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34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
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# ON THE PERIODIC NONLINEARITY AND THE MULTIPLICITY OF SOLUTIONS

Kung-Ching CHANG
Department of Mathematics
University of Wisconsin-Madison
Madison, WI
USA

UNIVERSITY OF WISCONSIN-MADISON
CENTER FOR THE MATHEMATICAL SCIENCES

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Kung-Ching Chang*

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ABSTRACT

We study the multiplicity of solutions for semilinear elliptic systems as well as Hamiltonian systems, in which the nonlinear terms are periodic in certain variables. The cuplength for cohomology rings of the torus is used. Our results generalize and unify several recent works by Conley-Zehnder, Rabinowitz, Mawhin-Willem, Pucci-Serrin etc. In particular, the resonance problems and indefinite problems are studied.

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*Also Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, and Peking University, People's Republic of China.

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## §1. Introduction

Inspired by the work of Conley and Zehnder [3] on the solution of the Arnold conjecture, the author presented a different proof of their statement, and noticed that the periodicity of the Hamiltonian function is the essence of the occurrence of multiple periodic solutions [1-2]. The main purpose of this paper is to apply the following theorem obtained in [1] to various different problems which are studied recently by many authors in dealing with periodic nonlinearities.

Let $H$ be a real Hilbert space, and let $A$ be a bounded self-adjoint operator defined on $H$. According to its spectral decomposition, $H = H_+ \oplus H_0 \oplus H_-$, where $H_+$, $H_0$, and $H_-$ are invariant subspaces corresponding to the positive, zero, and negative spectrum of $A$ respectively.

<u>Theorem 0</u>. Suppose that $A$ satisfies the following assumptions

$(H_1)$ $A_\pm \triangleq A|_\pm$ has a bounded inverse on $H_\pm$,

$(H_2)$ $\gamma \triangleq \dim(H_- \oplus H_0) < \infty$.

Let $V^n$ be a $C^2$ compact n-manifold without boundary, and let $g \in C^1(H \times V^n, R^1)$ be a function having a bounded and compact differential $dg(x)$. Assume that

$$g(P_0x,v) \to -\infty \quad \text{as} \quad \|P_0x\| \to \infty \quad \text{if} \quad \dim H_0 \neq 0$$

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where $P_0$ is the orthogonal projection onto $H_0$. Then the function

$$f(x,v) = \frac{1}{2}(Ax,x) + g(x,v)$$

possesses at least cuplength $(V^n) + 1$ distinct critical points.

If further, we assume that $g \in C^2(H \times V, R^1)$, and that $f$ is nondegenerate, then $f$ has at least $\sum_{i=0}^{n} \beta_i(V^n)$ critical points, where $\beta_i(V^n)$ is the $i^{th}$ Betti number of $V^n$, $i = 0,1,\ldots,n$.

Remark. In the statement of Theorem 8.3 in [1], the function $g$ was assumed to be $C^2$, however, in the proof of the first conclusion, $C^1$ is sufficient.

Most recent studies only concerned with the case where $A$ is positive definite, we shall give more applications where $A$ is semidefinite, i.e., the negative eigenspace as well as the null space are finite dimensional. They are used to study semilinear elliptic systems and the periodic solution problems for $2^{nd}$ order ODE. Theorem 2 generalizes and unifies the results due to Mawhin [7], Mawhin and Willem [8], Li [6], Jiang [5], Franks [4], Pucci and Serrin [9,10] and Rabinowitz [11].

Periodic solution problems for Hamiltonian systems reduce to case where $A$ is unbounded and indefinite. Theorem 4 is a generalization of Theorem 2. It implies the early results due to Conley and Zehnder [3] as special cases. In particular, the multiple periodic solutions of Hamiltonian systems with resonance are studied, where the Hamiltonian functions are only periodic in certain variables.

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## §2. Semi-definite functionals

A direct consequence of Theorem 0, is the following:

Theorem 1. Suppose that $A$ is a self-adjoint operator satisfying $(H_1)$ and $(H_2)$, defined on a Hilbert space $H$. Suppose that $\Phi \in C^1(H,R^1)$ is a function having a bounded and compact differential $d\Phi$, and satisfies the following periodicity condition:

(P) $\exists\, e_1,\ldots,e_r \in \ker A$, they are linearly independent, and $\exists (T_1,\ldots,T_r) \in R^r$ such that

$$\Phi(x + \sum_{j=1}^{r} m_j T_j e_j) = \Phi(x), \quad \forall x \in H,\ \forall (m_1,\ldots,m_r) \in Z^r$$

and the resonance condition:

(LL) $\Phi(x) \to -\infty$ if $\|x\| \to \infty$ and $x \in \ker(A) \cap \{e_1,\ldots,e_r\}^{\perp}$.

Then the equation

$$Ax + d\Phi(x) = 0 \tag{2.1}$$

possesses at least $r + 1$ distinct solutions.

If further, $\Phi \in C^2(H,R^1)$ and all solutions of (2.1) are nondegenerate, then (2.1) possesses at least $2^r$ solutions.

Proof. We consider the following functional

$$J(x) = \frac{1}{2}(Ax,x) + \Phi(x) .$$

According to (P),

$$J(x + \sum_{j=1}^{r} m_j T_j e_j) = J(x), \quad \forall (m_1,\ldots,m_r) \in Z^r .$$

However, we have an orthogonal decomposition

$$H = \ker A \oplus (\ker A)^{\perp}$$
$$= Z \oplus Y \oplus (\ker A)^{\perp}$$

where $Z = \mathrm{span}\{e_1,\dots,e_r\}$, and $Y = Z^\perp \cap \ker(A)$. If we restrict ourselves on the quotient space

$$T^r \times (Y \oplus (\ker A)^\perp)$$

where $T^r = Z/Z^r(T_1,\dots,T_r)$, $Z^r(T_1,\dots,T_r) := \{(m_1T_1,\dots,m_rT_r) | (m_1,\dots,m_r) \in Z^r\}$, the functionals

$$f(u,v) = J(x) ,$$

and

$$g(u,v) = \Phi(x) ,$$

are well defined, where $(v,u) \in T^r \times (Y \oplus (\ker A)^\perp)$ and $x = u + v$. The critical point of $f$ is a solution of (2.1). Since $f$ and $g$ satisfy all conditions in Theorem 0 with $H_0 = Y$ and $V = T^r$, the conclusion follows directly. We present here an application.

Theorem 2. Let $M$ be a compact manifold without boundary, let $(a_{ij}(x))$ be a symmetric $(N-r)$ matrix valued continuous function defined on $M$, and let

$$\ker(-\Delta \cdot I_{N-r} + (a_{ij}(x))\cdot) = \mathrm{span}\{\varphi_1,\dots,\varphi_k\} ,$$

where $0 < r < N$ are integers. Assume that $F \in C^1(M \times R^N, R^1)$ satisfies the following assumptions

(1) $F(x,u + \sum_{i=1}^{r} m_iT_ie_i) = F(x,u) \quad \forall(x,u) \in M \times R^N, \forall(m_1,\dots,m_r) \in Z^r$

where $e_i = (0,\dots,\underset{i\text{th}}{1},\dots,0)$, $i = 1,2,\dots,r$, and $(T_1,\dots,T_r) \in R^r$ is given,

(2) $\|F_u(x,u)\|_{L^\infty(M,R^N)} < \infty$,

(3) $F(x, \sum_{j=1}^{k} t_j\varphi_j(x)) \to \infty$ as $|t| = (\sum_{j=1}^{k} t_j^2)^{1/2} \to \infty$,

and that $h \in C(M,R^N)$, $h = (h_1,\dots,h_N)$ satisfies



$$\int_M h_i(x)dx = 0, \quad i = 1,2,\dots,r ,$$

and $h_j(x) = 0$, $j = r+1,\dots,N$. Then the elliptic system

$$-\Delta u + \tilde{a}(x) \cdot u - F_u(x,u) + h(x) = 0 \quad \text{on } M$$

has at least $r + 1$ solutions, where

$$\tilde{a}(x) = \begin{pmatrix} 0 & \\ & (a_{ij}(x)) \end{pmatrix}_{N\times N} .$$

Proof. Let $H = W^{1,2}(M,R^N)$, $A = I_N + (-\Delta)^{-1}\tilde{a}(x)$, and

$$\Phi(u) = \int_M -F(x,u(x)) + \langle h(x) \cdot u(x)\rangle_N .$$

Obviously,

$$\ker A = \mathrm{span}\{e_1,\dots,e_r, \varphi_1,\dots,\varphi_k\} ,$$

and $\Phi \in C^1(H,R^1)$, having a bounded and compact differential, satisfies the conditions (P) and (LL).

The conclusion follows immediately from Theorem 1.

Remark 2.1. In Theorem 2, we may replace the compact manifold $M$ by a smooth bounded domain $\Omega$ in $R^n$, in addition to the Neumann boundary value condition

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0 ,$$

where $\nu$ is the outward pointing normal of the boundary $\partial\Omega$.

Example 2.1. $M = S^1$, $r = N = 1$. This is just the periodic solution problem for ODE

$$\ddot{u} + F_u(t,u) = h(t) \tag{2.2}$$

where $F \in C^1(S^1 \times R^1, R^1)$ is periodic in $u$, and $h \in C(S^1,R^1)$ satisfies the zero mean condition $\int_{S^1} h(t)dt = 0$. Under these conditions, (2.2) has at least two solutions. It was shown by Mawhin and Willem [8], Li [6] and Franks [4].

Example 2.2. The case $M = S^1$, and $r = N$. The corresponding ODE system was studied by Jiang [5] and Rabinowitz [11]. In this case, the following system possesses at least $N + 1$ solutions

$$\ddot{u} + F_u(t,u) = h(t) \qquad (2.3)$$

where $F \in C^1(S^1 \times R^N, R^1)$ is periodic in $u = (u_1, \dots, u_N)$, and $h \in C(S^1, R^N)$, satisfies $\int_{S^1} h(t)dt = \theta$.

Example 2.3. The case $M = S^1$, $r < N$, with $(a_{ij}(t))_{(N-r)\times(N-r)}$ positive definite. The ODE system was studied by Mawhin [7]. The system

$$\ddot{u} - \tilde{a}(t)u + F_u(t,u) = 0$$

possesses at least $r + 1$ solutions, provided that $F \in C^1(S^1 \times R^N, R^1)$ is periodic in the first $r$ variables $(u_1, \dots, u_r)$, and $\|F_u(t,u)\|_{L^\infty} < \infty$.

Example 2.4. The case $M = T^n$, $r = N = 1$. The problem was studied by Pucci and Serrin [9,10]. The following equation

$$\Delta u + F_u(x,u) = 0 \quad \text{on} \quad T^n$$

possesses at least two solutions, provided that $F \in C^1(T^n \times R^1, R^1)$, and is periodic in $u$.

The Neumann problem for the elliptic equation (in case $r = N = 1$) was studied by Rabinowitz [11].

Remark 2.2. All the above examples deal only with functionals bounded from below, however, Theorem 2 implies more than that. The improvements are in two directions:

(1) The functional is semi-definite, i.e., it is bounded from below except on a finite dimensional subspace.

(2) The resonance case is studied, it only happens when $r < n$.



## §3. Indefinite functionals

In this section, we shall extend the results of §2 to indefinite functionals. The saddle point reduction argument will be applied.

Let $H$ be a Hilbert space, and let $A$ be a self-adjoint operator with domain $D(A) \subset H$ (unbounded). Assume that $F$ is a potential operator with $\Phi \in C^1(H, R^1)$, $F = d\Phi$ and $\Phi(\theta) = 0$. The following assumptions are made

(A) $\exists\ \alpha < 0 < \beta$ such that $\alpha, \beta \notin \sigma(A)$ and $\sigma(A) \cap [\alpha, \beta]$ consists of at most finitely many eigenvalues of finite multiplicities.

(F) $F$ is bounded and Gateaux differentiable, with

$$\|dF(u) - \frac{\alpha + \beta}{2} I\| < \frac{\beta - \alpha}{2}, \qquad \forall u \in H.$$

(D) For small $\varepsilon > 0$, with $-\varepsilon \notin \sigma(A)$, let $V = D(|(\varepsilon I + A)|^{1/2})$, assume that $\Phi \in C^2(V, R^1)$.

Theorem 4. Suppose that

(P) $\exists\ e_1, \dots, e_r \in \ker A$, they are linearly independent, and $\exists\ (T_1, \dots, T_r) \in R^r$, such that

$$\Phi(x + \sum_{j=1}^{r} m_j T_j e_j) = \Phi(x), \quad \forall (m_1, \dots, m_r) \in Z^r, \ \forall x \in H.$$

(LL) $\Phi(x) \to \pm\infty$ if $\|x\| \to \infty$ $\forall x \in \ker A \cap \mathrm{span}\{e_1, \dots, e_r\}^{\perp}$.

Then the equation

$$Ax + \Phi'(x) = 0$$

has at least $r + 1$ distinct solutions.

Proof. A saddle point reduction procedure is applied. Let

$$P_0 = \int_{\alpha}^{\beta} dE_\lambda, \quad P_+ = \int_{\beta}^{+\infty} dE_\lambda, \quad P_- = \int_{-\infty}^{\alpha} dE_\lambda$$

where $\{E_\lambda\}$ is the spectral resolution of $A$, and let

$$H_0 = P_0 H, \quad H_\pm = P_\pm H$$

and for small $\varepsilon > 0$, $-\varepsilon \notin \sigma(A)$, let

$$V_0 = |(\varepsilon I + A)|^{-1/2} H_0, \quad V_\pm = |(\varepsilon I + A)|^{-1/2} H_\pm .$$

For each $u \in H$, we have the decomposition

$$u = u_+ + u_0 + u_-$$

with $u_0 \in H_0$, $u_\pm \in H_\pm$. Let $x = x_+ + x_0 + x_- \in V$, where

$$x_0 = |(\varepsilon I + A)|^{-1/2} u_0, \quad x_\pm = |(\varepsilon I + A)|^{-1/2} u_\pm .$$

We define a function on the finite dimensional space $V_0$ as follows

$$a(z) = \frac{1}{2}(Ax(z), x(z)) + \phi(x(z))$$

where $x(z) = x_+(z) + x_-(z) + z$, $z = x_0 \in V_0$, and $x_\pm(z)$ are the solutions of the equations

$$x_\pm = -(\varepsilon I + A)^{-1} P_\pm(\varepsilon I + F)(x_+ + x_- + z) .$$

We shall prove that

1° $x_\pm(z + \sum_{j=1}^{r} T_j e_j) = x_\pm(z)$, $\quad \forall z \in V_0$.

In fact,

$$P_\pm(\varepsilon I + F)(x_+ + x_- + z + \sum_{j=1}^{r} T_j e_j) = P_\pm(\varepsilon I + F)(x_+ + x_- + z)$$

therefore

$$x_\pm(z) = x_\pm(z + \sum_{j=1}^{r} T_j e_j) .$$

2° $a(z + \sum_{j=1}^{r} T_j e_j) = a(z)$.

Claim:



$$a(z + \sum_{j=1}^{r} T_j e_j) = \frac{1}{2}(Ax(z + \sum_{j=1}^{r} T_j e_j), x(z + \sum_{j=1}^{r} T_j e_j)) + \phi(x(z + \sum_{j=1}^{r} T_j e_j))$$

$$= \frac{1}{2}(Ax(z), x(z) + \sum_{j=1}^{r} T_j e_j) + \phi(x(z) + \sum_{j=1}^{r} T_j e_j)$$

$$= \frac{1}{2}(Ax(z), x(z)) + \phi(x(z))$$

$$= a(z) .$$

3° $a$ satisfies the (PS) condition on $T^r \times (Y \oplus N(A)^\perp) \cap V_0$ where $Y = N(A) \cap \operatorname{span}\{e_1, \dots, e_r\}^\perp$.

Claim: Suppose that $\{z^k\}$ is a sequence along which

$$\{a(z^k)\} \text{ is bounded, and } \|a'(z^k)\| = o(1) .$$

According to Chang [1, p. 105],

$$\|Ax(z^k) + F(x(z^k))\|_H = o(1) .$$

Let $Q$ be the orthogonal projection onto $Y$, which is considered as a subspace of the Hilbert space $H = Y \oplus N(A)^\perp$. Thus on the space $H$,

$$(I - Q)x(z^k) = -A^{-1}(I - Q)F(x(z^k)) + o(1) ,$$

since $F$ is bounded, $\|(I - Q)x(z^k)\|$ is bounded. Noticing

$$\phi(Qx(z^k)) = \phi(x(z^k)) - \int_0^1 (F(x_t(z^k)), (I - Q)x(z^k))dt$$

$$= a(z^k) - \frac{1}{2}(Ax(z^k), x(z^k)) - \int_0^1 (F(x_t(z^k)), (I - Q)x(z^k))dt ,$$

where

$$x_t(z) = ((1 - t)I + tQ)x(z) ,$$

and

$$(Ax(z^k), x(z^k)) = (Ax(z^k), (I - Q)x(z^k)) = (-F(x(z^k)) + o(1), (I - Q)x(z^k)) ,$$

$\phi(Qx(z^k))$ must be bounded. According to the condition (LL), $Qx(z^k)$ is bounded. The compactness of $z^k$ now follows from the boundedness of $x(z^k)$ and the finiteness of the dimension of $V_0$.

4° If we decompose $V_0$ into $\operatorname{span}\{e_1,\dots,e_r\} \oplus (Y \oplus N(A)^{\perp}) \cap V_0$,

$$z = v + w,\ (v,w) \in \operatorname{span}\{e_1,\dots,e_r\} \oplus (Y \oplus N(A)^{\perp}) \cap V_0 ,$$

and let

$$g(w,v) = \frac{1}{2}(A\xi(w+v),\xi(w+v)) + \phi(x(w+v))$$

where

$$\xi(z) = x_+(z) + x_-(z)$$

then $g$ is well defined on $T^r \times (Y \oplus N(A)^{\perp}) \cap V_0$, and

$$dg(w,v) = P_0F(x(w+v))$$

which is bounded and then is compact on finite dimensional manifold. The function $a(z)$ now is written in the following form:

$$a(w,v) = \frac{1}{2}(Aw,w) + g(w,v) .$$

Noticing that $F$ is bounded, $\|\xi(z)\|$ is always bounded. If we denote $y$ the projection of $w$ onto $Y$ we have

$$g(y,v) = \frac{1}{2}(A\xi(y+v),\xi(y+v)) + \phi(y) + [\phi(\xi(y+v)+y+v) - \phi(y)] .$$

The first term and the third term are bounded, therefore

$$g(y,v) \to \pm\infty \quad \text{as} \quad \|y\| \to \infty .$$

The function $a(w,v)$ satisfies all assumptions of Theorem 0. Theorem 4 is proved.

Now we study the periodic solutions of the Hamiltonian systems, in which the Hamiltonian functions are periodic in some of the variables.

We use the following notations: $p,q \in R^N$,

$$p = (p_1,\dots,p_N),\quad q = (q_1,\dots,q_N),\quad 1 < r < s < t < N ,$$

$$\bar p = (p_1,\dots,p_r),\quad \bar q = (q_1,\dots,q_r) ,$$

$$\tilde p = (p_{r+1},\dots,p_S),\quad \tilde q = (q_{r+1},\dots,q_S) ,$$



$$\hat p = (p_{S+1},\dots,p_T),\quad \hat q = (q_{S+1},\dots,q_T) ,$$

$$\check p = (p_{T+1},\dots,p_N),\quad \check q = (q_{T+1},\dots,q_N) .$$

We assume

(I) $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are symmetric continuous matrix function on $S^1$, of order $(S-r)\times(S-r)$, $(T-s)\times(T-s)$, $(N-T)\times(N-T)$ and $(N-T)\times(N-T)$ respectively. Let $\tilde A = \int_{S^1} A(t)$, and $\hat B = \int_{S^1} B(t)$ be invertible.

(II) $\hat H \in C^2(S^1 \times R^{2N}, R^1)$ is periodic in the following variables $\bar p,\bar q,\hat p,\tilde q$, and $\hat H''$ is bounded.

(III) Let $\operatorname{span}\{\varphi_1,\dots,\varphi_m\} = \ker(-\check J \frac{d}{dt} - (C(t) \oplus D(t)))$ where $\check J = \begin{pmatrix} 0 & -I_{N-T} \\ I_{N-T} & 0 \end{pmatrix}$, and $\varphi_1,\dots,\varphi_m$ are linearly independent. And

$$\hat H(t, \sum_{j=1}^{m} \tau_j\varphi_j) \to \pm\infty \quad \text{as} \quad |\tau| = (\tau_1^2 + \dots + \tau_m^2)^{1/2} \to \infty .$$

(IV) $c,d \in C(S^1,R^T)$, with $c = (c_1,\dots,c_T)$, $d = (d_1,\dots,d_T)$ and

$$\int_{S^1} c_i(t) = \int_{S^1} d_j(t) = 0 ,$$

$i = 1,\dots,r,\ s+1,\dots,T,\ j = 1,\dots,s$ .

We define a Hamiltonian function as follows

$$H(t,p,q) = \frac{1}{2}A(t)\tilde p\cdot\tilde p + \frac{1}{2}B(t)\hat q\cdot\hat q + \frac{1}{2}(C(t)\check p\cdot\check p + D(t)\check q\cdot\check q)$$

$$+ \sum_{i=1}^{T}(c_i(t)p_i + d_i(t)q_i) + \hat H(t,p,q) .$$

<u>Theorem 5</u>. Under conditions (I)-(IV), the Hamiltonian system

$$\text{(HS)} \qquad -J\frac{d}{dt}z = H_z(t,z), \qquad t \in S^1$$

has at least $r + T + 1$ periodic solutions, where $z = (p,q) \in R^{2N}$.

**Proof.** Let

$$A(t) = \begin{pmatrix} 0 & & & & & & & \\ & A(t) & & & & & & \\ & & 0 & & & & & \\ & & & C(t) & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & B(t) & \\ & & & & & & & D(t) \end{pmatrix}$$

and let (the subscripts on $J$ coincide with those on $p$)

$$A = (-J\frac{d}{dt} - A(t))$$

$$= \left(-\bar{J}\frac{d}{dt}\right) \oplus \left(-\tilde{J}\frac{d}{dt} - \begin{pmatrix} A(t) & \\ & 0\end{pmatrix}\right) \oplus \left(-\hat{J}\frac{d}{dt} - \begin{pmatrix} 0 & \\ & B(t)\end{pmatrix}\right) \oplus \left(-\check{J}\frac{d}{dt} - \begin{pmatrix} C(t) & \\ & D(t)\end{pmatrix}\right).$$

We have

$$(\tilde{p},\tilde{q}) \in \ker\left(-\tilde{J}\frac{d}{dt} - \begin{pmatrix} A(t) & \\ & 0\end{pmatrix}\right),$$

$$\Longleftrightarrow \begin{cases} \dot{\tilde{q}} = A(t)\tilde{p} \\ \dot{\tilde{p}} = 0 , \end{cases}$$

$$\Longleftrightarrow \begin{cases} \tilde{q} = \int_0^t A(s)ds \cdot \tilde{c} + \tilde{d}, \quad \text{with } \tilde{q}(2\pi) = \tilde{q}(0) , \\ \tilde{p} = \tilde{c} , \end{cases}$$

(i.e., with $\tilde{A} \cdot \tilde{c} = \theta$). According to the assumption I, $\tilde{c} = \theta$. We have

$$\ker\left(-\tilde{J}\frac{d}{dt} - \begin{pmatrix} A(t) & \\ & 0\end{pmatrix}\right) = \{(\theta,\tilde{d}) | \tilde{d} \in R^{S-r}\} \cong R^{S-r} .$$

Similarly,

$$\ker\left(-\hat{J}\frac{d}{dt} - \begin{pmatrix} 0 & \\ & B(t)\end{pmatrix}\right) = \{(\hat{c},\theta) | \hat{c} \in R^{T-s}\} \cong R^{T-s} .$$

Thus

$$\ker(A) = R^{2r} \oplus R^{s-r} \oplus R^{T-s} \oplus \mathrm{span}\{\varphi_1,\dots,\varphi_m\} .$$



Let

$$\Phi(z) = \int_{S^1} \left[\hat{H}(t,z(t)) + \sum_{i=1}^{T} [c_i(t)p_i(t) + d_i(t)q_i(t)]\right]dt .$$

Then all the assumptions (A), (F), (D), (P) and (LL) are satisfied. The proof is complete.

Example 3.1. If the Hamiltonian function $H \in C^2(S^1 \times R^{2N}, R^1)$ is periodic in each variable, then (HS) has at least $2N + 1$ periodic solutions.

This is the case $r = s = T = N$.

This result related to the Arnold conjecture, was first obtained by Conley and Zehnder [3], see also Chang [2].

Example 3.2. If $H \in C^2(S^1 \times R^{2N}, R^1)$, where $H$ is periodic in the components of $q$, and that there is an $R > 0$ such that for $|p| > R$,

$$H(t,p,q) = \frac{1}{2} Mp \cdot p + a \cdot p$$

where $a \in R^N$, and $M$ is a symmetric nonsingular time independent matrix, then the corresponding (HS) possesses at least $N + 1$ distinct periodic solutions.

This is the case $r = 0$, $S = T = N$.

This is a result obtained by Conley and Zehnder [3], see also P. H. Rabinowitz [11].

Example 3.3. Let $H \in C^2(S^1 \times R^4, R^1)$ be periodic in $(p_1,q_1)$. Assume that $\exists R > 0$ such that

$$H(t,p_1,p_2,q_1,q_2) = \frac{1}{2}(cp_2^2 + dq_2^2) \pm A\sqrt{p_2^2 + q_2^2}$$

for $\sqrt{p_2^2 + q_2^2} > R$, where $cd = k^2 > 0$ for some $k \in Z$, and $A > 0$. Then the corresponding (HS) possesses at least 3 periodic solutions.

In fact,

$$\ker\left(-J\frac{d}{dt}-\begin{pmatrix}c & 0\\ 0 & d\end{pmatrix}\right) = \operatorname{span}\{(-\sqrt{\tfrac{d}{c}}\sin kt, \cos kt), (\sqrt{\tfrac{d}{c}}\cos kt, \sin kt)\}$$

it follows

$$\max\{\tfrac{d}{c}, 1\}(\lambda_1^2+\lambda_2^2) \geq \tfrac{d}{c}(-\lambda_1\sin kt + \lambda_2\cos kt)^2 + (\lambda_1\cos kt + \lambda_2\sin kt)^2$$
$$\geq \min\{\tfrac{d}{c}, 1\}(\lambda_1^2+\lambda_2^2) .$$

Therefore

$$\hat{H}(t,0,\sqrt{\tfrac{d}{c}}(-\lambda_1\sin kt + \lambda_2\cos kt),0,(\lambda_1\cos kt + \lambda_2\sin kt))$$
$$\geq \min\{\sqrt{\tfrac{d}{c}},1\}\sqrt{\lambda_1^2+\lambda_2^2} \to +\infty, \quad \text{or} \quad \to -\infty ,$$

as $\sqrt{\lambda_1^2+\lambda_2^2} \to \infty$.

Remark. In the assumption (I), if the operators $\tilde{A}$ and $\hat{B}$ are singular, then

$$(\tilde{p},\tilde{q}) \in \ker\left(-\tilde{J}\frac{d}{dt} - \begin{pmatrix}A(t) & \\ & 0\end{pmatrix}\right) \iff \tilde{p} \in \ker\tilde{A}, \quad \tilde{q} = \int_0^t A(s)ds\,\tilde{p} + \tilde{d} .$$

Thus,

$$\ker\left(-\tilde{J}\frac{d}{dt} - \begin{pmatrix}A(t) & \\ & 0\end{pmatrix}\right) = \{(\theta,\tilde{d})\,|\,\tilde{d}\in R^{s-r}\} \oplus \{(\tilde{c},\int_0^t A(s)ds\,\tilde{c})\,|\,\tilde{c}\in\ker\tilde{A}\}.$$

Similarly,

$$\ker\left(-\hat{J}\frac{d}{dt} - \begin{pmatrix}0 & \\ & B(t)\end{pmatrix}\right) = \{(\hat{c},\theta)\,|\,\hat{c}\in R^{T-S}\} \oplus \{(\int_0^t B(s)ds\hat{d},\hat{d})\,|\,\hat{d}\in\ker\hat{B}\}.$$

In order to apply Theorem 5, the assumption III is replaced by

$$\hat{H}(t,\tilde{c} + \int_0^t A(s)ds\tilde{c} + \int_0^t B(s)ds\hat{d} + \hat{d} + \sum_{j=1}^m \tau_j\varphi_j(t)) \to \pm\infty ,$$

as $|\tilde{c}| + |\hat{d}| + |\tau| \to \infty$, where $\tilde{c}\in\ker\tilde{A}$, $\hat{d}\in\ker\hat{B}$, and $\tau\in R^m$. The same theorem holds.

Example 3.4. Let $H \in C^2(S^1 \times R^4, R^1)$ be periodic in $(p_1,q_1,q_2)$. Assume that $\exists\, R > 0$ such that



$$H(t,p_1,p_2,q_1,q_2) = \frac{1}{2}\cos tp_2^2 \pm A\sqrt{1+p_2^2}$$

for $|p_2| > R$, where $A > 0$ is a constant, then the corresponding (HS) possesses at least 4 periodic solutions.

In fact,

$$\hat{H}(t,p_1,p_2,q_1,q_2) = \pm A\sqrt{1+c^2} \to \pm\infty, \quad \text{as} \quad |c| \to \infty .$$

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