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ON THE NASH POINT EQUILIBRIA IN THE CALCULUS OF VARIATION

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Technical Summary Report #

ABSTRACT

The existence and partial regularity of the Nash point equilibria for a pair of multiple integrals

$$J(u,v) = \int_\Omega F(x,u,v,\nabla u,\nabla v)dx ,$$

$$K(u,v) = \int_\Omega G(x,u,v,\nabla u,\nabla v)dx ,$$

are studied.

The conditions as well as the results are similar to those for local minima obtained by Acerbi, Fusco and Giaquinta, Giusti.

AMS (MOS) Subject Classifications: 49A45, 49A22, 49A50

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§1. Introduction

The problems of the existence and the regularity of the local minima are of fundamental importance in the theory of the calculus of variations. Although there has been a long history, several remarkable contributions appeared in recent years. Among them, I would like to mention two results in these directions.

(1) The existence of a local minimum of the variational integral

$$J(u) = \int_\Omega f(x,u(x),\nabla u(x))dx \quad \text{for} \quad u \in W_0^{1,r}(\Omega,R^N)$$

where $\Omega \subset R^N$ is a bounded open set, $1 < r < \infty$, and N is an integer. The function f is assumed satisfying

I. $f : \Omega \times R^{N(1+n)} \to R^1$, is a Caratheodory function, with the growth condition

$$|f(x,p,P)| \le a(x) + C(|p| + |P|)^r ,$$

where a is nonnegative, and is in $L^1(\Omega)$, and $C > 0$ is a constant.

II. (Coerciveness) $\exists$ constants $c_1, K > 0$ such that $c_1|P|^r - K \le f(x,p,P)$.

Under these conditions, a minimizing sequence exists, and possesses a weakly convergent subsequence. If we know that J is sequentially weakly lower semicontinuous (swlsc in short), then the local minimum exists. The following result due to Acerbi and Fusco [1] gives an answer to the swlsc.

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Theorem 0.1 (Acerbi, Fusco). If f is a nonnegative function satisfying (I) and

III. (Quasiconvexity in the Morrey sense) For a.e. $x \in \Omega$, $\forall p \in R^n$, $\forall O \subset \Omega$, bounded open subset, $\forall w \in C_0^\infty(O,R^N)$

$$f(x,p,P)\,\mathrm{mes}(O) \le \int_O f(x,p,P + \nabla w(y))dy \qquad \forall P \in R^{nN} .$$

Then J is swlsc.

(Actually, (III) is also a necessary condition for swlsc.)

Therefore J possesses a local minimum, if (I), (II) and (III) hold.

(2) Regularity. The following results were obtained by Giaquinta and Giusti [4].

Theorem 0.2. If f satisfies (I) and (II), and if u_0 is a local minimum of J, then $u_0 \in W^{1,s}_{loc}(\Omega,R^N)$ for some $s > r$.

Theorem 0.3. Assume that $\{A^{ij}_{hk}(x,p),\ i,j = 1,\dots,n,\ h,k = 1,\dots,N\}$ are bounded continuous functions in $\Omega \times R^N$, with $A^{ij}_{hk} = A^{ji}_{kh}$, satisfying

$$A^{ij}_{hk}(x,p)\xi_i\xi_j\eta^h\eta^k \ge \lambda|\xi|^2|\eta|^2 \quad \forall(\xi,\eta) \in R^n \times R^N, \quad \lambda > 0 ,$$

and

$$|P|^2 - K \le f(x,p,P) := A^{ij}_{hk}(x,p)P^h_iP^k_j \le C|P|^2 + K ,$$

$\forall P = \{P^h_i\} \in R^{nN}$, for some constants $C, K > 0$. Let $u \in W^{1,2}_{loc}(\Omega,R^N)$ be a local minimum of J, then $\exists$ an open set $\Omega_0 \subset \Omega$ such that $u \in C^{0,\alpha}(\Omega_0,R^N)$ for every $\alpha < 1$. Moreover, the Hausdorff measure $H^{n-q}(\Omega\setminus\Omega_0) = 0$ for some $q > 2$.

The purpose of this paper is to extend all these results to Nash point equilibria for variational integrals.

Let $E_1,E_2,\dots,E_m$ be m sets, and let $f_1,f_2,\dots,f_m : E_1 \times E_2 \cdots \times E_m \to R^1$ be m functions. A point $x = (x_1,x_2,\dots,x_m) \in \prod_{i=1}^m E_i$ is called a Nash point equilibrium, if

$$f_1(x_1,x_2,\dots,x_m) \ge f_1(y_1,x_2,\dots,x_m) ,$$
$$f_2(x_1,x_2,\dots,x_m) \ge f_2(x_1,y_2,\dots,x_m) ,$$
$$\dots \qquad \dots$$
$$f_m(x_1,x_2,\dots,x_m) \ge f_m(x_1,x_2,\dots,y_m)$$

$\forall(y_1,y_2,\dots,y_m) \in \prod_{i=1}^m E_i$.

The concept of Nash point equilibria is a natural extension of the local minima ($m = 1$, $f = -f_1$), and of the saddle points ($m = 2$, $f_1 = -f$, $f_2 = f$). In order to simplify the notations, we only consider $m = 2$.

In the following, we assume both the functions $F,G : \Omega \times R^{(N+M)(1+n)} \to R^1$ satisfying

(I) They are Caratheodory functions, with the growth conditions:

$$|F(x,p,q,P,Q)|,\ |G(x,p,q,P,Q)| \le a(x) + C(|p| + |q| + |P| + |Q|)^r ,$$

$\forall(x,p,q,P,Q) \in \Omega \times R^N \times R^M \times R^{nN} \times R^{nM}$, for $1 < r < \infty$.

And we define

$$J(u,v) = -\int_\Omega F(x,u(x),v(x),\nabla u(x),\nabla v(x))dx ,$$

$$K(u,v) = -\int_\Omega G(x,u(x),v(x),\nabla u(x),\nabla v(x))dx$$

for $(u,v) \in W_0^{1,r}(\Omega,R^{N+M})$. We introduce a new function $H : \Omega \times R^{2(N+M)(1+n)}$ as follows

$$H(x,p,q,P,Q;\ \bar p,\bar q,\bar P,\bar Q) := F(x,p,q,P,Q) + G(x,p,q,P,Q) - F(x,\bar p,q,\bar P,Q) - G(x,p,\bar q,P,\bar Q) .$$

Some assumptions similar to those for local minima are made:

(II) (Coerciveness) $\exists$ constants $C_1,C_2,C_3 > 0$ and some $0 < \hat r < r$, such that

$$H(x,p,q,P,Q;\ \bar p,\bar q,\bar P,\bar Q) \ge C_1(|P| + |Q|)^r - C_3(|\bar P| + |\bar Q|)^r - C_2(1 + |p| + |q| + |\bar p| + |\bar q|)^{\hat r} .$$

(III) (Quasi-convexity in the Morrey sense) For a.e. $x \in \Omega$, and $\forall$ $(p,q,\bar{p},\bar{q},\bar{P},\bar{Q}) \in \mathbb{R}^{(2+n)(N+M)}$, the function $(P,Q) \longmapsto H(x,p,q,P,Q;\bar{p},\bar{q},\bar{P},\bar{Q})$ is quasi-convex in the Morrey sense.

The main results in this paper read as follows.

<u>Theorem I.</u> In addition to assumptions (I), (II) and (III), we assume

(IV) The functions

$$(p,P) \longmapsto F(x,p,q,P,Q) \qquad \forall (x,q,Q) \in \Omega \times \mathbb{R}^{(1+n)M},$$

$$(q,Q) \longmapsto G(x,p,q,P,Q) \qquad \forall (x,p,P) \in \Omega \times \mathbb{R}^{(1+n)N}$$

are convex.

The functional pair (J,K) possesses a Nash point equilibrium $(u_0,v_0) \in W_0^{1,r}(\Omega,\mathbb{R}^{N+M})$.

<u>Theorem II.</u> Under the assumptions (I) and (II), if (u_0,v_0) is a Nash point equilibrium of (J,K), then $(u_0,v_0) \in W_{loc}^{1,s}(\Omega,\mathbb{R}^{N+M})$ for some $s > r$.

The partial regularity result is also extended. We restrict ourselves to perturbed quadratic functionals. Let

$$F = A_{hk}^{ij}(x,p,q)P_i^hP_j^k + B_{h\ell}^{ij}(x,p,q)P_i^hQ_j^\ell + C_{\ell m}^{ij}(x,p,q)Q_i^\ell Q_j^m + f(x,p,q,P,Q),$$

$$G = a_{hk}^{ij}(x,p,q)P_i^hP_j^k + b_{h\ell}^{ij}(x,p,q)P_i^hQ_j^\ell + c_{\ell m}^{ij}(x,p,q)Q_i^\ell Q_j^m + g(x,p,q,P,Q),$$

where A_{hk}^{ij}, $B_{h\ell}^{ij}$, $C_{\ell m}^{ij}$, a_{hk}^{ij}, $b_{h\ell}^{ij}$, $c_{\ell m}^{ij}$ $(i,j = 1,\dots,n,\ h,k = 1,\dots,N,\ \ell,m = 1,\dots,M)$, are bounded and uniformly continuous functions in $\Omega \times \mathbb{R}^{N+M}$, and f, g are Caratheodory functions.

We assume

(a) $A_{hk}^{ij} = A_{kh}^{ji}$, $a_{hk}^{ij} = a_{kh}^{ji}$, $C_{\ell m}^{ij} = C_{m\ell}^{ji}$, $c_{\ell m}^{ij} = c_{m\ell}^{ji}$.

(b) a is independent of q, and C is independent of p.

(c) $\exists\ \lambda > 0$ such that

$$\left.\begin{array}{l} A_{hk}^{ij}(x,p,q)P_i^hP_j^h \geq \lambda|P|^2 \\ C_{\ell m}^{ij}(x,p,q)Q_i^\ell Q_j^m \geq \lambda|Q|^2 \end{array}\right. \qquad \forall (P,Q) \in \mathbb{R}^{n(N+M)},$$

and $\exists\ \lambda' < 2\lambda$ such that

$$\|B_{h\ell}^{ij}(x,p,q)\|_{L^\infty},\ \|b_{h\ell}^{ij}(x,p,q)\|_{L^\infty} < \lambda', \qquad \forall i,j,h,\ell .$$

(d) $\exists\ 0 < r < 2,\ 2 < r + s < 2 + \frac{2s}{n}$ such that

$$|f|,|g| \leq c_4[(|p| + |q|)^{2n/n-2} + (|p| + |q|)^s(|P| + |Q|)^r] .$$

<u>Theorem III.</u> Under the above assumptions (a)-(d), let $(u,v) \in W_{loc}^{1,2}(\Omega,\mathbb{R}^{N+M})$ be a Nash point equilibrium of the pair (J,K). Then $\exists$ an open set $\Omega_0 \subset \Omega$ such that $(u,v) \in C^{0,\alpha}(\Omega_0,\mathbb{R}^{N+M})$ for every $\alpha < 1$. Moreover, $H^{n-q}(\Omega\backslash\Omega_0) = 0$ for some $q > 2$.

The same problems have been studied by Bensoussan and Frehse [2]. In their work, the differentiability of the functions F and G with respect to (p,q,P,Q), as well as the growth conditions in these derivatives are assumed. However, these kinds of assumptions are not natural in the theory of calculus of variations.

The main difference from theirs, is an assumption of the quasiconvexity of the function H (Assumption III), which replaces an ellipticity condition on (F,G) given in their paper. The advantage of this approach are twofold: (1) Only weakly sequential convergence rather than the strong convergence is used. It makes the argument clearer and more direct. (2) The functions F and G are no more quadratic, we may apply our theorems to a large class of functions.

A different approach, which improves [2] as well, is given by Zhang Ke-Wei [7].

The proofs of Theorems I, II and III are given in §2. In the third section, we present some examples, the first one, in some sense, is a comparison with Bensoussan and Frehse [2]. The second one compares with a study of saddle points due to the author [3]. And the third provides a new example.

§2. The proofs

Firstly, we modify the Ky Fan inequality to noncompact convex sets. It is the abstract framework of the existence proof.

<u>Lemma.</u> Let X be a closed convex set of a separable Banach space E; and let

$$\varphi : X \times X \to R^1$$

be a function satisfying the following conditions:

(1) $\forall y \in X$, $x \longmapsto \varphi(x,y)$ is swlsc,

(2) $\forall x \in X$, $y \longmapsto \varphi(x,y)$ is quasiconcave, and is lsc (in the strong topology).

(3) $\exists\, y_0 \in X$ such that the function $x \longmapsto \varphi(x,y_0)$ is coercive, i.e. $\varphi(x,y_0) \to +\infty$ as $\|x\| \to \infty$.

(4) $\varphi(x,x) \le 0 \quad \forall x \in X$.

Then there exists $x_0 \in X$ such that

$$\varphi(x_0,y) \le 0 \quad \forall y \in X .$$

<u>Proof 1°.</u> We consider a sequence of finite dimensional linear subspaces of E:

$$L_1 \subset L_2 \subset \cdots \subset L_n \subset \cdots$$

such that $\overline{\bigcup_{n=1}^{\infty} L_n} = E$ and $y_0 \in L_{k_0}$ for some k_0.

On each L_n, $n > k_0$, we define

$$\varphi_n = \varphi|_{X_n \times X_n}$$

where $X_n = L_n \cap X$. Then we have

(1) $\forall z \in X_n$, $w \longmapsto \varphi_n(w,z)$ is lsc.

(2) $\forall w \in X_n$, $z \longmapsto \varphi_n(w,z)$ is quasiconcave.

(3) $\varphi_n(w,y_0) \to +\infty$ as $\|w\|_{L_n} \to \infty$.

(4) $\varphi_n(w,w) = 0 \quad \forall w \in X_n$.

According to Ky Fan Minimax inequality, we obtain $w_n \in X_n$ such that

(5) $\varphi(w_n,z) \le 0 \quad \forall z \in X_n$.

2°. Let us define

$$K := \{x \in X \,|\, \varphi(x,y_0) \le 0\} .$$

Provided by the assumptions (1) and (3), K is bounded and sequentially closed, so it is sequentially weakly compact.

According to (5), one has $\{w_n\}_{n \ge k_0} \subset K$. This implies a subsequence $w_{n_j} \rightharpoonup x_0 \in K$.

Again by the assumption (1), we have

$$\varphi(x_0,z) \le 0 \quad \forall z \in \bigcup_{n=1}^{\infty} X_n .$$

However, the function $z \longmapsto \varphi(x_0,z)$ is assumed lower semicontinuous and since $\overline{\bigcup_{n=1}^{\infty} X_n} = X$ (in the strong topology)

$$\therefore \varphi(x_0,y) \le 0 \quad \forall y \in X .$$

<u>The proof of Theorem I.</u> Define a Banach space $E = W_0^{1,r}(\Omega,R^{N+M})$, and denote $\xi = (u,v) \in E$, where $u \in W_0^{1,r}(\Omega,R^N)$ and $v \in W_0^{1,r}(\Omega,R^M)$. We define a function $\varphi : X \times X \to R^1$ as follows:

$$\varphi(\xi,\eta) = \int_\Omega H(x,u(x),v(x),\nabla u(x),\nabla v(x);\ \bar u(x),\bar v(x),\nabla \bar u(x),\nabla \bar v(x))dx$$

where $\xi = (u,v)$, $\eta = (\bar u,\bar v) \in X$. The functional φ is continuous (strongly) in $E \times E$, and that

$$\forall \xi \in X, \quad \eta \longmapsto \varphi(\xi,\eta) \text{ is concave .}$$

These follow from the assumptions I and IV respectively.

Provided by the assumption II,

$$\varphi(\xi,\theta) = \int_\Omega H(x,u,v,\nabla u,\nabla v;\ 0,0,0,0)dx$$

$$\ge C_1 \int_\Omega (|\nabla u| + |\nabla v|)^r dx - C_2 \int_\Omega (|u| + |v| + 1)^{\hat r} dx \to +\infty$$

as $|\xi| \to \infty$. And obviously we have

$$\varphi(\xi,\xi) = 0 .$$

Therefore, in order to apply the lemma, we only want to verify the swlsc of the functionals: $\forall\eta = (\bar{u},\bar{v})$, $\xi = (u,v) \longmapsto \varphi(\xi,\eta)$, i.e. $\forall(\bar{u},\bar{v}) \in E$, the swlsc of the functional

$$(u,v) \longmapsto \int_\Omega H(x,u,v,\nabla u,\nabla v;\ \bar{u},\bar{v},\nabla\bar{u},\nabla\bar{v})dx .$$

Let

$$g(x,p,q) = C_2(1 + |p| + |q| + |\bar{u}(x)| + |\bar{v}(x)|)^{\hat{r}} + C_3(|\nabla\bar{u}(x)|^r + |\nabla\bar{v}(x)|^r)$$

and let

$$f(x,p,q,P,Q) = H(x,p,q,P,Q,\bar{u}(x),\bar{v}(x),\nabla\bar{u}(x),\nabla\bar{v}(x)) + g(x,p,q) .$$

The function f is quasiconvex in the Morrey sense with respect to (P,Q). Thanks to the theorem due to Acerbi and Fusco, we have

$$\lim_{k\to\infty} \int_\Omega f(x,u_k(x),v_k(x),\nabla u_k(x),\nabla v_k(x))dx$$

$$\geq \int_\Omega f(x,u(x),v(x),\nabla u(x),\nabla v(x))dx$$

as $\xi_k = (u_k v_k) \rightharpoonup (u,v)$ weakly in E. Provided by the Sobolev embedding theorem together with the continuity of the Nemytcki operator, we obtain

$$\lim_{k\to\infty} \int_\Omega g(x,u_k(x),v_k(x))dx = \int_\Omega g(x,u(x),v(x))dx .$$

It follows

$$\varphi(\xi_k,\eta) \to \varphi(\xi,\eta) \quad \text{as} \quad \xi_k \rightharpoonup \xi \quad \text{in} \quad E .$$

Therefore the lemma is applied, we have $\xi_0 = (u_0,v_0) \in E$ such that

$$\varphi(\xi_0,\eta) = \int_\Omega H(x,u_0,v_0,\nabla u_0,\nabla v_0;\ \bar{u},\bar{v},\nabla\bar{u},\nabla\bar{v})dx \leq 0$$

$\forall\eta = (\bar{u},\bar{v}) \in E$. The last inequality is equivalent to

$$J(u_0,v_0) \geq J(\bar{u},v_0), \qquad K(u_0,v_0) \geq K(u_0,\bar{v}) \qquad \forall(\bar{u},\bar{v}) \in E .$$

The theorem is proved.

<u>The proof of Theorem II.</u> Since the Nash point equilibrium $(u_0,v_0) \in E$ satisfies the inequality

$$\int_\Omega H(x,u_0(x),v_0(x),\nabla u_0(x),\nabla v_0(x);\ u(x),v(x),\nabla u(x),\nabla v(x))dx \leq 0 .$$

$\forall(u,v) \in W_0^{1,r}(\Omega,R^{N+M})$. The coerciveness assumption II implies

$$C_1 \int_\Omega (|\nabla u_0|^r + |\nabla v_0|^r)dx \leq C_2 \int_\Omega (|\nabla u|^r + |\nabla v|^r)dx$$

$$+ C_3 \int_\Omega (1 + |u| + |v| + |u_0| + |v_0|)^{\hat{r}}dx$$

$\forall(u,v) \in W_0^{1,r}(\Omega,R^{N+M})$. According to the Poincaré inequality, (u_0,v_0) turns out to be a Q-minimum of the generalized Dirichlet integral

$$\int_\Omega (1 + |\nabla u_0|^r + |\nabla v_0|^r)dx \leq Q \int_\Omega (1 + |\nabla u|^r + |\nabla v|^r)dx$$

for suitable $Q > 0$. A result due to Giaquinta and Giusti [4,5] is applied, see also Chang [3].

<u>Remark.</u> Theorems I and II can be generalized to the case where F and G are inhomogeneous with respect to P and Q. Namely, say F, G are r-power growth in P, and t-power growth in Q. Under suitably modified assumptions I and II, these two theorems hold as well. The proof of Theorem I is similar. As to Theorem II, we refer to Chang [3].

<u>The proof of Theorem III.</u> We use the following conventional notations

$A,B,\ldots,C$ stand for bilinear forms $A_{hk}^{ij}(x,p,q)$,

$B_{h\ell}^{ij}(x,p,q),\ldots C_{\ell m}^{ij}(x,p,q)$ etc.

$\bar{A},\bar{B},\ldots,\bar{C}$ stand for bilinear forms $A_{hk}^{ij}(x,\bar{p},q)$,

$B_{h\ell}^{ij}(x,\bar{p},q),\ldots C_{\ell m}^{ij}(x,\bar{p},q)$ etc.

$\hat{A},\hat{B},\dots,\hat{C}$ stand for bilinear forms $A_{hk}^{ij}(x,p,\tilde{q})$,

$B_{h\ell}^{ij}(x,p,\tilde{q}),\dots C_{\ell m}^{ij}(x,p,\tilde{q})$ etc.

$P = (P_i^h)$, $Q = (Q_i^\ell)$,

$AP \cdot P = A_{hk}^{ij}(x,p,q)P_i^h P_j^k, \dots\dots$ etc.

$u_{x_0,R} = \fint_{B_R(x_0)} u(x)dx, \quad v_{x_0,R} = \fint_{B_R(x_0)} v(x)$,

where $B_R(x_0)$ is the ball with radius $R > 0$ and center x_0, and $\fint$ stands for the mean value.

Let us denote $\omega : R^+ \to R^+$, the continuity modulus of the functions $A,B,\dots,C$. It is increasing, concave, continuous and satisfies $\omega(0) = 0$, $\omega(t) < M$, a const., and

$|A(x,p,q) - A(x',p',q')|, |B(x,p,q) - B(x',p',q')|,\dots|C(x,p,q) - C(x',p',q')|$

$\le \omega(|x - x'|^2 + |p - p'|^2 + |q - q'|^2)$.

Assume that (u,v) is a Nash point equilibrium of the function pair (J,K). The proof is based on the following estimate

$$\int_{B_\rho(x_0)} (1 + |\nabla u|^2 + |\nabla v|^2)$$

$$\le C\Big[\left(\tfrac{\rho}{R}\right)^n + \omega\Big(R^2 + C_1 R^{2-n}\int_{B_R(x_0)} (|\nabla u|^2 + |\nabla v|^2)\Big)^{1-2/s}$$

$$+ C_2\Big(\int_{B_R(x_0)} |\nabla u|^2 + |\nabla v|^2\Big)^{2/n-2}\Big]\int_{B_{2R}(x_0)} (1 + |\nabla u|^2 + |\nabla v|^2)$$

$$+ C_3 R^n(|u_{x_0,R}| + |v_{x_0,R}|)^{2n/n-2} + C_4 R^n \tag{1}$$

$\forall x_0 \in \Omega$, $\forall\, 0 < \rho < R < \frac{1}{2}\,\mathrm{dist}(x_0,\partial\Omega)$, where $C,C_1,C_2,C_3,C_4 > 0$ are constants, and $s > 2$ is a suitable constant. Once it is established, the conclusion follows directly from Giaquinta and Giusti, cf. Giaquinta [6, Thm. 1.1, Ch. VI].

Let $\dot{A},\dot{B},\dots,\dot{C}$, denote the constant coefficient bilinear forms $A_{hk}^{ij}(x_0,u_{x_0,R},v_{x_0,R})$, $B_{h\ell}^{ij}(x_0,u_{x_0,R},v_{x_0,R}),\dots,C_{\ell m}^{ij}(x_0,u_{x_0,R},v_{x_0,R})$ etc.

Let $(\bar{u},\bar{v})$ be the solutions of the following constant coefficient elliptic boundary value problems

$$\int_{B_R(x_0)} \dot{A}\nabla\bar{u} \cdot \nabla\varphi + \frac{1}{2}\dot{B}\nabla\varphi \cdot \nabla v = 0, \quad \forall\varphi \in W_0^{1,2}(B_R(x_0),R^N)$$

and

$$\int_{B_R(x_0)} \dot{C}\nabla\bar{v} \cdot \nabla\psi + \frac{1}{2}\dot{b}\nabla u \cdot \nabla\psi = 0 \quad \forall\psi \in W_0^{1,2}(B_R(x_0),R^M)$$

with $\bar{u}|_{\partial B_R(x_0)} = u|_{\partial B_R(x_0)}$, $\bar{v}|_{\partial B_R(x_0)} = v|_{\partial B_R(x_0)}$. And let $(u_0,v_0) \in W_0^{1,2}(B_R(x_0),R^{N+M})$ be the solutions of the following equations

$$\int_{B_R(x_0)} \dot{A}\nabla u_0 \cdot \nabla\varphi - \frac{1}{2}\dot{B}\nabla\varphi \cdot \nabla v = \int_{B_R(x_0)} \dot{C}\nabla v_0 \cdot \nabla\psi - \frac{1}{2}\dot{b}\nabla u \cdot \nabla\psi = 0$$

$\forall\varphi \in W_0^{1,2}(B_R(x_0),R^N)$, $\forall\psi \in W_0^{1,2}(B_R(x_0),R^M)$, we have

$$\int_{B_R(x_0)} \dot{A}\nabla(\bar{u} + u_0) \cdot \nabla\varphi = 0, \qquad \int_{B_R(x_0)} \dot{C}\nabla(\bar{v} + v_0) \cdot \nabla\psi = 0 ,$$

$\forall\varphi \in W_0^{1,2}(B_R(x_0),R^N)$, $\forall\psi \in W_0^{1,2}(B_R(x_0),R^M)$.

According to condition (c), the Caciappolli inequality and the L^p theory of elliptic systems imply that $\exists$ constant C_5 and $C_p > 0$ such that for $0 < \rho < R$,

$$\int_{B_\rho(x_0)} (|\nabla(\bar{u} + u_0)|^2 + |\nabla(\bar{v} + v_0)|^2)$$

$$\le C_5\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} (|\nabla(\bar{u} + u_0)|^2 + |\nabla(\bar{v} + v_0)|^2)$$

and $\forall\ 1 < p < \infty$,

$$\int_{B_R(x_0)} (|\nabla(\tilde{u} + u_0)|^p + |\nabla(\bar{v} + v_0)|^p)$$

$$< C_p \int_{B_R(x_0)} (|\nabla(u + u_0)|^p + |\nabla(v + v_0)|^p) .$$

(Cf. [6, Ch VI, p. 206-210].)

These inequalities imply that

$$\int_{B_\rho(x_0)} (|\nabla\tilde{u}|^2 + |\nabla\bar{v}|^2)$$

$$< C_\varepsilon (\tfrac{\rho}{R})^n \int_{B_R(x_0)} (|\nabla u|^2 + |\nabla v|^2) + (1 + \varepsilon) \int_{B_R(x_0)} (|\nabla u_0|^2 + |\nabla v_0|^2) \qquad (2)$$

and

$$\int_{B_R(x_0)} (|\nabla\tilde{u}|^p + |\nabla\bar{v}|^p) < C_p \int_{B_R(x_0)} (|\nabla u|^p + |\nabla v|^p) \qquad (3)$$

where $\varepsilon > 0$ is an arbitrary positive number, and C_ε is a constant depending on ε.

Let $w = u - \tilde{u}$, $z = v - \bar{v}$. According to the Plancherel identity and because $(\tilde{u}, \bar{v})$ are the solutions of the above BVPs, we obtain

$$\lambda \int_{B_R(x_0)} (|\nabla w|^2 + |\nabla z|^2) < \int_{B_R(x_0)} \dot{A}\nabla w \cdot \nabla w + \dot{c}\nabla z \cdot \nabla z$$

$$= \int_{B_R(x_0)} \dot{A}\nabla(u+\tilde{u}) \cdot \nabla(u-\tilde{u}) + \dot{B}\nabla w \cdot \nabla v + \dot{b}\nabla u \cdot \nabla z + \dot{c}\nabla(v+\bar{v})\nabla(v-\bar{v})$$

$$= \int_{B_R(x_0)} (\dot{A}-\bar{A})(\nabla u\cdot\nabla u-\nabla\tilde{u}\cdot\nabla\tilde{u}) + (\bar{B}-\dot{B})\nabla\tilde{u} \cdot \nabla v + (\hat{b}-\dot{b})\nabla u \cdot \nabla\bar{v}$$

$$+ (\dot{c}-\hat{c})(\nabla v\cdot\nabla v-\nabla\bar{v}\cdot\nabla\bar{v})$$

$$+ \int_{B_R(x_0)} (\bar{A}-A)\nabla u \cdot \nabla u + (\dot{B}-B)\nabla u \cdot \nabla v + (\dot{b}-b)\nabla u \cdot \nabla v + (\hat{c}-c)\nabla v \cdot \nabla v$$

$$+ \int_{B_R(x_0)} A\nabla u \cdot \nabla u + (B+b)\nabla u \cdot \nabla v + c\nabla v \cdot \nabla v$$

$$- \int_{B_R(x_0)} \bar{A}\nabla\tilde{u} \cdot \nabla\tilde{u} + \bar{B}\nabla\tilde{u} \cdot \nabla v + \hat{b}\nabla u\, \nabla\bar{v} + \hat{c}\nabla\bar{v} \cdot \nabla\bar{v}$$

$$= I_1 + I_2 + I_3 + I_4 . \qquad (4)$$

Noticing

$$I_1 + I_2 < C_6 \int_{B_R(x_0)} \omega(R^2 + 2[|u - u_{x_0,R}|^2 + |v - v_{x_0,R}|^2 +$$

$$+ |u - \tilde{u}|^2 + |v - \bar{v}|^2])[|\nabla u|^2 + |\nabla v|^2 + |\nabla\tilde{u}|^2 + |\nabla\bar{v}|^2]$$

and that ω is bounded, we have for some $s > 2$

$$\int_{B_R(x_0)} \omega \cdot (|\nabla u|^2 + |\nabla v|^2) < C_7[\int_{B_R(x_0)} |\nabla u|^s + |\nabla v|^s]^{2/s}[\int_{B_R(x_0)} \omega]^{1-\frac{2}{s}}$$

$$< C_8 \int_{B_{2R}(x_0)} (1 + |\nabla u|^2 + |\nabla u|^2) \cdot [\fint_{B_R(x_0)} \omega]^{1-\frac{2}{s}}$$

by Theorem II. By the concavity of ω, we have

$$\fint_{B_R(x_0)} \omega(R^2 + 2(|u - u_{x_0,R}|^2 + |v - v_{x_0,R}|^2 + |w|^2 + |z|^2])$$

$$< \omega(R^2 + 2 \fint_{B_R(x_0)} (|u - u_{x_0,R}|^2 + |v - v_{x_0,R}|^2) + 2 \fint_{B_R(x_0)} (|w|^2 + |z|^2))$$

$$< \omega(R^2 + C_9 R^{2-n} \int_{B_R(x_0)} (|\nabla u|^2 + |\nabla u|^2)) .$$

The last inequality follows from the Poincaré inequality and the inequality (3). Similarly, we have

$$\int_{B_R(x_0)} \omega\cdot(|\nabla\tilde u|^2+|\nabla\tilde v|^2) \le c_{10}\Big[\int_{B_R(x_0)} (|\nabla\tilde u|^s+|\nabla\tilde v|^s)\Big]^{\frac{2}{s}}\Big[\int_{B_R(x_0)}\omega\Big]^{1-\frac{2}{s}}$$

$$\le c_{11}\Big[\int_{B_R(x_0)} (|\nabla u|^s+|\nabla v|^s)\Big]^{\frac{2}{s}}\Big[\int_{B_R(x_0)}\omega\Big]^{1-\frac{2}{s}}$$

$$\le c_{12}\int_{B_{2R}(x_0)}(1+|\nabla u|^2+|\nabla v|^2)\Big[⨍_{B_R(x_0)}\omega\Big]^{1-\frac{2}{s}}.$$

It follows

$$I_1+I_2 \le c_{13}\omega\Big(R^2+c_9R^{2-n}\int_{B_R(x_0)}(|\nabla u|^2+|\nabla v|^2)\Big)^{1-\frac{2}{s}}\int_{B_{2R}(x_0)}(1+|\nabla u|^2+|\nabla v|^2).$$

If we write the integrand of the summation I_3+I_4 in the form

$$H(x,u,v,\dots,\nabla\tilde u,\nabla\tilde v)+h(x,u,v,\dots,\nabla\tilde u,\nabla\tilde v),$$

then

$$|h(x,p,q,\dots,\tilde P,\tilde Q)| \le c_{14}\Big[(|p|+|q|+|\tilde p|+|\tilde q|)^{\frac{2n}{n-2}} + (|p|+|q|+|\tilde p|+|\tilde q|)^s(|P|+|Q|+|\tilde P|+|\tilde Q|)^r\Big]$$

where $0<r<2$, $2<r+s<2+\frac{2s}{n}$, provided by the assumption (d).

However, (u,v) is a Nash point equilibrium, we have

$$\int_{B_R(x_0)} H(x,u,v,\nabla u,\nabla v;\ \tilde u,\tilde v,\nabla\tilde u,\nabla\tilde v)\le 0.$$

(By extending $(\tilde u,\tilde v)=(u,v)$ outside the ball $B_R(x_0)$.) It follows

$$I_3+I_4\le c_{15}\int_{B_R(x_0)}\Big[(|u|+|v|+|\tilde u|+|\tilde v|)^{\frac{2n}{n-2}} + (|u|+|v|+|\tilde u|+|\tilde v|)^s(|\nabla u|+|\nabla v|+|\nabla\tilde u|+|\nabla\tilde v|)^r\Big]$$

$$\le c_{16}\Big(\int_{B_R(x_0)}|\nabla u|^2+|\nabla v|^2\Big)^{\frac{2n}{n-2}} + c_{17}R^n(|u_{x_0,R}|+|v_{x_0,R}|)^{\frac{2n}{n-2}}+c_{18}R^n$$

cf. [3, Thm. 3]. In summary

$$\int_{B_R(x_0)}(|\nabla w|^2+|\nabla z|^2)\le c_{19}\Big[\int_{B_{2R}(x_0)}(1+|\nabla u|^2+|\nabla v|^2)\Big\{\omega\Big(R^2 + c_9R^{2-n}\int_{B_R(x_0)}(|\nabla u|^2+|\nabla v|^2)\Big)^{1-\frac{2}{s}} + \Big(\int_{B_R}(|\nabla u|^2+|\nabla v|^2)\Big)^{\frac{2}{n-2}}\Big\} + R^n(|u_{x_0,R}|+|v_{x_0,R}|)^{\frac{2n}{n-2}}+R^n\Big]. \quad (5)$$

Using the relation

$$\int_{B_\rho(x_0)}(|\nabla u|^2+|\nabla v|^2)\le(1+\varepsilon)\int_{B_\rho(x_0)}(|\nabla\tilde u|^2+|\nabla\tilde v|^2) + C_\varepsilon\int_{B_\rho(x_0)}(|\nabla w|^2+|\nabla z|^2), \quad (6)$$

and since $(u_0,v_0)\in W_0^{1,2}(B_r(x_0),R^{n+m})$ satisfying

$$\int_{B_R(x_0)}\dot A\nabla u_0\nabla\varphi-\frac12\dot B\nabla\varphi\cdot\nabla v=0,\qquad \int_{B_R(x_0)}\dot C\nabla v_0\cdot\nabla\psi-\frac12\dot b\nabla u\cdot\nabla\psi=0.$$

One sees that from condition (d)

$$\lambda\int_{B_R(x_0)}|\nabla u_0|^2\le\frac12\lambda'\int_{B_R(x_0)}|\nabla v|^2 \quad\text{and}$$

$$\lambda\int_{B_R(x_0)}|\nabla v_0|^2\le\frac12\lambda'\int_{B_R(x_0)}|\nabla u|^2. \quad (7)$$

Now we substitute (2) and (7) into (6), it follows

$$\int_{B_\rho(x_0)}(|\nabla u|^2+|\nabla v|^2)\le C'_\varepsilon\Big(\frac{\rho}{R}\Big)^n\int_{B_R(x_0)}(|\nabla u|^2+|\nabla v|^2) + (1+\varepsilon)^2\int_{B_\rho(x_0)}(|\nabla u|^2+|\nabla v|^2)+C_\varepsilon\int_{B_R(x_0)}(|\nabla w|^2+|\nabla z|^2).$$

Choosing $\varepsilon>0$ sufficiently small, (1) is obtained by (5).

The rest of the proof is essentially the same as Theorem 1.1, in [6, Ch. IV] and Theorems 1.1, 1.2 in [6, Ch. VI].

§3. Examples

In this section, we shall present several examples to illustrate the conditions stated in the theorems.

<u>Example 1.</u> Suppose that $A^{ij}_{hk}, B^{ij}_{hk}, \ldots\ldots, C^{ij}_{\ell m}$ etc. are bounded Caratheodory functions defined on $\Omega \times R^{N+M}$.

Let

$$\tilde{C} = C(x,p,q) - C(x,\bar{p},q),$$

$$\tilde{a} = a(x,p,q) - a(x,p,\bar{q})$$

and let

$$Q = (A + \tilde{a})P \cdot P + (B + b)P \cdot Q + (\tilde{C} + c)Q \cdot Q$$

where we use the abbreviation notations as in §2 Theorem II.

Assume that Q is positive definite in (P,Q) $\forall(x,p,q,\bar{p},\bar{q}) \in \Omega \times R^{2(N+M)}$, i.e. $\exists\ \lambda > 0$ such that

$$Q > \lambda(|P|^2 + |Q|^2) \ . \tag{3.1}$$

Suppose that $f,g : \Omega \times R^{(1+n)(N+M)} \to R^1$ are Caratheodory functions which are linear in (P,Q), and satisfy the growth condition

$$|f|,|g| < (1 + |p|^{\hat{r}} + |q|^{\hat{r}} + |P| + |Q|), \quad \hat{r} < 2 \ . \tag{3.2}$$

Then the functions

$$F(x,p,q,P,Q) = AP \cdot P + BP \cdot Q + CQ \cdot Q + f(x,p,q,P,Q) \ ,$$

$$G(x,p,q,P,Q) = aP \cdot P + bP \cdot Q + cQ \cdot Q + g(x,p,q,P,Q)$$

satisfy the assumptions I, II, III of Theorem I with $r = 2$.

We verify the assumption II

$$H = (A + \tilde{a})P \cdot P + (B + b)P \cdot Q + (c + \tilde{C})Q \cdot Q$$
$$+ f + g - \bar{f} - \hat{g} - (\bar{A}P \cdot P + \bar{B}\bar{P} \cdot Q + \hat{b}P \cdot \bar{Q} + \hat{c}\bar{Q} \cdot \bar{Q})$$

where we use the abbreviation notations of §2 Theorem III. Thus

$$H \geq \lambda(|P|^2 + |Q|^2) - M(|\bar{P}|^2 + |\bar{Q}|^2) - 2C(1 + |p|^{\hat{r}} + |q|^{\hat{r}} + |\bar{p}|^{\hat{r}} + |\bar{q}|^{\hat{r}})$$

$$- \frac{\lambda}{4} |Q|^2 - \frac{M}{4\lambda} |\bar{P}|^2 - \frac{\lambda}{4} |P|^2 - \frac{M}{4\lambda} |\bar{Q}|^2 - \frac{\lambda}{4} (|Q|^2 + |P|^2) - \frac{C^2}{4\lambda} (|\bar{P}|^2 + |\bar{Q}|^2)$$

$$\geq \frac{\lambda}{2} (|P|^2 + |Q|^2) - M_1(|\bar{P}|^2 + |\bar{Q}|^2) - C_1(|p| + |q| + |\bar{p}| + |\bar{q}| + 1)^{\hat{r}} .$$

As to assumption III, we observe that the function

$$(P,Q) \longmapsto H(x,p,q,P,Q; \bar{p},\bar{q}, \bar{P},\bar{Q}) = Q(x,p,q,\bar{p},\bar{q}; P,Q) + \text{linear terms of } (P,Q)$$

is convex. So that is quasiconvex in the Morrey sense.

Furthermore, we assume that the assumption IV hold, i.e. the function $(p,P) \longmapsto F(x,p,\ldots,Q)$ and $(q,Q) \longmapsto G(x,p,\ldots,Q)$ are convex.

Then Theorems I and II hold true for this pair (F,G).

<u>Remark</u>. This is just the example given by Bensoussan and Frehse in [2]. Although the abstract assumptions of theirs are different from ours, this example is a common model. An obvious advantage in this paper is that neither the differentiable conditions nor the growth conditions of the differentials of the functions F, G are needed.

<u>Example 2</u>. Suppose that F is a function satisfying the assumption I and

Assumption II'. $\exists$ constants $r > \hat{r} > 0$ and $C, C_1, C_2 > 0$ such that

$$F(x,p,q,P,Q) \geq C|P|^r - C_1|Q|^r - C_2(|p| + |q| + 1)^{\hat{r}} ,$$

$$-F(x,p,q,P,Q) \geq C|Q|^r - C_1|P|^r - C_2(|p| + |q| + 1)^{\hat{r}} .$$

Assumption IV'. $\forall (x,p,P) \in \Omega \times R^{(1+n)N}$, $(q,Q) \longmapsto -F(x,p,q,P,Q)$, and $\forall (x,q,Q) \in \Omega \times R^{(1+n)M}$, $(p,P) \longmapsto F(x,p,q,P,Q)$ are convex functions.

Then the function pair $(F,-F)$ satisfies the assumptions I-IV. In fact, Assumptions I and IV are obviously true. And

$$H(x,p,q,P,Q; \bar{p},\bar{q},\bar{P},\bar{Q}) = -F(x,\bar{p},q,\bar{P},Q) + F(x,p,\bar{q},P,\bar{Q})$$

$$\geq C(|P|^r + |Q|^r) - C_1(|\bar{P}|^r + |\bar{Q}|^r) - C_2(|p| + |q| + |\bar{p}| + |\bar{q}| + 1)$$

it follows the assumption II. Since, now, the function H is convex in

(P,Q) according to the assumption IV', it is quasiconvex in the Morrey sense i.e. the assumption III holds.

The corresponding Nash point equilibrium (u_0,v_0) of the functional pair $(J,-J)$, where

$$J(u,v) = - \int_\Omega F(x,u,v,\nabla u,\nabla v)dx ,$$

is just the saddle point of J.

The existence Theorem I and partial regularity Theorems II, III imply the corresponding results in Chang [3].

However, the above assumptions for the saddle point problem do not satisfy the assumptions given in Bensoussan and Frehse [2], particularly, the ellipticity condition.

We present here an example. Assume

$$f(t) = \frac{1}{r} |t|^r$$

where $1 < r < 2$, and let

$$F(P,Q) = f(P) - f(Q) .$$

Obviously, the assumptions I, II', IV' are all satisfied, but there is no positive constant $c_0 > 0$ such that

$$(F_P(P,Q) - F_P(\bar{P},\bar{Q}))(P-\bar{P}) - (F_Q(P,Q) - F_Q(\bar{P},\bar{Q}))(Q-\bar{Q}) \geq c_0(|P-\bar{P}|^2 + |Q-\bar{Q}|^2) .$$

<u>Example 3</u>. We present here some high power functionals which have not been studied in [2]. Let $n = M = N = 1$,

$$F(P,Q) = A'P^4 + AP^2 + BPQ + CQ^2 ,$$

$$G(P,Q) = aP^2 + bPQ + cQ^2 + c'Q^4$$

where $A,B,\ldots,c,A',c'$ are constants

$$H(P,Q,\bar{P},\bar{Q}) = A'P^4 + AP^2 + (B + b)PQ + cQ^2 + c'Q^4$$

$$- A'\bar{P}^4 - A\bar{P}^2 - B\bar{P}Q - bP\bar{Q} - c\bar{Q}^2 - c'\bar{Q}^4 .$$

Therefore, if $A,A',c,c' > 0$ and $Ac > \frac{1}{4} (B + b)^2$, all the assumptions I-IV hold, and Theorems I and II are applied.

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