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COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS
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ON A CLASS OF NONLINEAR PROBLEMS AT RESONANCE

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1. Introduction

In these notes we shall present a class of "resonant" nonlinear boundary value problems which possess an infinite number of positive solutions. As it will be seen, these solutions appear as a consequence of the special oscillatory nature of the branch of positive solutions which bifurcates from infinity at the first eigenvalue of the corresponding linear problem, when we deal with nonlinear perturbations which include certain types of bounded domains $\Omega \subset \mathbb{R}^N$ and periodic nonlinearities of zero mean.

The main results presented here are contained in the paper [CJSS]. For more details, we refer the reader to that paper and the related work [SS], [LS], [S], [W].

Let us consider the "resonant" problem

$$(*) \qquad \begin{cases} \Delta u + \lambda_1 u + g(u) = h(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $h:\bar{\Omega} \to \mathbb{R}$, $g:\mathbb{R} \to \mathbb{R}$ are given Hölder continuous functions. We are letting λ_1 denote the first eigenvalue of the problem $\Delta u + \lambda u = 0$ in Ω, $u = 0$ on $\partial\Omega$, and ϕ an associated eigenfunction with $\phi > 0$ in Ω. And we shall assume that the given h, g satisfy the following conditions:

$$(h_1) \qquad \int_\Omega h\phi \, dx = 0,$$

$$(g_1) \qquad \lim_{s \to +\infty} \frac{g(s)}{s} = 0,$$

$$(g_2) \qquad \begin{cases} G(s) \le G(s_n), & 0 \le s \le s_n \\ G(s) \ge G(t_n), & 0 \le s \le t_n, \end{cases}$$

where $G(s) = \int_0^s g(\tau)d\tau$ and $\{s_n\}$, $\{t_n\}$ are monotone unbounded sequences.

In the ODE case $(N=1)$ with g a periodic function of zero mean it was shown by Ward [W], using variational arguments, that $(*)$ has a solution. Subsequently, Solimini [S] and Lupo-Solimini [LS] extended Ward's method and result to the PDE case $(N \ge 2)$. On the other hand, using methods from global bifurcation theory, Schaaf and Schmitt [SS] showed that $(*)$ has in fact infinitely many positive and infinitely many negative solutions under the hypotheses in [W]. And recently, extending the method of [SS] to the PDE case, Costa-Jeggle-Schaaf-Schmitt [CJSS] showed that this infinite multiplicity result remains true in dimensions $N \ge 2$ provided the domain Ω satisfies a suitable geometric condition, which we here denote by (q_+). For the time being, let us only say that (q_+) is a requirement on the positivity of a certain quadratic form involving the eigenfunction ϕ and let us remark that

(i) The condition (q_+) is satisfied in the following situations: N=2 and Ω a disc of arbitrary radius; $N \ge 2$ and $\Omega = \{x \mid a < |x| < b\}$ an annular domain with $(b-a)$ small in regard to a.

(ii) The condition (q_+) is not satisfied in dimension $N \ge 3$ when Ω is a ball.

(iii) Numerical experiments (cf. [CJSS]) seem to indicate that the infinite multiplicity result fails for balls in dimension greater than three.

2. Motivation - The Linear Problem

Let us consider the following simple one parameter linear problem

$$(**)_\lambda \qquad \begin{cases} \Delta u + \lambda u = h(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$ and we denote by $0 < \lambda_1 < \lambda_2 < \ldots$ the distinct eigenvalues of the homogeneous problem $\Delta u + \lambda u = 0$ in Ω, $u=0$ on $\partial\Omega$. As is well-known, if $\lambda \neq \lambda_j$ for all j, then $(**)_\lambda$ has a unique solution for any given $h \in L^2(\Omega)$; on the other hand, if $\lambda = \lambda_k$ for some k, then $(**)_{\lambda_k}$ has a solution if and only if the given h is L^2-orthogonal to the λ_k-eigenspace N_k; in this case $(h \perp N_k)$, one obtains in fact a "continuum" of solutions given by $u = u_0 + N_k$, with $u_0 \perp N_k$. We can illustrate these results by the following "bifurcation diagrams" where, for our purpose here, we restrict λ (the horizontal axis) to a neighborhood of λ_1 of the form $(0, \mu_1)$ with $\lambda_1 < \mu_1 < \lambda_2$, and let the vertical axis denote the sup norm u_{max} of u (say h is Hölder continuous).

Case 1 $\int_\Omega h\phi \, dx \neq 0$

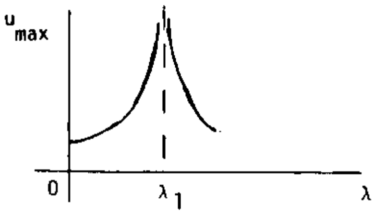

Fig. 1

Case 2 $\int_\Omega h\phi \, dx = 0$

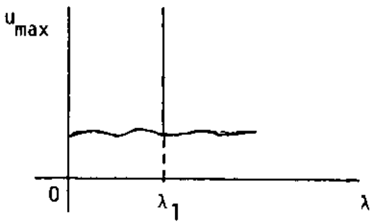

Fig. 2

Now, the "bifurcation approach" to our-original problem $(*)$ is the following: embed $(*)$ into the one parameter (nonlinear) problem

$$(*)_\lambda \qquad \begin{cases} \Delta u + \lambda u + g(u) = h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and study the λ_1-section of the solution set S of $(*)_\lambda$. Since the nonlinear problem $(*)_\lambda$ is a "perturbation" (by g) of the linear problem $(**)_\lambda$, it is natural to expect that the "bifurcation diagrams" for $(*)_\lambda$ in Cases 1 and 2 are "perturbations" of the corresponding bifurcation diagrams for the linear case. In fact, one obtains a result on "bifurcation from ∞" at $\lambda = \lambda_1$, which we present next.

3. On Bifurcation from Infinity

In this section we state an abstract result about bifurcation from infinity and a corollary which we shall use in our problem. For their proofs the reader is referred to [SS] and [CJSS] (see also [PS] and [R]).

Consider the equation

$$u = K(\lambda)u + k(\lambda, u), \quad u \in X, \tag{1}$$

where X is a Banach space with norm $\|\cdot\|$, $K: [a,b] \subset \mathbb{R} \to B(X)$ is a differentiable family of compact linear operators on X and $k: [a,b] \times X \to X$ is a completely continuous mapping satisfying

$$\frac{k(\lambda,u)}{\|u\|} \to 0 \quad \text{as} \quad \|u\| \to \infty,$$

uniformly for $\lambda \in [a,b]$. Then we have the following

Lemma 1. Let $\lambda_1 \in (a,b)$ be such that

(i) $\ker(id - K(\lambda_1)) = \text{span}\, \phi$, $\|\phi\| = 1$,

(ii) $K'(\lambda_1)\phi \notin \text{range}(id - K(\lambda_1))$, and $P \subset X$ be an open cone containing ϕ. Then there exists $\varepsilon_0 > 0$ and a continuum (i.e., a closed, connected set) $C \subset [a,b] \times P$ of solutions of (1) with the property that for any $0 < \varepsilon \leq \varepsilon_0$ we can find a subcontinuum $C_\varepsilon \subset C$ such that

$$C_\varepsilon \subset U_\varepsilon := \{(\lambda,u) \mid |\lambda - \lambda_1| < \varepsilon, \ \|u\| > 1/\varepsilon\},$$

and C_ε connects (λ_1,∞) to ∂U_ε. Moreover, if $(\lambda_n,u_n) \in C \cap U_\varepsilon$ is suth that $\|u_n\| \to \infty$, then

$$\lambda_n \to \lambda_1 \quad \text{and} \quad \frac{u_n}{\|u_n\|} \to \phi.$$

Corollary 2. Let the assumptions of lemma 1 hold and assume that $K(\lambda)$, $k(\lambda,.)$ map X continuously into a Banach space $Y \subset X$ which is compactly embedded in X and that $K: [a,b] \to B(X,Y)$, $k: [a,b] \times X \to Y$ are continuous with

$$\frac{k(\lambda,u)}{\|u\|} \to 0 \quad \text{in} \quad Y \quad \text{as} \quad \|u\| \to \infty,$$

uniformly on $[a,b]$. Then, if $(\lambda_n,u_n) \in C \cap U_{\varepsilon_0}$ is such that $\|u_n\| \to \infty$, we get

$$\lambda_n \to \lambda_1 \quad \text{and} \quad \left\| \frac{u_n}{\|u_n\|} - \phi \right\|_Y \to 0.$$

In particular, if $\tilde{P} \subset Y$ is any open cone containing ϕ, then, by decreasing ε_0 if necessary, we obtain that $C \subset [a,b] \times \tilde{P}$.

4. The Landesman-Lazer Situation

In 1970 Landesman and Lazer [LL] considered the resonant problem (*) under the assumptions that the limits $g(\pm\infty) = \lim_{s \to \pm\infty} g(s)$

exist as finite numbers and g, h satisfy

(2) $$g(-\infty) \leq g(s) \leq g(+\infty) \quad \text{for all } s,$$

(3) $$g(-\infty) \int_\Omega \phi \, dx < \int_\Omega h\phi \, dx < g(+\infty) \int_\Omega \phi \, dx.$$

Then they proved the existence of at least one solution for (*). This nice result can be understood, a posteriori, by inferring that the Landesman-Lazer condition (3) causes the bifurcation diagrams of Figs. 1 and 2 to deform into diagrams which roughly have the shape in Figs. $\tilde{1}$ and $\tilde{2}$ below. In other words, condition (3) must imply that the branch which bifurcates from infinity at λ_1 will cross the "hyperplane" $\lambda = \lambda_1$ at least once.

__Case 1__ $\int_\Omega h\phi \, dx \neq 0$

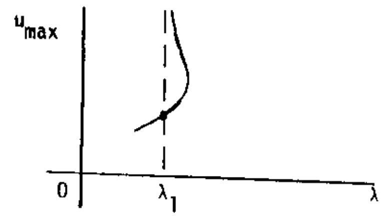

Fig. $\tilde{1}$

__Case 2__ $\int_\Omega h\phi \, dx = 0$

Fig. $\tilde{2}$

5. The Periodic Situation

We now consider the resonant problem (*) with the nonlinearity g satisfying the hypotheses

$$(4) \qquad g(s+T) = g(s) \quad \text{for all} \quad s,$$

$$(5) \qquad \int_0^T g(s)ds = 0.$$

As already mentioned in the Introduction, Ward [W] in the ODE case and Solimini, Lupo-Solimini [S,LS] in the PDE case proved the existence of at least one solution for (*) by means of variational methods. We shall now prove, following Schaaf--Schmitt [SS] in the ODE case (cf. also Costa-Jeggle-Schaaf--Schmitt [CJSS] for the PDE case) that (*) has in fact infinitely many positive (and negative) solutions. The proof consists in showing that conditions (4), (5) [or, more generally, conditions (g_1), (g_2) in the Introduction] cause the bifurcation diagram of Fig. 2 to deform into a diagram which roughly has the shape in Fig. 3 below. In other words, the branch of positive solutions which bifurcates from infinity at $\lambda = \lambda_1$ will be shown to cross the 'hyperplane' $\lambda = \lambda_1$ infinitely many times in this case

$\left(\int h\phi \, dx = 0\right)$

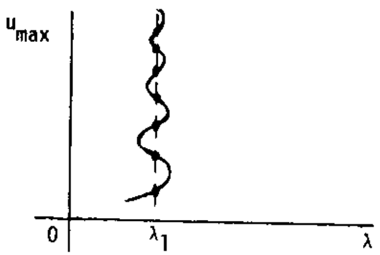

Fig. 3

Indeed, let us consider the one-parameter problem

$$(*)_\lambda \qquad \begin{cases} u'' + \lambda u + g(u) = h(x), & 0 < x < \pi \\ u(0) = u(\pi) = 0, \end{cases}$$

and study the λ_1-section of its solution set. Here $\lambda_1 = 1$ and $h: [0,\pi] \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$ are continuous functions such that g satisifes conditions (4), (5) [or (g_1), (g_2)] and h is L^2-orthogonal to $\phi(x) = \sin x$, i.e., h satisfies

$$(6) \qquad \int_0^\pi h(x)\sin x \, dx = 0.$$

Now let $K: C[0,\pi] \to C[0,\pi]$ be the operator defined by $Kf = u$ if and only if u is the solution of $u'' + f = 0$, $0 < x < \pi$, $u(0) = u(\pi) = 0$. Clearly K is a bounded linear operator from $C[0,\pi]$ into $C_0^2[0,\pi]$ so that, in particular, $K: C[0,\pi] \to C[0,\pi]$ is compact. Our problem $(*)_\lambda$ above is then equivalent to the operator equation

$$u = \lambda Ku + K(g(u) - h)$$

in the space $X = C_0[0,\pi] = \{u \in C[0,\pi] \mid u(0) = u(\pi) = 0\}$. We may hence apply lemma 1 and corollary 2 with $K(\lambda) = \lambda K$, $k(\lambda,u) = K(g(u)-h)$ and letting $Y = C_0^1[0,\pi]$, $P = \{u \in X \mid \int_0^\pi u\phi \, dx > 0\}$, $\tilde{P} = \{u \in Y \mid u > 0 \text{ in } (0,\pi),\ u'(0) > 0,\ u'(\pi) < 0\}$. We obtain $\varepsilon_0 > 0$ and a continuum $C \subset \mathbb{R} \times \tilde{P}$ of solutions of $(*)_\lambda$ such that $C \cap U_\varepsilon \neq \emptyset$ for any $0 < \varepsilon \leq \varepsilon_0$ and such that if $(\lambda_n,u_n) \in C$ with $|\lambda_n - \lambda_1| < \varepsilon_0$ and $\|u_n\| = \max|u_n| \to \infty$, then

$$\lambda_n \to \lambda_1 \quad \text{and} \quad \frac{u_n}{\max u_n} \to \phi = \sin x \quad \text{in} \quad C_0^1[0,\pi].$$

In fact, by regularity we obtain that

$$(7) \qquad \frac{u_n}{\max u_n} \to \phi \quad \text{in} \quad C_0^2[0,\pi].$$

Now for a given (λ,u) in the branch C above we obtain, multiplying equation $(*)_\lambda$ by ϕ and integrating by parts, that

$$(1-\lambda)\int_0^\pi u\phi\, dx = \int_0^\pi g(u)\phi dx.$$

Since $u \in P$ it follows that the right hand side of above determines the sign of $1-\lambda$. Letting $\alpha = \max u$, $G(s) = \int_0^s g$ and integrating by parts we obtain

$$\int_0^\pi g(u)\phi dx = \int_0^\pi \frac{d}{dx}\left[G(u(x)) - G(\alpha)\right]\frac{\phi}{u'}\,dx$$

$$= \int_0^\pi \left[G(u(x))-G(\alpha)\right]\frac{\phi' u' - \phi u''}{(u')^2}\,dx := \int_0^\pi AQdx.$$

Therefore, for $(\lambda,u) \in C$ we have that

(8) $$\operatorname{sgn}(1-\lambda) = \operatorname{sgn}\int_0^\pi AQ\, dx,$$

and we shall now study the signs of A and Q.

In view of lemma 1, corollary 2 and keeping (7) in mind, it follows that

$$\operatorname{sgn}\left[\phi' u' - \phi u''\right] = \operatorname{sgn}\left[(\phi')^2 + \phi^2\right]$$

for $\alpha = \max u$ sufficiently large, hence

(9) $$Q > 0 \quad \text{if} \quad \alpha \text{ is large.}$$

Finally, we choose a sequence (α_n) with

$$1 \ll \alpha_1 < \alpha_2 < \ldots \to \infty$$

and solutions $(\lambda_n,u_n) \in C$ with

$$\max u_n = \alpha_n$$

and

$$\begin{cases} G(s) \le G(\alpha_{2n}), & 0 \le s \le \alpha_{2n} \\ G(s) \ge G(\alpha_{2n+1}), & 0 \le s \le \alpha_{2n+1} \end{cases}$$

(This can be done in view of (g_2), which is implied by (4), (5)). It then follows that

(10) A alternates in sign infinitely often as $\alpha \to \infty$.

Therefore, combining (8), (9), (10) we conclude that, for $(\lambda,u) \in C$, $1-\lambda$ alternates in sign infinitely often as $\alpha = \max u \to \infty$. Since C is a continuum there exists a sequence $\tilde{u}_n$ with $\max u_n \le \max \tilde{u}_n \le \max u_{n+1}$ such that $(\lambda_1,\tilde{u}_n) \in C$. The proof is complete.

For the PDE case we obtain the following

Theorem ([CJSS]) Let $h:\bar{\Omega} \to \mathbb{R}$, $g:\mathbb{R} \to \mathbb{R}$ be Hölder continuous functions satisfying the conditions (h_1), (g_1), (g_2) stated in the introduction and suppose the domain $\Omega \subset \mathbb{R}^N$ verifies the "geometric condition"

(q_+) $$\inf_\Omega Q(\phi) > 0$$

where $Q(\phi) = |\nabla\phi|^2 - \phi\Delta\phi - 2(N-1)\phi|\nabla\phi|H$ and $H(x)$ denotes the mean curvature at $y = x$ of the level surface $\{y \,|\, \phi(y) = \phi(x)\}$ with respect to the unit normal vector $-\nabla\phi/|\nabla\phi|$. Then, the problem (*) possesses an infinite number of solutions $\{u_n\} \subset C_0^{2+\mu}(\bar{\Omega})$ with $u_n > 0$ in Ω, $\partial u_n/\partial\nu < 0$ on $\partial\Omega$ and such that $\max u_n \to \infty$ and $u_n/\max u_n \to \phi$ in $C_0^{2+\mu}(\bar{\Omega})$ as $n \to \infty$.

6. Final Comments

As previously mentioned, the proofs in [W,S,LS] of existence of one solution for (*) are variational ones.

The variational approach to problem (*) is the following. One considers the Sobolev space $E = H_0^1(\Omega)$ and the functional $I:E \to \mathbb{R}$ defined by

$$I(u) = \int_\Omega (\frac{1}{2} |\nabla u|^2 - \frac{1}{2} \lambda_1 u^2 + hu - G(u))dx = J(u) + N(u),$$

where

$$J(u) = \int_\Omega (\frac{1}{2} |\nabla u|^2 - \frac{1}{2} \lambda_1 u^2 + hu)dx, \quad u \in E.$$

It can be easily shown (under the hypotheses (4), (5) on g) that $I \in C^1(E,\mathbb{R})$ with derivative $I'(u) \in E^* = H^{-1}(\Omega)$ given by

$$I'(u)\cdot v = \int_\Omega (\nabla u \cdot \nabla v - \lambda_1 uv + hv - g(u)v)dx$$

for $u,v \in E$. So, $u \in E$ is a critical point of I if and only if u is a weak solution of (*). And, therefore, one looks for the critical points of the functional I.

Now, assuming (as we are) that h is orthogonal to ϕ, we obtain that J is bounded from below on E and, hence, that the functional I is also bounded from below on E (as G is T-periodic by (4), (5)). In the ODE case, Ward [W] shows that either the functional I attains its infimum $m^* = \inf_E I(u)$, or else there must still exist a critical point at a higher level $c > m^*$. This critical level is obtained by using a Saddle Point Theorem of Rabinowitz [R_2] and, to that end, it is shown in [W] that I satisfies the Palais-Smale condition (PS) at the level c (i.e, whenever $\{u_n\} \subset E$ is a sequence satisfying $I(u_n) \to c$, $I'(u_n) \to 0$, then $\{u_n\}$ has a convergent subsequence). On the other hand, for the PDE case (and under more general conditions), Solimini and Lupo-Solimini [S,LS] also show the existence of a critical point for I by means of a suitable version of the Saddle Point Theorem of Rabinowitz. And it is explicitly noted in [S,LS] that I does not satisfy completely the Palais-Smale condition: more precisely, I satisfies the (PS) condition at levels $c \in \mathbb{R} \setminus J(\bar{u})$ where $\bar{u}$ is a solution of $\Delta u + \lambda_1 u = h$ in Ω, $u = 0$ on $\partial\Omega$ (recall that $h \in \phi^\perp$); and, moreover, it satisfies the following condition

(PS)' Every sequence $\{u_n\} \subset E$ such that $(u_n,\phi)_E$ is bounded and $I'(u_n) \to 0$ has a converging subsequence.

It would be very nice (and certainly not an easy task) to find a proof for the infinite multiplicity result of Schaaf-Schmitt [SS] (ODE case) by means of variational methods. Such a proof, aside from its own interest, might help to decide whether a geometric condition like (q_+) on the domain Ω is indeed necessary for the infinite multiplicity result to hold true in the PDE case. Of course, besides the main difficulty already mentioned that the functional I does not satisfy completely the condition (PS), one would also have to deal with the fact that, under conditions (4), (5), the eventual critical levels of I belong necessarily to a bounded interval (although the critical points themselves do not remain bounded). Indeed, since

$$I(u) = J(u)+N(u) = J(u_1)+N(u)$$

where $u_1 = Qu$ is the orthogonal projection of u onto $\phi^\perp$, and since $N(u)$ is uniformly bounded, we have that $I(u)$ becomes unbounded if and only so does $J(u_1)$ or, equivalently, if the projection u_1 become unbounded (Note that J is coercive on $\phi^\perp$). On the other hand, if $u \in E$ is a critical point of I we obtain

$$0 = I'(u)\cdot u_1 = \|u_1\|_E^2 - \lambda_1 |u_1|_{L^2}^2 + (h-g(u),u_1)_{L^2},$$

from which (using that $\|u_1\|_E^2 \geq \lambda_2 |u_1|_{L^2}^2 \ \forall u_1 \in \phi^\perp$) we obtain $\|u_1\|_E \leq C$, for some constant C independent of u. Therefore, the critical levels of I remain bounded.

References

[CJSS] D. Costa, H. Jeggle, R. Schaaf and K. Schmitt, Periodic perturbations of linear problems at resonance, Resultate der Mathematik (to appear).

[LL] E.M. Landesman and A.C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19 (1970), 609-623.

[LS] D. Lupo and S. Solimini, A note on a resonant problem, preprint.

[PS] H. Peitgen and K. Schmitt, Global analysis of two--parameter elliptic eigenvalue problems, Trans. Amer. Math. Soc. 283 (1984), 57-95.

[R_1] P.H. Rabinowitz, On bifurcation from infinity, J. Diff. Eqs. 14 (1973), 462-475.

[R_2] P.H. Rabinowitz, Some minimax theorems and applications to nonlinear partial differential equations, in Nonlinear Analysis: A Collection of Papers in Honor of Erich H. Rothe (Cesari, Kannan and Weinberger, ed.) Academic Press, 1978, 161-177.

[S] S. Solimini, On the solvability of some elliptic partial differential equations with linear part at resonance, preprint.

[SS] R. Schaaf and K. Schmitt, A class of nonlinear Sturm--Liouville problems with infinitely many solutions, Trans. Amer. Math. Soc. (to appear).

[W] J. Ward, A boundary value problem with a periodic nonlinearity, Nonl. Anal. 10 (1986), 207-213.