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COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS
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INTRODUCTORY REMARKS ON VARIATIONAL THEORY

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Introductory remarks on variational theory.

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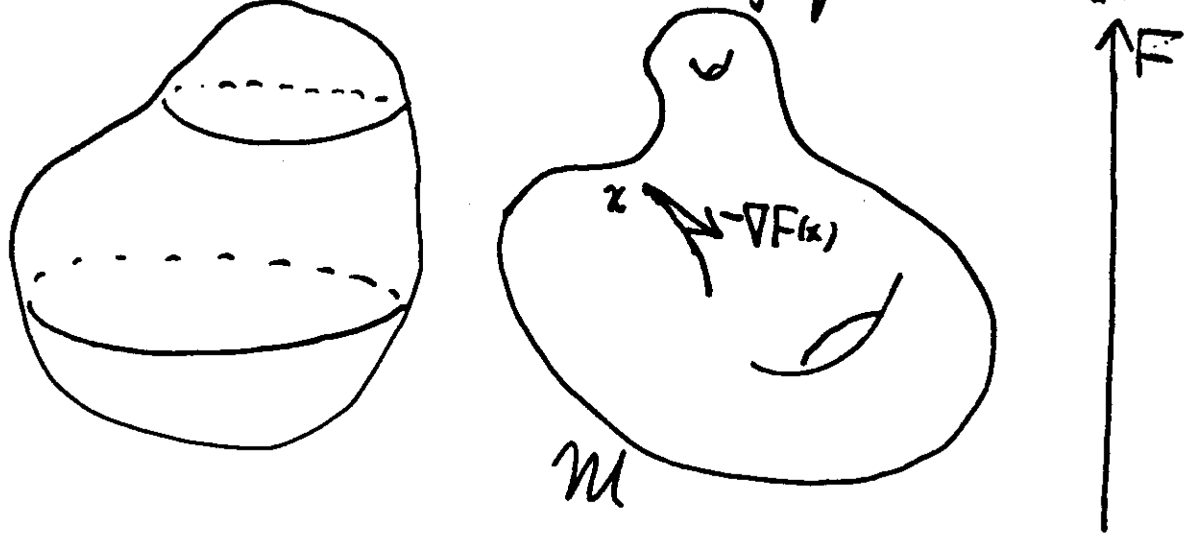
1. Central to variational theory is the topology of a real-valued function $F: M \to \mathbb{R}$. That is true conceptually, from the viewpoint of methods, and with respect to physical applications. In this introductory lecture I shall attempt to illustrate some of the main ideas by way of examples — with particular reference to the syllabus set out in the second Bulletin of the College.

2. Suppose that M is a compact surface, and think of F as the height function (relative to some embedding of M in $\mathbb{R}^3$).



Let ∇F denote the <u>gradient vector field</u> of F. Its coordinate description near a point $x \in M$ is $\left(\frac{\partial F(x)}{\partial x_1}, \frac{\partial F(x)}{\partial x_2}\right)$.

Its <u>trajectories</u> are the paths $\gamma : \mathbb{R} \times M \to M$ satisfying

$$\left.\begin{aligned} \frac{d\,\gamma(t,x)}{dt} &= -\nabla F(\gamma(t,x)) \\ \gamma(0,x) &= x \end{aligned}\right\}$$

Say that $a \in M$ <u>is a critical point of</u> F if $\nabla F(a) = 0$.

The trajectories partition M. With each critical point a we associate $U_a = \{x \in M : \gamma(t,x) \to a \text{ as } t \to -\infty\}$.

Then U_a is contractible in M, and contains exactly one critical point of F.

Let $\mathrm{Cat}\, M$ = least integer k such that M can be covered by k sets contractible in M. $\mathrm{Cat}\, M$ is a topological invariant of M, called its <u>Lusternik-Schnirelmann category</u>.

Then

$\mathrm{Cat}\, M \leqslant$ <u>number of critical points of</u> F.

3. A critical point a is <u>nondegenerate</u> if the symmetric matrix

$$\left(\frac{\partial^2 F(a)}{\partial x_i \partial x_j}\right) \text{ has the form } \begin{pmatrix} \pm & 0 \\ 0 & \pm \end{pmatrix}$$

with diagonal entries $\neq 0$. The nondegenerate critical points are isolated on M.

a <u>is a local minimum</u> if $\begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}$; let

$\mu_0(F)$ = number of these;

a <u>is a saddle point</u> if $\begin{pmatrix} \pm & 0 \\ 0 & \mp \end{pmatrix}$; let

$\mu_1(F)$ = number of these;

a <u>is a local maximum</u> if $\begin{pmatrix} - & 0 \\ 0 & - \end{pmatrix}$; let

$\mu_2(F)$ = number of these.

<u>Kronecker theorem. Let M be a surface with p handles; and $F: M \to \mathbb{R}$ a function having only nondegenerate critical points.</u>

<u>Then</u> $2-2p = \mu_0(F) - \mu_1(F) + \mu_2(F)$.

Note: The left member is a topological invariant of M. The right member contains basic qualitative information about F.

Since $1 \leq \mu_0(F)$, $1 \leq \mu_2(F)$ we obtain

$2p \leq \mu_1(F)$. These are a simple instance of the <u>Morse inequalities</u>.

Observe that any nondegenerate function has at least $2+2p$ critical points.

4. If M is non-compact, then we can obtain the same sort of results by requiring further that $F: M \to \mathbb{R}$ be <u>proper</u> (i.e., $F^{-1}(C)$ is compact whenever $C \subset \mathbb{R}$ is compact).

It is only a technical matter (elementary homology theory) to extend these ideas/results to manifolds M of any finite dimension.

There are substantial obstructions to extending the theory to manifolds of infinite dimension. Yet that is the setting of an important part of variational methods in geometrical and physical problems.

Again, we give an example.

5. Let V be a Euclidean vector space, and consider the space $\mathcal{L}_1^2(S^1, V)$ of maps $x : S^1 \to V$ of a circle into V. This is a Hilbert space of infinite dimension, with algebraic operations defined pointwise, and inner product

$$\langle x, y\rangle = \int_{S^1} \left(\langle x(s), y(s)\rangle + \langle x'(s), y'(s)\rangle\right) ds$$

Define $F : \mathcal{L}_1^2(S^1, V) \to \mathbb{R}$ by $F(x) = \frac{1}{2}\int_{S^1} |x'(s)|^2 ds$.

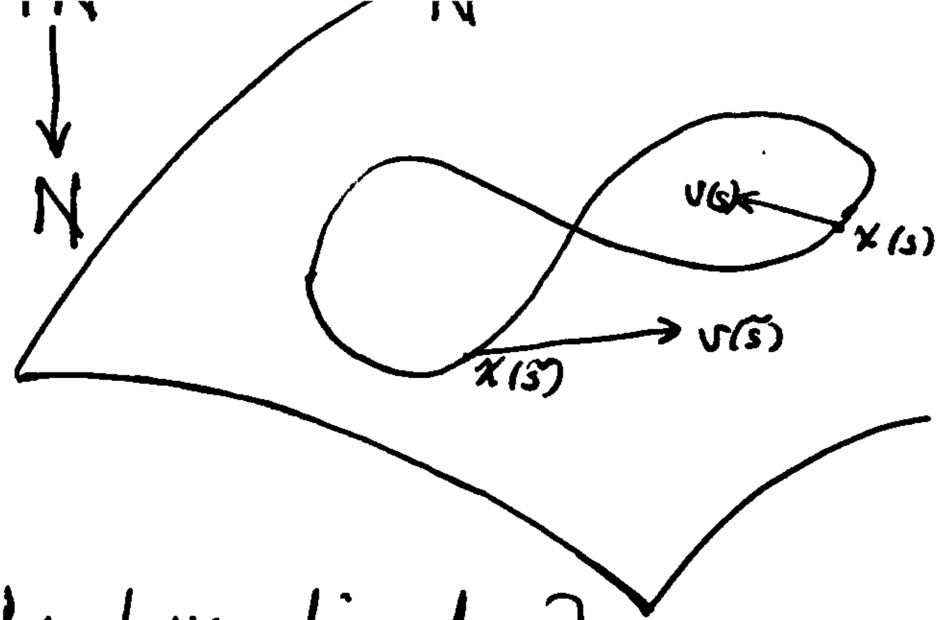
That is a quadratic form, and is a smooth function.

Now take a compact submanifold $N \subset V$; and let $\mathcal{M} = \{x \in \mathcal{L}_1^2(S^1, V) : x(S^1) \subset N\}$.

Then (1) $\mathcal{M}$ is a smooth manifold locally homeomorphic to an infinite dimensional Hilbert space. Thus $\mathcal{M}$ is not locally compact.

[Indeed, the tangent space $T_x\mathcal{M}$ at x is the Hilbert space of $\mathcal{L}_1^2$-maps $v : S^1 \to TN$ (= the tangent vector bundle of N) such that for every $s \in S^1$, $v(s)$ is a tangent vector to N at $x(s)$:

$$\begin{tikzcd} & TN \arrow[d] \\ S^1 \arrow[ur, "v"] \arrow[r, "x"'] & N \end{tikzcd}$$



$T_x\mathcal{M}$ is the space of variations of x.]

(2) The critical points of F are the closed geodesics of N:

For any path $(x_h)\subset \mathcal{M}$,

$$\frac{d}{dh}F(x_h)\Big|_{h=0} = -\int_{S^1}\left\langle \frac{Dx'(s)}{ds}, \frac{dx_h}{dh}\Big|_{h=0}\right\rangle ds.$$

Now x_0 is a critical point of F iff the left member vanishes for all paths (x_h) with $x_h|_0 = x_0$. The right member vanishes for all such paths (x_h) iff $\frac{Dx'(s)}{ds}\equiv 0$. That is the equation for a closed geodesic ($\frac{D}{ds}$ being the covariant differential along x).

(3) F has the following compactness property (Palais–Smale): Any sequence $(x_i) \subset \mathcal{M}$ on which F is bounded and for which $\nabla F(x_i) \to 0$ has a subsequence converging to a point $a \in \mathcal{M}$. a is a critical point of F.

Application There is a non-constant closed geodesic on N. (Fet's theorem).

The technique of Lusternik–Schnirelman and Morse theory apply.

(4) We could replace this Lagrangian formulation by its Hamiltonian analogue — and employ the symplectic geometry of the cotangent bundle $T^*(\mathcal{M})$.

6. If we try to make the analogous constructions with S^1 replaced by the 2-sphere S^2 (or any other compact surface), the natural function $F: \mathcal{L}_1^2(S^2, N) \to \mathbb{R}$

is $F(\varphi) = \frac{1}{2} \int_{S^2} |d\varphi|^2 ds$.

We can define its critical points. Those which are continuous are minimal surfaces $\varphi: S^2 \to N$.

However, $\mathcal{L}_1^2(S^2, N)$ is not a smooth manifold; and F does not have the required compactness to obtain Lusternik-Schnirelmann or Morse theory. (That situation improves if we replace S^2 by a surface with boundary, and pose a Plateau boundary value problem.)

7. We can formulate the same variational problem

$$F : \mathcal{L}_1^2(S^3, N) \to \mathbb{R} .$$

The critical points can be identified with harmonic maps. A new phenomenon: <u>There are definitely no minima of F except the constant maps.</u> However, there may be a large and interesting collection of critical points.

If N is a surface, the theory of harmonic maps $\varphi : S^3 \to N$ is the simplest variational problem associated with liquid crystals.