is possible to construct a set Ω for wich there are not trajectories with only N bounce points. For N=1 this is obvious. For N=2 we refer to [5,11] for such a cotroexample.

Remark (1.3). The result of Theorem (1.1) is somewhat surprising. In fact analougous problems exibit a more complicated fenomenology.

For example the Cauchy problem has a solution (in general non unique) provided that the concept of solution is generalized to include trajectories which spend some time lying on the boundary (see [6,7,8, 13] and Remark (2.14)).

The illumination problem (i.e. existence of bounce trajectories with prescribed extreme points) may not have any solution even in a generalized sense (see [14,16] for controexamples and [9,12] for some recent results).

We refer also to [10,12] where the existence of periodic trajectories of special type has been proved in some particular cases.

Theorem (1.1) can be obtained as a consequence of a more general result. Perhaps now it is convenient to give some rigorous definitions.

Let $V \in C^1(\bar{\Omega},\mathbb{R})$, $\nabla V(x)$ the gradient of V at x and $\nu(x)$ the exterior unit normal to $\bar{\Omega}$ in $x \in \partial\Omega$.

DEFINITION (1.4). A loop γ from S^1 to $\bar{\Omega}$ is called a periodic bounce trajectory with respect to the potential V if:

(i) $\gamma \in C^2(S^1)$ except for at most a finite number of instants $t_1,\dots,t_l$ for wich $\gamma(t_j) \in \partial\Omega$;

(ii) $\gamma''(t)+\nabla V(\gamma(t))=0$ for every $t_1,\dots,t_l$;

(iii) for every $t\in(t_1,\dots,t_l)$ there exist the limits $\lim_{s\to t^{\pm}}\gamma'(s):=\gamma'_{\pm}(t)$ and

(1.5) $$\gamma'_+(t)-\langle\gamma'_+(t),\nu(\gamma(t))\rangle\nu(\gamma(t))= =\gamma'_-(t)-\langle\gamma'_-(t),\nu(\gamma(t))\rangle\nu(\gamma(t)),$$

(1.6) $$\langle\gamma'_+(t),\nu(\gamma(t))\rangle=-\langle\gamma'_-(t),\nu(\gamma(t))\rangle\neq 0;$$

(iv) the set $(t_1,\dots,t_l)$ is not empty.

The instants $t_1,\dots,t_l$ for wich (1.5) and (1.6) hold are called bounce instants, while the points $\gamma(t_j)$ are called bounce points.

Notice that $\gamma(t_j)\in\partial\Omega$ does not implies that $\gamma(t_j)$ is a bounce point according to our definition. In fact it may happen that $\langle\gamma'_+(t),\nu(\gamma(t))\rangle=-\langle\gamma'_-(t),\nu(\gamma(t))\rangle=0$.

Using the above definition we can enunciate the following

THEOREM (1.7). Let $\Omega\subset\mathbb{R}^N$ be an open bounded set with boundary of class C^2 and $V\in C^2(\bar\Omega,\mathbb{R})$. Then for every $T>0$ there exists a T-periodic nonconstant bounce trajectory having at most N+1 bounce points.

The proof is based on an approximation scheme introduced in [2]. A bounce trajectory is obtained as limit of regular solutions of a Lagrangian system constrained in a potential well. The approximating problem is studied with variational methods. The number of the bounce points is related to the Morse index of an approximating trajectory. However for technical reason it is convenient to use a generalization of the Conley index (see [3]) and a theorem related to it (see [4]).

2. The approximation scheme.

In this section we show how the existence of a bounce trajectory (in a generalized sense) can be obtained as limit of regular solutions of a Lagrangian system.

Let $\Omega\subset\mathbb{R}^N$ be an open bounded set with boundary $\partial\Omega$ of class C^2 and ν the exterior unit normal to $\bar\Omega$. Let $h\in C^2(\bar\Omega)$ be a function having the following properties:

(2.1)
$$\begin{cases} \text{(i)} & h(x)=\mathrm{dist}(x,\partial\Omega) \text{ if } \mathrm{dist}(x,\partial\Omega)\le d_o;\\ \text{(ii)} & h(x)>d_o \text{ if } \mathrm{dist}(x,\partial\Omega)>d_o;\\ \text{(iii)} & h(x)\le 1 \text{ for every } x\in\bar\Omega;\\ \text{(iv)} & |\nabla h(x)|\le 1 \text{ for every } x\in\Omega,\ h(x)=1 \text{ far from } \partial\Omega;\\ \text{(v)} & \lim_{x\to x_o} -\nabla h(x)=\nu(x_o) \text{ for every } x_o\in\partial\Omega;\\ \text{(vi)} & h_o:=\sup_{x\in\Omega,\,y\neq 0}\dfrac{\langle h''(x)y,y\rangle}{|y|^2}<\infty; \end{cases}$$

where d_o is a constant sufficiently small.

Let $U\in C^2(\Omega,\mathbb{R}^+)$ be defined ás follows:

(2.2) $$U(x)=\frac{1}{h^2(x)}-1,$$

(the term -1 has been added so that $U(x)=0$ for any x far from $\partial\Omega$: this will semplify the notation) and let $V\in C^2(\bar\Omega,\mathbb{R}^+)$.

Now we shall prove a proposition which shows that a bounce solution can be obtained by a suitable approximation scheme. The proposition is somewhat more general of what we need. It uses a "concept" of generalized solution used in [6,7,8,9,13] which allows solutions which may spend some time lying on $\partial\Omega$.

(2.3) PROPOSITION. Let $T>0$ and $\varepsilon>0$. Let $\gamma_\varepsilon\in C^2([0,T],\Omega)$ a T-periodic solution of the Lagrangian system:

(2.4) $$\gamma_\varepsilon''+\nabla V(\gamma_\varepsilon)+\varepsilon\nabla U(\gamma_\varepsilon)=0$$
such that:

(2.5) $$E(\gamma_\varepsilon):=\frac{1}{2}|\gamma_\varepsilon'|^2+V(\gamma_\varepsilon)+\varepsilon U(\gamma_\varepsilon)\le K \quad [2]$$
where K is a real constant independent of ε.

Then γ_ε has a subsequence convergent in $H^1(S^1,\bar\Omega)$ [3] to a curve $\gamma\in H^1(S^1,\bar\Omega)$ satisfying the following properties:

(2.6) γ is Lipschitz continuous;

there is a positive finite real Borel measure μ on $[0,T]$ with $\operatorname{supt}\mu\subseteq C(\gamma):=\{t\in[0,T]:\gamma(t)\in\partial\Omega\}$ such that $\gamma''=-\nabla V(\gamma)-\nu(\gamma)\mu$ in the distributions sense, i.e.

(2.7) $$\int_0^T\langle\gamma',v'\rangle dt-\int_0^T\langle\nabla V(\gamma),v\rangle dt=\int_{C(\gamma)}\langle\nu(\gamma),v\rangle d\mu$$
for every $v\in C^\infty([0,T],\mathbb{R}^N)$ such that $v(0)=v(T)$;

γ has left and right derivative in every $t\in[0,T]$ and

(2.8) $$\frac{1}{2}|\gamma_\pm'(t_2)|^2-\frac{1}{2}|\gamma_\pm'(t_1)|^2=V(\gamma(t_1))-V(\gamma(t_2))$$
for every $t_1,t_2\in[0,T]$;

(2.9) $$\gamma_+'(t)-\langle\gamma_+'(t),\nu(\gamma(t))\rangle\nu(\gamma(t))=$$
$$=\gamma_-'(t)-\langle\gamma_-'(t),\nu(\gamma(t))\rangle\nu(\gamma(t))$$
for every $t\in C(\gamma)$;

(2.10) $$\langle\gamma_+'(t),\nu(\gamma(t))\rangle=-\langle\gamma_-'(t),\nu(\gamma(t))\rangle$$
for every $t\in C(\gamma)$.

Proof. By (2.4) we have

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[2] Notice that $E(\gamma_\varepsilon)$ is a constant of the motion, i.e. the energy.

[3] Here $H^1(S^1,\bar\Omega)=\{q\in AC(0,T;\bar\Omega):q'\in L^2(0,T;\mathbb{R}^N),q(0)=q(T)\}$.

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(2.11) $$\int_0^T\langle\gamma_\varepsilon',v'\rangle dt-\int_0^T\langle\nabla V(\gamma_\varepsilon),v\rangle dt-\varepsilon\int_0^T\langle\nabla U(\gamma_\varepsilon),v\rangle dt=0$$
for every $v\in H^1(S^1;\mathbb{R}^N)$.

Let $v_\varepsilon=-\nabla h(\gamma_\varepsilon)$. By (2.5) γ_ε' is bounded in L^∞ because we have supposed $U(x)\ge0$, $V(x)\ge0$ for every $x\in\Omega$. Moreover by (2.1)(vi) $v_\varepsilon'=-h''(\gamma_\varepsilon)\gamma_\varepsilon'$ is bounded in L^∞. Since also $\langle\nabla V(\gamma_\varepsilon),v_\varepsilon\rangle$ is bounded in L^∞, by (2.11) we get that
$$\varepsilon\int_0^T\langle\nabla U(\gamma_\varepsilon),v_\varepsilon'\rangle dt=2\varepsilon\int_0^T\frac{|\nabla h(\gamma_\varepsilon)|^2}{h^3(\gamma_\varepsilon)}dt$$
is bounded independently of ε. By (2.1)(v) $|\nabla h(x)|\ge\frac{1}{2}$ in a neighbourhood of $\partial\Omega$, therefore there exists M_0 independent of ε such that

(2.12) $$\int_0^T\frac{2\varepsilon}{h^3(\gamma_\varepsilon)}dt\le M_0.$$

Then $\varepsilon\langle\nabla U(\gamma_\varepsilon)=\dfrac{-2\varepsilon\nabla h(\gamma_\varepsilon)}{h^3(\gamma_\varepsilon)}$ is bounded in L^1, hence, by (2.4), γ_ε'' is bounded in L^1.

Since for every $1<p<+\infty$ $H^{1,1}([0,T];\mathbb{R}^N)$ is compactly embedded in L^p, up to a subsequence, there exists $\gamma\in H^1(S^1;\mathbb{R}^N)$ such that $\gamma_\varepsilon\to\gamma$ in H^1 (and uniformly). Obviously $\gamma(t)\in\bar\Omega$ $\forall t\in[0,T]$, $\gamma(0)=\gamma(T)$ and γ is Lipchitz continuous.

By (2.12), the sequence of positive real functions $\dfrac{2\varepsilon}{h^3(\gamma_\varepsilon)}$ converges (up to a subsequence) in L^1-weak*. Since $[L^1(S^1;\mathbb{R})]^*\subset[C^0(S^1;\mathbb{R})]^*$ (where $[\ \ ]^*$ denotes the dual space) we get that
$$\frac{2\varepsilon}{h^3(\gamma_\varepsilon)}\longrightarrow\mu\in[C^0(S^1;\mathbb{R})]^*\ \text{weakly}.$$

By well known theorems, μ is a positive finite Borel measure. Moreover if $\bar t\notin C(\gamma)$ we have that $\varepsilon U(\gamma_\varepsilon)\longrightarrow0$ uniformly in a neighbourhood of $\bar t$, therefore $\operatorname{supt}\mu\subset C(\gamma)$.

Since (2.1)(v) holds, when ε tends to 0 by (2.11) we get (2.7).

By (2.7) $\gamma'\in BV(S^1;\mathbb{R}^N)$ [4] and (2.9) holds.

To prove (2.8) we shall need the following property:

(2.13) $$\lim\varepsilon U(\gamma_\varepsilon(t))=0\quad\text{a.e. in }[0,T],$$

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[4] then γ has left and right derivative in every $t\in S^1$ which are left continuous and right continuous respectively.

up to a subsequence.

Since $\gamma'_\varepsilon \to \gamma'$ in L^2, up to a subsequence, $\gamma'_\varepsilon \to \gamma'$ a.e. in $[0,T]$. Since $\varepsilon U(x) \geq 0$ $\forall x \in \Omega$, the real number $E(\gamma_\varepsilon)$ defined at (2.5) is bounded independly of ε, therefore there exists $w \in L^\infty([0,T];\mathbb{R}^N)$ such that

$$\varepsilon U(\gamma_\varepsilon(t)) \to w(t) \quad \text{a.e. in } [0,T].$$

We claim that $w(t)=0$ a.e. Indeed

$$\varepsilon U(\gamma_\varepsilon(t)) = \frac{\varepsilon}{h^2(\gamma_\varepsilon(t))}$$

and

$$\varepsilon \nabla U(\gamma_\varepsilon(t)) = \frac{-2\nabla h(\gamma_\varepsilon(t))}{h(\gamma_\varepsilon(t))} \varepsilon U(\gamma_\varepsilon(t)).$$

Therefore if $w(t) \neq 0$ on a set $E \subset [0,T]$ having positive Lebesgue measure, we have $|\varepsilon \nabla U(\gamma_\varepsilon(t))| \to +\infty$ $\forall t \in E$, hence, by Fatou Lemma,

$$\liminf_{\varepsilon \to 0} \int_E |\varepsilon \nabla U(\gamma_\varepsilon(t))| dt = +\infty$$

in contradiction with the boundness of $\varepsilon \nabla U(\gamma_\varepsilon(t))$ in L^1.

By (2.13) and (2.5)

$$\frac{1}{2}|\gamma'(t_2)|^2 - \frac{1}{2}|\gamma'(t_1)|^2 = V(\gamma(t_1)) - V(\gamma(t_2))$$

for almost every $t_1, t_2 \in [0,T]$. Since the left derivative of γ is left continuous and the right derivative ia right continuous we get (2.8).

By (2.8) with $t_1 = t_2$ we get $|\gamma'_+(t)| = |\gamma'_-(t)|$ $\forall t \in [0,T]$. Then, since (2.9) holds, it must be

$$|\langle \gamma'_+(t), \nu(\gamma(t)) \rangle| = |\langle \gamma'_-(t), \nu(\gamma(t)) \rangle|$$

for every $t \in C(\gamma)$. If $\langle \gamma'_+(t), \nu(\gamma(t)) \rangle \neq 0$ it must be $\langle \gamma'_+(t), \nu(\gamma(t)) \rangle = -\langle \gamma'_-(t), \nu(\gamma(t)) \rangle$ because $\gamma(t) \in \bar{\Omega}$ $\forall t$. Then (2.10) is proved.

■

(2.14) Remark. For every couple $(\gamma_o, p_o) \in \Omega \times \mathbb{R}^N$ the Cauchy problem has at least one solution, i.e. there exists a curve γ with initial conditions

$$(2.15) \qquad \begin{cases} \gamma(t_o) = \gamma_o \\ \gamma'(t_o) = p_o \end{cases}$$

which satisfies (2.7)–(2.10).

Proof. It is easy to check that the equation (2.4) has alwais a unique solution γ_ε satisfying (2.15) for every $t \in \mathbb{R}$ and its energy is

$$\frac{1}{2}p_o^2 + V(\gamma_o) + \varepsilon U(\gamma_o).$$

For any $T>0$ by (2.4) we have

$$\int_{-T}^{T} \langle \gamma''_\varepsilon + \nabla V(\gamma_\varepsilon) + \varepsilon \nabla U(\gamma_\varepsilon), v \rangle = 0$$

for every $v \in H^1([-T,T];\mathbb{R}^N)$. Therefore

$$\int_{-T}^{T} \langle \gamma'_\varepsilon, v' \rangle dt - \int_{-T}^{T} \langle \nabla V(\gamma_\varepsilon), v \rangle dt - \varepsilon \int_{-T}^{T} \langle \nabla U(\gamma_\varepsilon), v \rangle dt =$$

$$= \langle \gamma'_\varepsilon(T), v(T) \rangle - \langle \gamma'_\varepsilon(-T), v(-T) \rangle$$

for every $v \in H^1([-T,T];\mathbb{R}^N)$.

At this point, since γ'_ε is bounbed in L^∞ independently of ε, as in the proof of Proposition (2.3) we get the conclusion.

■

3. The existence of a solution of the approximating problem.

To enunciate the abstract theorem which we use to study the approximating problem we recall the Palais–Samale condition, the notion of linking, and the definition of Morse index.

Let X be a real Hilbert space with norm $\| \ \|$ and scalar product $\langle \ , \ \rangle$ and let Λ be an open set in X. If $J \in C^1(\Lambda, \mathbb{R})$, J' will denote its Frechet derivative which can be identified, by virtue of $\langle \ , \ \rangle$ with a function from Λ to X.

(3.1) DEFINITION. We say that J satisfies the Palais–Smale condition (P.S.) on Λ if every sequence γ_n such that $J(\gamma_n)$ is bounded and $J'(\gamma_n) \to 0$ has a subsequence which concerges to $\bar{\gamma} \in \Lambda$.

(3.2) DEFINITION. Let S be a closed set in X and let $Q \subset X$ an Hilbert manifolds with boundary ∂Q. We say that S and ∂Q

link if

(i) $S\cap\partial Q=\emptyset$;

(ii) if $h:Q\cup\partial Q\to X$ is a continuous map such that $h(u)=u$ for every $u\in\partial Q$, then $h(Q)\cap S\neq\emptyset$.

(3.3) DEFINITION. Let $J\in C^2(\Lambda,\mathbb{R})$ and $\gamma\in\Lambda$ such that $J'(\gamma)=0$. We call Morse index of γ the dimension of the space spanned by the eigenvectors of $J''(\gamma)$ corresponding to the strictly negative eigenvalues.

We denote by $m(\gamma)$ the morse index of γ.

(3.4) LEMMA. Let Λ be an open subset of the real Hilbert space X. Let $J\in C^2(\Lambda,\mathbb{R})$, $0\in\Lambda$, $J(0)\leq 0$. Assume that:

(J_1) if $\gamma_n\to\gamma_o\in\partial\Lambda$ then $J(\gamma_n)\to-\infty$;

(J_2) J satisfies (P.S.) on Λ;

(J_3) there exists an N-dimensional space E_N $(N\geq 1)$ such that:

(i)
$$J_{|E_N\cap\Lambda}\leq 0$$

(ii) there exist $\rho>0,\alpha>0$ such that $B_\rho:=\{\gamma\in X:\|\gamma\|\leq\rho\}\subset\Lambda$ and
$$\inf_S J>\alpha,$$
where $S=\partial B_\rho\cap E_N^\perp$ and $E_N^\perp=\{v\in X:\langle v,w\rangle=0\ \forall w\in E_N\}$;

(iii) there exists $e\in E_N^\perp\setminus\{0\}$ such that the set
$$Q_\Lambda=\{y+re:\ y\in E_N, r\geq 0\}\cap\Lambda$$
is bounded .

Then if $\beta<+\infty$ is such that
$$\sup_{Q_\Lambda} J<\beta,$$

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J has a critical point γ[5] such that:
$$\alpha<J(\gamma)<\beta$$
and
$$m(\gamma)\leq N+1.$$

The existence of a critical point γ such that $\alpha<J(\gamma)<\beta$ can be obtained by a slight variant of the linking theorems (see 2.3, [1,15]) and its proof can be carried out in a similar way.

Indeed if we put $J(\gamma)=-\infty$ $\forall\gamma\in X\setminus\Lambda$, because of (J_1), (J_3)(i) and (J_3)(iii), there exists $R>0$ such that
$$Q_\Lambda\subset Q:=\{y+re:\ y\in E_N, \|y\|\leq R, 0\leq r\leq R\},$$
$$\sup_{\partial Q} J\leq 0 \text{ and } \sup_Q J<\beta.$$
Moreover S and ∂Q link (see Proposition (2.2) of [1]), so using (J_1) and (J_2) we are able to prove the existence of a critical point $\gamma\in\Lambda$ such that $\alpha<J(\gamma)<\beta$.

To get the estimate on the Morse index of the critical point γ, we use a generalization of the Morse–Conley index (see [3]). In fact Lemma (3.4) can be obtained as Corollary of Theorem (3.14) of [4]. We must only pay attention to the fact that in Theorem (3.14) of [4] J is defined on X while here J is defined in an open subset of X.

Now we are able to prove the esistence of a solution for the approximating problem. The approximation scheme which we use has been introduced in [2]. Here the situation is simpler because J satisfies (P.S) on Λ and $J(\gamma_n)$ tends to $-\infty$ when γ_n approaches $\partial\Lambda$. By Lemma (3.4) we get also an estimate of the Morse index of the our solution of the approximating problem. This estimate will be used to give the estimate of the bounce points of the solution.

(3.5) PROPOSITION. Let $T>0$, $\Omega\subset\mathbb{R}^N$ be an open bounded set with

(5) i.e. $J'(\gamma)=0$.

boundary $\partial\Omega$ of class C^2, $V\in C^2(\bar{\Omega},\mathbb{R}^+)$ and $U\in C^2(\Omega,\mathbb{R}^+)$ be the function defined at (2.2).

Then there exists $E^-, E^+, \alpha, \beta \in \mathbb{R}^+\setminus\{0\}$ and $m\in\mathbb{N}\setminus\{0\}$ such that for every $\varepsilon>0$ there exists $\gamma_\varepsilon\in C^2(\mathbb{R},\Omega)$, T/m-periodic solution of the Lagrangian system (2.4), verifying the following properties:

(i)
$$0<E^-\le E(\gamma_\varepsilon)\le E^+$$
where the energy $E(\gamma_\varepsilon)$ is defined at (2.5);

(ii)
$$0<\alpha\le J_\varepsilon(\gamma_\varepsilon)\le\beta$$
where $J_\varepsilon\in C^2(\Lambda,\mathbb{R})$ is the functional

(3.6)
$$J_\varepsilon(\gamma)=\frac{1}{2}\int_0^{T/m}|\gamma'|^2dt-\int_0^{T/m}V(\gamma)dt-\varepsilon\int_0^{T/m}U(\gamma)dt,$$

and

$$\Lambda=\{\gamma\in H^1(0,T/m;\mathbb{R}^N):\gamma(0)=\gamma(T/m),\gamma(t)\in\Omega\ \forall t\in[0,T/m]\};$$

(iii)
$$\frac{1}{2}\int_0^{T/m}|\gamma_\varepsilon'|^2dt\ge\alpha>\frac{\alpha}{2}\ge\int_0^{T/m}\langle\nabla V(\gamma_\varepsilon),\gamma_\varepsilon\rangle dt;$$

(iv)
$$m(\gamma_\varepsilon)\le N+1.$$

In order to prove Proposition (3.5) applying Lemma (3.4), we need some preliminary notations and results. Let

$$X=\{\gamma\in H^1(0,T/m;\mathbb{R}^N):\gamma(0)=\gamma(T/m)\}$$

with inner product

$$\langle v,w\rangle_X=\int_0^{T/m}\langle v',w'\rangle dt+\langle\int_0^{T/m}v\,dt,\int_0^{T/m}w\,dt\rangle$$

where $\langle\ ,\ \rangle$ is the standard inner product in $\mathbb{R}^N$.

Let Λ be that the statement of Proposition (3.5), that is

$$\Lambda=\{\gamma\in X:\gamma(t)\in\Omega\ \forall t\in[0,T/m]\}.$$

It is easy to check that

$$J_\varepsilon'(\gamma)v=\int_0^{T/m}\langle\gamma',v'\rangle dt-\int_0^{T/m}\langle\nabla V(\gamma),v\rangle dt-\varepsilon\int_0^{T/m}\langle\nabla U(\gamma),v\rangle dt$$

for every $\gamma\in\Lambda$, for every $v\in X$.

As known if γ_ε is a critical point for J (that is $J'(\gamma_\varepsilon)v=0\ \forall v\in X$) then γ_ε is the restriction to the interval $[0,T/m]$ of a T/m-periodic solution of (2.4).

(3.7) LEMMA. Let $(\gamma_n)\subset\Lambda$ be such that γ_n converges to γ weakly in H^1. Assume that $\gamma\in\partial\Lambda$. Then

$$\lim_{n\to+\infty}\int_0^{T/m}\frac{1}{h^2(\gamma_n(t))}dt=+\infty.$$

Proof. Since $\gamma\in\partial\Lambda$, there exists $t_0\in[0,T/m]$ such that $\gamma(t_0)\in\partial\Omega$. Obviously we can suppose $t_0=0$. We have

$$|\gamma_n(t)-\gamma_n(0)|\le\int_0^t|\gamma_n'|ds\le t^{1/2}\left(\int_0^{T/m}|\gamma_n'|^2ds\right)^{1/2}\le t^{1/2}\|\gamma_n\|_X.$$

Since (2.1)(iv) holds and $\|\gamma_n\|_X\le C$ for some $C>0$, we have

$$|h(\gamma_n(t))-h(\gamma_n(0))|\le|\gamma_n(t)-\gamma_n(0)|\le t^{1/2}\|\gamma_n\|_X\le t^{1/2}C.$$

Since γ_n converges to γ weakly in H^1, γ_n converges to γ also in L^∞. In particular $\gamma_n(0)\to\gamma(0)\in\partial\Omega$. Then $h(\gamma_n(0))\to0$. Let $b_n=h(\gamma_n(0))$. We have

$$h(\gamma_n(t))\le b_n+t^{1/2}C.$$

Then

$$\frac{1}{h^2(\gamma_n(t))}\ge\frac{1}{(b_n+t^{1/2}C)^2}\ge\frac{1}{2}\left[\frac{1}{b_n^2+C^2t}\right]$$

hence

$$\int_0^{T/m}\frac{1}{h^2(\gamma_n(t))}dt\ge\frac{1}{2}\int_0^{T/m}\frac{1}{b_n^2+C^2t}dt=\frac{1}{2C^2}\log\left[1+\frac{C^2(T/m)}{b_n^2}\right].$$

Since $b_n\to0$ we get the thesis. ■

(3.8) LEMMA. Let $(\gamma_n)\subset\Lambda$ such that $J_\varepsilon(\gamma_n)$ is bounded from above and $J_\varepsilon'(\gamma_n)\to0$.

Then there exists a subsequence $\gamma_{n_k}\to\bar{\gamma}\in\Lambda$. In particular J_ε satisfies (P.S.) on Λ.

Proof. Since

$$\lim_{\substack{x\to x_o\\ x\in\Omega}} \frac{\langle \nabla U(x), -\nabla h(x)\rangle}{U(x)} = +\infty$$

for every $x_o\in\partial\Omega$, for every $\delta>0$ there exists $a_\delta\in\mathbb{R}^+$ such that

(3.9) $$U(x)\le\delta\langle\nabla U(x),-\nabla h(x)\rangle + a_\delta$$

for every $x\in\Omega$.

Since $J'(\gamma_n)\to 0$ we have

(3.10) $$\int_0^{T/m}\langle\gamma_n',v'\rangle dt - \int_0^{T/m}\langle\nabla V(\gamma_n),v\rangle dt - \varepsilon\int_0^{T/m}\langle\nabla U(\gamma_n),v\rangle dt =$$

$$= a_n\|v\|_X$$

for every $v\in X$, where $a_n\to 0$.

Because of (2.1)(vi) $-\nabla h(\gamma_n)\in X$, then by (3.9), (3.10), (2.1)(vi) and (2.1)(iv) we get

$$\varepsilon\int_0^{T/m}U(\gamma_n)dt \le \delta\varepsilon\int_0^{T/m}\langle\nabla U(\gamma_n),-\nabla h(\gamma_n)\rangle dt + (T/m)a_\delta \le$$

$$\le\delta\left[h_o\int_0^{T/m}|\gamma_n'|^2dt + (T/m)\sup_{\bar\Omega}|\nabla V| +\right.$$

$$\left.+|a_n|\left(h_o\left(\int_0^{T/m}|\gamma_n'|^2dt\right)^{1/2}+(T/m)\right)\right]+(T/m)a_\delta.$$

Then there exists M_1 independent of n such that

$$\varepsilon\int_0^{T/m}U(\gamma_n)dt \le \delta\left[2h_o\int_0^{T/m}|\gamma_n'|^2dt + M_1\right]+(T/m)a_\delta =$$

$$=\delta\left[4h_oJ(\gamma_n)+4h_o\int_0^{T/m}V(\gamma_n)dt + 4h_o\varepsilon\int_0^{T/m}U(\gamma_n)dt + M_1\right]+(T/m)a_\delta.$$

Since $J(\gamma_n)$ is bounded from above there exists M_2 independent of n such that

$$\varepsilon\int_0^{T/m}U(\gamma_n)dt \le\delta\left[4h_o\varepsilon\int_0^{T/m}U(\gamma_n)dt + M_2\right]+(T/m)a_\delta.$$

Then if $4h_o\delta=\frac{1}{2}$ we have

(3.11) $$\frac{1}{2}\int_0^{T/m}U(\gamma_n)dt\le M$$

where M is a constant independent of n.

Now $J(\gamma_n)$ is bounded from above, therefore, by (3.11) $\int_0^{T/m}|\gamma_n'|^2dt$ is bounded. Then, up to a subsequence, γ_n is weakly convergent in H^1 (and strongly in L^∞) to $\bar\gamma\in X$ such that $\bar\gamma(t)\in\bar\Omega$ for every $t\in[0,T/m]$.

By (3.11) and Lemma (3.7) $\bar\gamma(t)\in\Omega$ $\forall t\in[0,T/m]$. At this

point by standard argument we can easily prove that the subsequence γ_n is strongly convergent in H^1 to $\bar\gamma\in\Lambda$. ■

Proof of Proposition (3.5). By Lemma (3.7) and Lemma (3.8) J_ε satisfies (J_1) and (J_2).

Obviously we can suppose $0\in\Omega$. Let us pose

$$E_N=\{\gamma\in X:\gamma \text{ is constant}\},$$

$$E_N^\perp=\{\gamma\in X:\int_0^{T/m}\gamma\,dt=0\},$$

and

$$S_\rho=\{\gamma\in E_N^\perp: \|\gamma\|_X=\rho\}$$

where $\rho>0$.

Since $0\in\Omega$ we can suppose that there exists $\rho_o>0$ such that the function h defined at (2.1) is equal to 1 for every x such that $|x|\le\rho_o$. Then we have

(3.12) $$U(x)=0\quad\forall x: |x|\le\rho_o.$$

Moreover

$$J_\varepsilon(0)\le 0 \text{ and } J|_{E_N\cap\Lambda}\le 0.$$

If $\gamma\in E_N^\perp$ we have

$$|\gamma(t)|\le\int_0^{T/m}|\gamma'|ds$$

for every $t\in[0,T/m]$, therefore

$$\|\gamma\|_{L^\infty}\le(T/m)^{1/2}\left(\int_0^{T/m}|\gamma'|^2ds\right)^{1/2}\quad\forall\gamma\in E_N^\perp$$

and

(3.13) $$\|\gamma\|_{L^\infty}\le(T/m)^{1/2}\rho\quad\forall\gamma\in S_\rho.$$

Let $\rho=1$, m such that $(T/m)^{1/2}\le\rho_o$, and $S=S_1$. By (3.12) and (3.13) we have

$$U(\gamma(t))=0\quad\forall t\in[0,T/m],\forall\gamma\in S.$$

Then for every $\gamma\in S$

$$J_\varepsilon(\gamma)=\frac{1}{2}\int_0^{T/m}|\gamma'|^2dt-\int_0^{T/m}V(\gamma)dt\ge\frac{1}{2}-(T/m)\sup_{\bar\Omega}V.$$

Therefore if m is such that $\frac{1}{2}-(T/m)\sup_{\bar\Omega}V \ge \frac{1}{4}$ we get

(3.14) $$J_\varepsilon(\gamma)\ge\frac{1}{4}:=\alpha\quad\forall\gamma\in S.$$

Moreover we can choose m such that

$$\frac{\alpha}{2}=\frac{1}{8}\geq (T/m)(\sup_{x\in\bar\Omega}|\nabla V(x)|)(\sup_{x\in\bar\Omega}|x|)$$

so we get

(3.15) $$\frac{\alpha}{2}\geq\int_0^{T/m}\langle\nabla V(\gamma),\gamma\rangle dt \quad \forall\gamma\in\Lambda.$$

Let $e\in\mathbb{R}^N$ with $\|e\|=1$ and

$$Q_\Lambda=(E_N^1+r\sin(\frac{2\pi m}{T}t)e\ :r\geq 0)\cap\Lambda.$$

Obviously Q_Λ is bounded in X. Moreover if $\gamma\in Q_\Lambda$

$$\gamma=y+r\sin(\frac{2\pi m}{T}t)e\in\Omega\ \forall t\in[0,T/m],$$

where $y\in\mathbb{R}^N$. Therefore

$$|y|<d,\ r<2d,\ \text{where } d=\sup_{x\in\bar\Omega}|x|.$$

Then

(3.16) $$J_\varepsilon(\gamma)\leq\frac{1}{2}\int_0^{T/m}|\gamma'|^2dt\leq\frac{4d^2\pi^2}{T/m}:=\beta$$

for every $\gamma\in Q_\Lambda$.

Then by Lemma (3.4) J_ε has a critical point γ_ε[(o)] such that

(3.17) $$0<\alpha\leq J_\varepsilon(\gamma_\varepsilon)\leq\beta$$

and

(3.18) $$m(\gamma_\varepsilon)\leq N+1.$$

Since $V(x)\geq 0$ and $U(x)\geq 0$ $\forall x\in\Omega$, by (3.17) we have

$$\frac{1}{2}\int_0^{T/m}|\gamma_\varepsilon'|^2dt\geq\alpha.$$

hence by (3.15), (iii) of Proposition (3.5) follows.

It remains to prove the estimate for $E(\gamma_\varepsilon)$. Now $E(\gamma_\varepsilon)$ is a constant of the motion, therefore

(3.19) $$(T/m)E(\gamma_\varepsilon)=\frac{1}{2}\int_0^{T/m}|\gamma_\varepsilon'|^2dt+\int_0^{T/m}V(\gamma_\varepsilon)dt+\varepsilon\int_0^{T/m}U(\gamma_\varepsilon)dt.$$

Since $V(x)\geq 0$ and $U(x)\geq 0$ $\forall x\in\Omega$, by (3.17) and (3.19) we get

$$\alpha\leq(T/m)E(\gamma_\varepsilon)\leq\beta+2(T/m)\sup_{x\in\bar\Omega}V(x)+2\varepsilon\int_0^{T/m}U(\gamma_\varepsilon)dt.$$

Moreover, as in the proof of Lemma (3.8) we get that

(o) which is the restriction to $[0,T/m]$ of a T/m–periodic solution of class C^2 of (2.4).

$\varepsilon\int_0^{T/m}U(\gamma_\varepsilon)dt$ is bounded from above by a constant M independent of ε. Then Proposition (3.5) holds with $E^-=\frac{\alpha}{T/m}$ and $E^+=\frac{\beta}{T/m}+2\sup_{x\in\bar\Omega}V(x)+\frac{2M}{T/m}$. ∎

4. Proof of the main result.

Now we want to find a bounce trajectory with at most N+1 bounce points (where N is the dimension of the space), using the approximation scheme introduced in section 2 and the Lemma (3.4).

To prove Theorem (1.7) obviously we can suppose $V(x)\geq 0$ $\forall x\in\bar\Omega$.

For every $\varepsilon>0$ let γ_ε the curve found in Proposition (3.5). By Proposition (2.3), up to a subsequence, γ_ε is convergent in $H^1(S^1,\bar\Omega)$ to a curve $\gamma:[0,T/m]\to\bar\Omega$ which verify (2.6), (2.7), (2.8), (2.9) and (2.10) and which is the restriction to $[0,T/m]$ of a T/m–periodic curve.

By (ii) of Proposition (3.5) γ is not constant (because V and U are positive on Ω).

To prove that γ has at most N+1 bounce points it is useful to introduce the following notions of "nonregular point for γ ".

(4.1) DEFINITION. Let γ as above. We say that $\bar t\in[0,T/m]$ is a "nonregular instant for γ" if there exists $\bar\delta>0$ such that for every $\delta\in(0,\bar\delta)$ the weak equation

(4.2) $$\int_{\bar t-\delta}^{\bar t+\delta}\langle\gamma',v'\rangle dt-\int_{\bar t-\delta}^{\bar t+\delta}\langle\nabla V(\gamma),v\rangle dt=0\quad\forall v\in H_0^1(\bar t-\delta,\bar t+\delta;\mathbb{R}^N)$$

is not verified.

We call "nonregular points for γ" the points $\gamma(\bar t)\in\partial\Omega$ such that $\bar t$ is a nonregular instants for γ.

Remark (4.3). Notice that if we prove that γ has at most N+1 nonregular points, by Proposition (2.3) we get that they are bounce points i.e. γ verifies (i),(ii) and (iii) of Definition (1.4), with $l \leq N+1$.

To prove Theorem (1.7) we need also the following Lemmas.

(4.4) LEMMA. Let $\gamma(\bar{t})$ be a nonregular point for γ and $I_\delta = [\bar{t}-\delta, \bar{t}+\delta]$ with $\delta \in (0, T/2m)$. Then

$$\liminf_{\varepsilon \to 0} \varepsilon \int_{I_\delta} \frac{1}{h^3(\gamma_\varepsilon)} dt > 0.$$

Proof. Since γ_ε satisfies (2.4) and $U(x)$ is defined by (2.2) we have

$$J'_\varepsilon(\gamma_\varepsilon)v = \int_0^{T/m} \langle \gamma'_\varepsilon, v' \rangle dt - \int_0^{T/m} \langle \nabla V(\gamma_\varepsilon), v \rangle dt + 2\varepsilon \int_0^{T/m} \frac{\langle \nabla h(\gamma_\varepsilon), v \rangle}{h^3(\gamma_\varepsilon)} dt = 0$$

for every $v \in H^1(S^1, \mathbb{R}^N)$.

If, up to a subsequence, $\lim_{\varepsilon \to 0} \varepsilon \int_{I_\delta} \frac{1}{h^3(\gamma_\varepsilon)} dt = 0$, going to the limit in ε we get

$$\int_{I_\delta} \langle \gamma', v' \rangle dt - \int_{I_\delta} \langle \nabla V(\gamma), v \rangle dt = 0$$

for every $v \in H^1_0(I_\delta, \mathbb{R}^N)$, which contradicts the hypothesis. ■

(4.5) LEMMA. Let $B = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < r_o\}$ where r_o is such that

$$\operatorname{dist}(x, \partial\Omega) < r_o \text{ implies } |\nabla h(x)| \geq \tfrac{1}{2} \text{ [7]}.$$

If $\gamma(t_o) \in \partial\Omega$ there exist $\varepsilon_o > 0$ and $\delta_o > 0$ such that:

$$\forall \delta < \delta_o, \ \forall \varepsilon < \varepsilon_o, \ \forall t \in (t_o - \delta, t_o + \delta), \quad \gamma_\varepsilon(t) \in B$$

Proof. Let ε_o be such that

$$\operatorname{dist}(\gamma_\varepsilon(t_o), \gamma(t_o)) \leq r_o/2 \quad \forall \varepsilon < \varepsilon_o.$$

By (i) of Proposition (3.5) $\frac{\rho}{2}|\gamma'_\varepsilon|^2 \leq E^+$ $\forall t \in [0, T/m], \forall \varepsilon > 0$, then it suffices to choose $\delta_o = \left(\frac{r_o}{4E^+}\right)^2$. ■

[7] notice that r_o exists because of (2.1)(iv).

Proof of Theorem (1.7). Assume, by contradiction, that there exist $t_1 < t_2 < \ldots < t_{N+2} \in [0, T/m]$ such that $\gamma(t_j)$, $(j = 1, \ldots, N+2)$ are distinct nonregular points for γ.

For every j let δ_j be as in Lemma (4.5) and such that (4.1) is not verified for every $\delta < \delta_j$ with $\bar{t} = t_j$.

Let $\delta_o \leq \min(\delta_1, \ldots, \delta_{N+2})$ such that $\forall \delta < \delta_o$, we have $t_{j+1} - t_j > 2\delta$ for every $j = 1, \ldots, N+1$, and $(T/m) + t_1 - t_{N+2} > 2\delta$.

Let $I_j = [t_j - \delta, t_j + \delta]$ and $I'_j = \left[t_j - \frac{\delta}{2}, t_j + \frac{\delta}{2}\right]$ with $\delta \in (0, \delta_o)$.

Moreover for every j let ε_j be as in Lemma (4.5), $\varepsilon_o \leq \min(\varepsilon_1, \ldots, \varepsilon_{N+2})$ and $\varepsilon < \varepsilon_o$.

For every $j = 1, \ldots, N+2$ let $\varphi_j \in C^1([0, T/m], [0, 1])$ such that

$$\varphi_j(t) = 0 \quad \forall t \in [0, T/m] \setminus I_j$$
$$\varphi_j(t) = 1 \quad \forall t \in I'_j.$$

Let $v_{\varepsilon j}(t) = -\varphi_j(t) \nabla h(\gamma_\varepsilon(t))$. We have

$$\langle J''_\varepsilon(\gamma_\varepsilon) v_{\varepsilon j}, v_{\varepsilon j} \rangle = \int_0^{T/m} |v'_{\varepsilon j}|^2 dt - \int_0^{T/m} \langle V''(\gamma_\varepsilon) v_{\varepsilon j}, v_{\varepsilon j} \rangle dt +$$
$$+ 2\varepsilon \int_0^{T/m} \frac{\langle h''(\gamma_\varepsilon) v_{\varepsilon j}, v_{\varepsilon j} \rangle}{h^3(\gamma_\varepsilon)} dt - 6\varepsilon \int_0^{T/m} \frac{\langle \nabla h(\gamma_\varepsilon), v_{\varepsilon j} \rangle^2}{h^4(\gamma_\varepsilon)} dt.$$

Since $\int_0^{T/m} |\gamma'_\varepsilon|^2 dt$ is bounded from above by a constant independent of ε, by (2.1)(vi) also $\int_0^{T/m} |v'_{\varepsilon j}|^2 dt$ is. Under our hypotheses $V \in C^2(\bar{\Omega}, \mathbb{R})$, therefore $\int_0^{T/m} \langle V''(\gamma_\varepsilon) v_{\varepsilon j}, v_{\varepsilon j} \rangle dt$ is bounded independently of ε. By (2.1)(vi) and (2.12) also $2\varepsilon \int_0^{T/m} \frac{\langle h''(\gamma_\varepsilon) v_{\varepsilon j}, v_{\varepsilon j} \rangle}{h^3(\gamma_\varepsilon)} dt$ is bounded by a constant independent of ε. Moreover we have

$$\varepsilon \int_0^{T/m} \frac{\langle \nabla h(\gamma_\varepsilon), v_{\varepsilon j} \rangle^2}{h^4(\gamma_\varepsilon)} dt \geq \varepsilon \int_{I_j} \varphi_j \frac{|\nabla h(\gamma_\varepsilon)|^4}{h^4(\gamma_\varepsilon)} dt \geq \varepsilon \int_{I'_j} \varphi_j^2 \frac{|\nabla h(\gamma_\varepsilon)|^4}{h^4(\gamma_\varepsilon)} dt \geq$$

$$\geq (\text{by Lemma } (4.5)) \frac{1}{16} \varepsilon \int_{I'_j} \frac{1}{h^4(\gamma_\varepsilon)} dt \geq$$

$$\geq (\text{by Hölder inequality}) \frac{1}{16} \varepsilon \left(\frac{1}{\delta}\right)^{1/3} \left(\int_{I'_j} \frac{1}{h^3(\gamma_\varepsilon)} dt\right)^{4/3} =$$

$$=\frac{1}{16}\left(\frac{1}{\delta}\right)^{1/3}\left[\varepsilon\int_{I'_j}\frac{1}{h^3(\gamma_\varepsilon)}dt\right]\left(\int_{I'_j}\frac{1}{h^3(\gamma_\varepsilon)}dt\right)^{1/3}.$$

Now by Lemma (3.7) and Hölder inequality

$$\lim_{\varepsilon\to 0}\int_{I'_j}\frac{1}{h^3(\gamma_\varepsilon)}dt=+\infty,$$

therefore, since $\gamma(t_j)$ is a nonregular point for γ, by Lemma (4.4)

$$\lim_{\varepsilon\to 0}\langle J''_\varepsilon(\gamma_\varepsilon)v_{\varepsilon j},v_{\varepsilon j}\rangle=-\infty.$$

Let $\bar\varepsilon$ be such that $\langle J''_\varepsilon(\gamma_\varepsilon)v_{\varepsilon j},v_{\varepsilon j}\rangle\le -1$ for every $\varepsilon\le\bar\varepsilon$ and for every $j=1,\dots,N+2$.

Since the curves $v_{\varepsilon j}$ are mutually orthogonal in X the bilinear form $J''_\varepsilon(\gamma_\varepsilon)$ is negative in a linear subspace of X having dimension at least N+2. Consequently $J''_\varepsilon(\gamma_\varepsilon)$ has at least N+2 strictly negative eigenvalues, hence

$$m(\gamma_\varepsilon)\ge N+2\qquad\forall\varepsilon\le\bar\varepsilon,$$

and this contradicts (iv) of Proposition (3.5). Then γ has at most N+1 nonregular points.

Because of Remark (4.3) it remains to prove that γ has at least a bounce point. By contradiction if γ has not bounce points, $\gamma\in C^2(S^1,\bar\Omega)$ and

$$\gamma''+\nabla V(\gamma)=0\qquad\forall t\in S^1.$$

Then $\langle\gamma''+\nabla V(\gamma),\gamma\rangle=0$ $\forall t\in[0,T/m]$ and since γ is T/m-periodic

$$\frac{1}{2}\int_0^{T/m}|\gamma'|^2dt=\int_0^{T/m}\langle\nabla V(\gamma),\gamma\rangle dt$$

and this contradicts (iii) of Proposition (3.5).

Theorem (1.7) is so completely proved. ■

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