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REGULARITY AND UNIQUENESS OF HARMONIC MAPS INTO AN ELLIPSOID

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This paper concerns harmonic maps from an open set Ω of $\mathbb{R}^N$ into an n-dimensional ellipsoid

$$N=\{(u_I,u_{II})\in\mathbb{R}^n\times\mathbb{R}/\ u_I^2+\frac{u_{II}^2}{a^2}=1\}\text{ where } a\in[0,1].$$

First one shows that any harmonic map whose image lies in a compact subset of the open upper hemisphere of the ellipsoid N_+ is smooth. Then one proves the uniqueness of any harmonic map whose image lies in a compact subset of N_+, and whose boundary data are prescribed. Using these results, on computes the minimizing map from a ball of dimension n into N whose boundary data are the equator map $\frac{x}{|x|}$.

Let $N=\{(u_I,u_{II})\in\mathbb{R}^n\times\mathbb{R}/u_I^2+\frac{u_{II}^2}{a^2}=1\}$ be a n-dimensional ellipsoid, where $n\geq 1$ and $a>0$.

$N_+=\{u=(u_I,u_{II})\in\mathbb{R}^n\times\mathbb{R}/u_{II}>0\}$ is the upper hemisphere.

Let Ω be a bounded open set with regular boundary $\partial\Omega$ of $\mathbb{R}^N$ where $N \geq 1$.

We consider the space $H^1(\Omega, N)$ of functions u of $H^1(\Omega, \mathbb{R}^{n+1})$ which verifies a.e. $u(x) \in N$. $H^1_0(\Omega, \mathbb{R}^{n+1})$ will be the closure of $C^\infty_c(\Omega, \mathbb{R}^{n+1})$ in $H^1(\Omega, \mathbb{R}^{n+1})$.

We define on $H^1(\Omega, N)$ the energy functionnal E by :

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx = \frac{1}{2}\int_\Omega \sum_{\alpha=1}^{N}\sum_{i=1}^{n+1}\left(\frac{\partial u_i}{\partial n_\alpha}\right)^2 .$$

We will study the critical points of E, i.e. the weakly harmonic maps. They satisfy in the sense of distributions the following equations :

$$(0.1) \qquad \begin{cases} \Delta u_I + \lambda u_I = 0 & (a) \\ \Delta u_{II} + \lambda \dfrac{u_{II}}{a^2} = 0 & (b) \end{cases}$$

where $\lambda = [|\nabla u_I|^2 + \frac{|\nabla u_{II}|^2}{a^2}] \frac{a^4}{a^4 u_I^2 + u_{II}^2}$, see e.g. A. Baldes [B].

We will only consider the case of a flattened ellipsoïd, i.e. we will assume that $a \in (0,1]$.

The first result is a regularity theorem. We will show that if $N \geq 3$, every harmonic map u whose image belongs to a compact subset of N_+ is smooth inside of Ω. This theorem is a variant of a result of S. Hildebrandt, H. Kaul and K-O. Widman who have proved in [HKW] the same result with a geodesic ball of radius M of a Riemannian manifold M instead of N_+, with the condition that $M < \frac{\pi}{2\sqrt{K}}$ where K is an upper bound of the sectionnal curvature of M. The two results coincide when $a = 1$, but the result of this paper is new in the case $a \in (0,1)$. However the following proof is a variant of [HKW].

The second result is a uniqueness principle. If $N \geq 1$ we show that for every φ in $H^1(\Omega, N)$ such that $\varphi(x)$ lies in a compact subset of N_+ for a.e. x in $\partial\Omega$, there exists a unique smooth harmonic map whose image belongs to N_+ and which agrees with φ on $\partial\Omega$. This theorem is a variant of a result of W. Jäger and H. Kaul who have obtained the same result in [JK1] with a geodesic ball whose radius is strictly bounded by $\frac{\pi}{2\sqrt{K}}$. The two results coïncide only when $a = 1$ but the proof here uses the same ideas of [JK1].

In the third part, we will use these two first results to study the following problem :

Let us suppose $n = N \geq 3$ and $\Omega = B^n = \{x \in \mathbb{R}^n / |x| < 1\}$, we consider the equator map $u_*(x) = (\frac{x}{|x|}, 0)$ of $H^1(B^n, N)$, and we want to find minimizing maps u of $H^1(B^n, N)$ which agree with the equator map on ∂B^n, i.e. a map u in $H^1(B^n, N)$ with the boundary condition $u_{|\partial B^n} = u_{*|\partial B^n}$, and such that $E(u) \leq E(v)$ for any v in $H^1(B^n, N)$ such that $v_{|\partial B^n} = u_{*|\partial B^n}$.

By a calculation of the hessian $\delta^2 E(u_*)$ of the energy functionnal E on u_*, A. Baldes showed in [B] that u_* cannot be a minimizing map if $a^2 < \frac{4(n-1)}{(n-2)^2}$, and that

$\partial^2 E(u_*)(v,v) > 0$ for any v in $H^1_o(B^n,\mathbb{R}^{n+1}) \cap L^\infty(B^n,\mathbb{R}^{n+1})$ such that $v(x) \in T_{u_*(x)} N$ a.e. and $v \neq 0$ provided that $a^2 \geq \frac{4(n-1)}{(n-2)^2}$. Note that this does not imply that u_* is local minimizing. He proved also that if u_* is minimizing, then u_* is the unique minimizer. Here we will show that :

- if $a^2 < \frac{4(n-1)}{(n-2)^2}$, there is a smooth minimizing map.
- if $a^2 \geq \frac{4(n-1)}{(n-2)^2}$, u_* is the unique minimizer.

The proof here uses the movement of a point on a cycloid and is a variant of the method of W. Jäger and H. Kaul in [JK2] who solved this problem in the case of a sphere, i.e. when $a = 1$, using the movement of a pendulum.

I want to express my gratitude to J.M. Coron for his helpful advice.

I - REGULARITY.

Let us formulate our first result.

Theorem 1. Suppose that $N \geq 3$, assume that $u \in H^1(\Omega, N)$, that the image of u belongs to a compact subset of N_+, and that u verifies weakly (0.1).

Then u is smooth on Ω.

Remark. If $N = 1$, the regularity is trivial, if $N = 2$ Morrey showed in [M] regularity for minimizing maps.

Notations. In all the paper, we will note by the dot the scalar product in $\mathbb{R}^{n+1}$, and by the bracket the scalar product in the dual space of $\mathbb{R}^N$, so that :

If $\varphi \in H^1(\Omega,\mathbb{R}^{n+1})$, $\beta \in H^1(\Omega,\mathbb{R})$, $\langle \nabla\beta, \nabla\varphi \rangle = \sum_{\alpha=1}^{N} \frac{\partial\beta}{\partial x^\alpha}\frac{\partial\varphi}{\partial x^\alpha}$

If $\varphi, \psi \in H^1(\Omega,\mathbb{R}^{n+1})$, $\nabla\varphi.\psi = \sum_{i=1}^{n+1} (\nabla\varphi^i)\psi^i$.

Proof of theorem 1. We will prove only continuity. Smoothness then follows from [LU]. See also [BG] or [S].

Let us consider a weak harmonic map u whose image belongs to a compact subset of N_+ So there exists $\alpha_o > 0$ with :

(1.1) $$u_{II} \geq \alpha_o \text{ a.e. on } \Omega .$$

We consider the equation for f in $H^1(\Omega,\mathbb{R})$:

(1.2) $$\begin{cases} -\Delta\zeta = f \\ \zeta \in H^1_o(\Omega,\mathbb{R}). \end{cases}$$

There is an associate Green function G which gives the solution of (1.2) by

(1.3) $$\forall y \in \Omega,\ \zeta(y) = \int_\Omega f(x)\ G(x,y)dx .$$

For every point y of Ω, there exists σ_o such that

$$B_{\sigma_o}(y) = \{z \in \mathbb{R}^N / |z-y| < \sigma_o\} \subset \Omega .$$

Then for $\sigma \in]0,\sigma_o]$, let us consider

(1.4) $$G^\sigma(x,y) = \frac{1}{\text{mes}[B_\sigma(y)]} \int_{B_\sigma(y)} G(x,z)dz = ⨍_{B_\sigma(y)} G(x,z)dz.$$

From (1.3) we deduce

(1.5) $$\int_\Omega f(x)G^\sigma(x,y)dy = ⨍_{B_\sigma(y)} \zeta(z)dz .$$

We use too the following properties which are in [LSW] or in [HKW] : there exists strictly positiv constants K_1, K_2, K_3 such that

(1.6) $$0 \leq G(x,y) \leq K_1 |x-y|^{2-N}$$

(1.7) $$G(x,y) \geq K_2 |x-y|^{2-N} \text{ , if } |x-y| \leq \frac{3}{4} d(y,\partial\Omega))$$

(1.8) $$|\nabla_x G(x,y)| \leq K_3 |x-y|^{1-N}$$

(1.9) $$G^\sigma(x,y) \leq 2^{N-2} K_1 |x-y|^{2-N} \text{ , if } \sigma < \frac{1}{2}|x-y| .$$

(1.10) if $d(y,\partial\Omega) > \sigma, [x \mapsto G^\sigma(x,y)] \in H^1_0(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R})$.

where x and y are in Ω.

We take an arbitrary point of Ω and show that u is continuous on this point. It is always possible using translation in $\mathbb{R}^N$ to assume that this point is 0 . By (1.10) we may use the map

$[x \mapsto (0,G^\sigma(x,0))]$ as a test function in (0.1), for σ small enough. This gives

(1.11) $$\int_\Omega [\Delta u_{II}(x)G^\sigma(x,0) + \lambda \frac{u_{II}(x)}{a^2} G^\sigma(x,0)]\, dx = 0 .$$

Let w in $H^1(\Omega,\mathbb{R})$ be the solution of the equation

$$\begin{cases} -\Delta w = 0 \\ w_{|\partial\Omega} = u_{II}{}_{|\partial\Omega} \end{cases} .$$

So $u_{II} - w \in H^1_0(\Omega,\mathbb{R})$ and by (1.5)

$$\int_\Omega \Delta u_{II}(x)G^\sigma(x,y)dx = \int_\Omega \Delta(u_{II}-w)G^\sigma(x,y)dx = - ⨍_{B\sigma(0)} (u_{II}-w)(z)dz .$$

Replacing this in (1.11).

$$\int_\Omega \lambda \frac{u_{II}}{a^2} G^\sigma(x,0)dx = ⨍_{B\sigma(0)} (u_{II}-w)(z)dz \leq a .$$

Using (1.1)

$$\int_\Omega \lambda\, G^\sigma(x,0)dx \leq \frac{a^3}{\alpha_0} .$$

Obviously $\lambda \geq a^2|\nabla u|^2$ and so

(1.12) $$\int_\Omega |\nabla u|^2 G^\sigma(x,0)dx \leq \frac{a}{\alpha_0} .$$

Now, using Fatou's lemma and passing to the limit in (1.12) when $\sigma \to 0$, we obtain

(1.13) $$\int_\Omega |\nabla u(x)|^2 G(x,0)dx \leq \frac{a}{\alpha_0} .$$

Using (1.7), for ε_0 sufficiently small :

$$\int_{B_{\varepsilon_0}(0)} |\nabla u(x)|^2 |x|^{2-N} \leq \frac{a}{\alpha_0 K_2} .$$

Hence, by Lebesgue's theorem, if $\varepsilon \to 0$

(1.14) $$\int_{B_\varepsilon(0)} |\nabla u(x)|^2 |x|^{2-N} \to 0 .$$

And it follows that

(1.14)bis $$\frac{1}{\varepsilon^{N-2}} \int_{B_\varepsilon(0)} |\nabla u(x)|^2 dx \to 0 .$$

Let us remark, using the results of R. Schoen and K . Uhlenbeck in [SU] , that if u would be a minimizing map , then (1.14)bis would be enough to prove regularity.

Now, for $R < \frac{\sigma_o}{2}$ let us define :

$T_{2R} = \{x \in \Omega \ / R < |x| < 2R\}$.

$B_{2R} = \{x \in \Omega \ / |x| < 2R\}$.

We note for $\alpha > 0$ $N_\alpha = N \cap \{u/u_{II} > \alpha\}$. So (1.1) implies that $u(x) \in N_{\alpha_o}$ a.e. .

We consider the point u_R of N_{α_o} which is defined by :

$$\begin{cases} u_{RI} = ⨍_{T_{2R}} u_I(x)dx \in \mathbb{R}^n \\ u_{RII} = a\sqrt{1-(u_{RI})^2} \end{cases}$$

So we have the following inequality using Poincaré's inequality

$$(1.15)\quad \|u-u_R\|_{L^2(T_{2R})} \leq K_{\alpha_o}\|u_I-u_{RI}\|_{L^2(T_{2R})}$$
$$\leq K_{\text{Poincaré}}\, K_{\alpha_o} R\, \|\nabla u_I\|_{L^2(T_{2R})}$$
$$\leq K_4\, R\, \|\nabla u\|_{L^2(T_{2R})} \quad .$$

We consider the constant speed parametrage of the unique geodesic in N_{α_o} the extremities of which are u_R and the north pole $P = (0,a)$,

$$[0,1] \to N_{\alpha_o}\ ,\ t \mapsto u_{R,t}\ ,\ \text{with } u_{R,0} = P,\ u_{R,1} = u_R\ .$$

For the sake of commodity we note $v = u_{Rt}$.

We define the linear mapping $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $(u_I, u_{II}) \mapsto (u_I, \frac{u_{II}}{a^2})$ and we use the following test function

$$A[u(x)-v]\ G^\sigma(x,y)\eta(x)$$

where $y \in B_{\frac{R}{2}}$, $\sigma < \frac{R}{2} - |y|$, $\eta \in C_c^\infty(B_{2R}, \mathbb{R})$, $\eta \equiv 1$ on B_R and $|\nabla\eta| \leq K_5 R^{-1}$. This gives :

$$\int_{B_{2R}} \{-\langle\nabla(u-v).\nabla A(u-v)\rangle G^\sigma\eta - \langle\nabla(u-v),\nabla\eta\rangle.A(u-v)G^\sigma$$
$$-\langle\nabla(u-v),\nabla G^\sigma\rangle.A(u-v)\eta + \lambda A(u).A(u-v)G^\sigma\eta\}dx = 0$$

$$\Leftrightarrow (1.16)\quad -\int_{B_{2R}} \langle\nabla[(u-v).A(u-v)\eta],\nabla G^\sigma\rangle\, dx$$
$$+\int_{B_{2R}} \langle\nabla\eta,\nabla G^\sigma\rangle (u-v).A(u-v)dx$$
$$-2\int_{B_{2R}} \langle\nabla(u-v),\nabla\eta\rangle.A(u-v)G^\sigma\, dx$$
$$+\ 2\int_{B_{2R}} [\lambda A(u).A(u-v) - \langle\nabla u,\nabla[A(u)]\rangle]\ G^\sigma\eta dx = 0$$

. Since $\sigma < \frac{R}{2} - |y|$, the first integral is equal to

$$-⨍_{B_\sigma(y)} (u(z)-v).A[u(z)-v]dz\ .$$

. In the second integral we make the decomposition

$$(u-v).A(v-u) = (v-u_R).A(v-u_R)$$
$$+[(u-v).A(u-v)-(v-u_R).A(v-u_R)]$$

and $|(u-v).A(u-v)-(v-u_R).A(v-u_R)| \leq K_6\,||u-v|-|v-u_R||$
$$\leq K_6|u-u_R|\ .$$

hence

$$\left|\int_{B_{2R}} \langle\nabla\eta,\nabla G^\sigma\rangle[(u-v).A(u-v)-(v-u_R).A(v-u_R)]\ dx\right|$$
$$\leq\ K_6\int_{T_{2R}} |\langle\nabla\eta,\nabla G^\sigma(x-y)\rangle|\,|u-u_R|dx$$
$$\leq\ K_5K_6\ 2^{n-2}K_3\ R^{-N}\int_{T_{2R}} |u-u_R|dx\ .$$

$$\leq K_3 K_4 K_5 K_6 2^{n-2}[R^{2-N}\int_{T_{2R}}|\nabla u|^2 dx]^{\frac{1}{2}}$$

$$\leq K_7[\int_{B_{2R}}|\nabla u(x)|^2 R^{2-N} dx]^{\frac{1}{2}}$$

using (1.8) and (1.15). So the second integral is equal to

$$(v-u_R).A(v-u_R) + O([\int_{B_{2R}}|\nabla u(x)|^2 R^{2-N} dx]^{\frac{1}{2}}) .$$

. In the third integral, we use the Cauchy-Schwarz inequality

$$|-2\int_{B_{2R}} < \nabla(u-v),\nabla\eta > .A(u-v)G^\sigma dx|$$

$$\leq 4\sqrt{\frac{1}{a^2}+1}[\int_{T_{2R}}|\nabla u|^2 G^\sigma(x,y)dx]^{\frac{1}{2}}[\int_{T_{2R}}|\nabla\eta|^2 G^\sigma(x,y)dx]^{\frac{1}{2}}$$

$$\leq 4\sqrt{\frac{1}{a^2}+1}[\int_{T_{2R}}K_1|\nabla u|^2(\frac{R}{2})^{2-N}dx]^{\frac{1}{2}} 2^{N-2}(K_1)^{\frac{1}{2}} K_5 \text{ mes } (T_2)$$

$$\leq K_8[\int_{B_{2R}}|\nabla u|^2 R^{2-N}dx]^{\frac{1}{2}} .$$

. In the fourth integral, we remark that $\lambda = \frac{<\nabla u.\nabla[A(u)]>}{|A(u)|^2}$, so $\lambda A(u).A(u-v) - <\nabla u,\nabla[A(u)]> = -A(u).A(v)\lambda$.
Hence (1.16) becomes

$$(1.17) ⨍_{B_\sigma(y)}(u(z)-v).A(u(z)-v)dz \leq (v-u_R).A(v-u_R)+$$
$$K_9[\int_{B_{2R}}|\nabla u|^2 R^{2-N}dx]^{\frac{1}{2}} -2\int_{B_{2R}}\lambda A(u).A(v)G^\sigma \eta dx .$$

Now, let $\sigma \to 0$ and use $\lambda \leq \frac{|\nabla u(x)|^2}{a^2}$, (1.9), (1.14) and Lebesgue's theorem. We find for a.e. $y \in B_{\frac{R}{2}}$

$$(1.18)\ (u(y)-v).A(u(y)-v) \leq (v-u_R).A(v-u_R)+$$
$$+ K_9[\int_{B_{2R}}|\nabla u|^2 R^{2-N}dx]^{\frac{1}{2}} - 2\int_{B_{2R}}\lambda A(u).A(v)G(x,y)\eta(x)dx .$$

Now, we take $\rho < \frac{R}{2}$ and $y \in B_\rho$, and we write

$$\int_{B_{2R}}\lambda A(u).A(v)G(x,y)\eta(x)dx$$
$$\geq -\int_{B_{2R}\setminus B_{2\rho}} a^2|\nabla u|^2 G(x,y)\eta(x)dx + \operatorname{Inf}_{B_{2\rho}}[A(u).A(v)] \int_{B_{2\rho}} a^2|\nabla u|^2 G(x,y)\eta(x)dx.$$

And since $2(x-y) \geq |x-y|+\rho \geq (x-y)+(y) \geq (x)$, and from (1.6),

$$|\int_{B_{2R}\setminus B_{2\rho}} a^2|\nabla u|^2 G(x,y)\eta(x)dx| \leq K_{10}\int_{B_{2R}\setminus B_{2\rho}} |\nabla u|^2|x|^{2-N}dx$$
$$\leq K_{10}\int_{B_{2R}}|\nabla u|^2|x|^{2-N} dx .$$

So (1.18) becomes

$$(1.19)\ (u(y)-v).A(u(y)-v) \leq (v-u_R).A(v-u_R)+$$
$$+K_{11}[\int_{B_{2R}}|\nabla u|^2(x)^{2-N}dx]^{\frac{1}{2}}$$
$$-2 \operatorname{Inf}_{B_{2\rho}}[A(u).A(v)] \int_{B_{2\rho}} a^2|\nabla u|^2 G(x,y)\eta(x)dx .$$

Now, we define $h : [0,1] \to \mathbb{R}+$ by

$$h(t) = \limsup_{R\to 0}[\limsup_{\rho\to 0} \sup_{B_\rho}(u(y)-v).A[u(y)-v]] .$$

Let us remember that $v = u_{R,t}$. It is standard that h is continuous.

Furthermore, $h(0)\le 2(1-\frac{\alpha_o}{a})$. In fact we have for every t in [0,1], $h(t)\le 2(1-\frac{\alpha_o}{a})$. Let us suppose that it is false, then, there exists t_o in]0,1] such that $h(t_o)=2(1-\frac{\alpha_o}{a})+\varepsilon$ for ε sufficiently small, such that $2(1-\frac{\alpha_o}{a})+\varepsilon<2$.

It follows that for small enough R and $\rho<\rho_o(R)$, we have :

for a.e. $y\in B_{2\rho}$, $(u(y)-u_{R,t_o}).A(u(y)-u_{R,t_o})<2$.

or for a.e. $y\in B_{2\rho}$, $A(u(y)).u_{R,t_o}>0$

which implies that for a.e. $y\in B_{2\rho}$, $A(u(y)).A(u_{R,t_o})>0$.

Now, we use (1.19), and find for a.e. $y\in B_2$,

$$(u(y)-u_{Rt_o}).A(u(y)-u_{Rt_o})\le(u_{Rt_o}-u_R).A(u_{Rt_o}-u_R) + K_{11}\Big[\int_{2R}|\nabla u|^2|x|^{N-2}dx\Big]^{\frac{1}{2}} .$$

And this implies

$$h(t_o)\le(u_{Rt_o}-u_R).A(u_{R,t_o}-u_R)\le 2(1-\frac{\alpha_o}{a})$$

which gives us a contradiction with $h(t_o)=2(1-\frac{\alpha_o}{a})+\varepsilon$. Hence we conclude that $\forall t\in[0,1]$, $h(t)\le 2$, and so for small enough R and $\rho<\rho_o(R)$, we have $A(u(y)).A(u_{Rt})\geq 0$, for a.e. $y\in B_{2R}$. We apply this in (1.19) with t=1, this gives

$$(u(y)-u_R).A(u(y)-u_R)\le K_{11}\Big[\int_{B_{2R}}|\nabla u|^2|x|^{2-N}dx\Big]^{\frac{1}{2}}$$

$$\Rightarrow (1.20)\quad |u(y)-u_R|^2\le K_{12}\Big[\int_{B_{2R}}|\nabla u|^2|x|^{2-N}dx\Big]^{\frac{1}{2}} ,$$

for almost everywhere y sufficiently small, and R sufficiently small. And using (1.14) the continuity of u on y is proved.

II. UNIQUENESS.

We want to solve the following Dirichlet problem : Let φ be a function of $H^1(\Omega,N)$ whose values on $\partial\Omega$ lies on a compact subset of N_+ , let us find a harmonic map u which agrees with φ on $\partial\Omega$ and whose values are in N_+ . The existence is standard by finding a minimizing map. We will show that this solution is unique using theorem 1 and the following result :

<u>Theorem</u> 2. If $u^1, u^2 \in C^\infty(\Omega,N_+)\cap H^1(\Omega,N_+)$ are harmonic, and if $u^1(\partial\Omega)$ and $u^2(\partial\Omega)$ belong to a compact subset of N_+ , then the function :

$\theta = \frac{1}{2}\,\frac{|u^1-u^2|^2}{u^1_{II}u^2_{II}}$ checks the maximum principle :

<u>Proof</u>. For r = 1,2, since $u^i(\partial\Omega)$ is contained in a compact subset of N_+ , there exists $\alpha_o>0$ with $u^i_{II}(x)\geq\alpha$ if $x\in\partial\Omega$.

Using (0.1) (b),we have

$$-\Delta u^i_{II}(x) = \lambda^i\,\frac{u^i_{II}(x)}{a^2} \geq 0 .$$

And so by the maximum principle

(2.1) $\qquad u^i_{II}(x)\geq\alpha$ on Ω .

The function $\Phi = -\mathrm{Log}(u^1_{II})-\mathrm{Log}(u^2_{II})$ is smooth and bounded on Ω . We will construct the elliptic operator $\mathcal{L}$ with Φ by : $\forall f\in H^1(\Omega,\mathbb{R})$, $\mathcal{L}(f) = \mathrm{div}[e^{-\Phi}\mathrm{grad}\ f]$.

We will show that $\mathcal{L}(\theta) \geq 0$, which assures the result.
Noting $\psi = \frac{1}{2}|u^1-u^2|^2$ so that $\theta = e^{\Phi}\psi$ and :

$\mathcal{L}(\theta) = \Delta\psi + \psi\Delta\Phi + \langle \nabla\psi, \nabla\Phi \rangle$, we have

$$\nabla\Phi = -\frac{\nabla u^1_{II}}{u^1_{II}} - \frac{\nabla u^2_{II}}{u^2_{II}}$$

$$\Delta\Phi = -\frac{\Delta u^1_{II}}{u^1_{II}} - \frac{\Delta u^2_{II}}{u^2_{II}} + \left|\frac{\nabla u^2_{II}}{u^1_{II}}\right|^2 + \left|\frac{\nabla u^2_{II}}{u^2_{II}}\right|^2 .$$

Hence

$$(2.1)\ \mathcal{L}(\theta) = \Delta\psi + \psi\left[\frac{-\Delta u^1_{II}}{u^1_{II}} - \frac{\Delta u^2_{II}}{u^2_{II}} + \left|\frac{\nabla u^1_{II}}{u^1_{II}}\right|^2 + \left|\frac{\nabla u^2_{II}}{u^2_{II}}\right|^2\right]$$

$$- \left\langle \frac{\nabla u^1_{II}}{u^1_{II}} + \frac{\nabla u^2_{II}}{u^2_{II}}, \nabla\psi \right\rangle .$$

Using Young's inequality

$$- \left\langle \frac{\nabla u^i_{II}}{u^i_{II}}, \nabla\psi \right\rangle \geq -\psi\left|\frac{\nabla u^i_{II}}{u^i_{II}}\right|^2 - \frac{|\nabla\psi|^2}{\psi} \quad , i = 1,2 .$$

We obtain :

$$(3.2)\quad \mathcal{L}(\theta) \geq \Delta\psi - \frac{|\nabla\psi|^2}{2\psi} - \psi\left[\frac{\Delta u^1_{II}}{u^1_{II}} + \frac{\Delta u^2_{II}}{u^2_{II}}\right] ,$$

and by the Cauchy-Schwarz's inequality

$$\frac{|\nabla\psi|^2}{2\psi} = \frac{\langle (u^1-u^2).\nabla(u^1-u^2), (u^1-u^2).\nabla(u^1-u^2)\rangle}{(u^1-u^2).(u^1-u^2)} \leq |\nabla(u^1-u^2)|^2 .$$

Inequality (3.2) gives

$$(3.3)\ \mathcal{L}(\theta) \geq \Delta\psi - \psi\left[\frac{\Delta u^1_{II}}{u^1_{II}} + \frac{\Delta u^2_{II}}{u^2_{II}}\right] - |\nabla(u^1-u^2)|^2 .$$

Now, $\Delta\psi = |\nabla(u^1-u^2)|^2 + (u^1-u^2).\Delta(u^1-u^2)$ implies using (0.1)

$$\Delta\psi = |\nabla(u^1-u^2)|^2 - (u^1-u^2).[\lambda_1 A(u^1) - \lambda_2 A(u^2)]$$
$$= |\nabla(u^1-u^2)|^2 - (\lambda_1+\lambda_2)\ [1-u^1.A(u^2)] .$$

Furthermore, $\frac{-\Delta u^1_{II}}{u^1_{II}} = \frac{\lambda_1}{a^2}$, $\frac{-\Delta u^2_{II}}{u^2_{II}} = \frac{\lambda_2}{a^2}$.

So, $\mathcal{L}(\theta) \geq |\nabla(u^1-u^2)|^2 - (\lambda_1+\lambda_2)\ [1-u^1.A(u^2)] +$
$+ \frac{|u^1-u^2|^2}{2}\ \frac{\lambda_1+\lambda_2}{a^2} - |\nabla(u^1-u^2)|^2$

or $\mathcal{L}(\theta) \geq \frac{\lambda_1+\lambda_2}{a^2}\left[\frac{|u^1-u^2|^2}{2} - a^2 + a^2 u^1.A(u^2)\right]$

or $\mathcal{L}(\theta) \geq \frac{\lambda_1+\lambda_2}{2a^2}\ (1-a^2)\ |u^1_I - u^2_I|^2 \geq 0$.

This termines the proof.

III. THE PROBLEM WITH THE EQUATOR MAP.

Now we choose $n = N \geq 3$, $B^n = \{x \in \mathbb{R}^n / |x| < 1\}$.
$u_*(x) = \frac{x}{|x|}$. We show the following result.

Theorem 3. u_* is the unique minimizing map if and only if $a^2 \geq \frac{4(n-1)}{(n-2)^2}$. If $a^2 < \frac{4(n-1)}{(n-2)^2}$, there is a smooth and radially symmetric minimizing map.

Proof. We will only prove that u_* is minimizing when

$a^2 \geq \frac{4(n-1)}{(n-2)^2}$. Unicity is then a consequence of theorem 5 of A. Baldes in [B].

We begin to look for the solution of the following Dirichlet problem: let be $\tau \in (0,1)$ and u_τ a map of $H^1(B^n, N_+)$ which verifies : $\forall x \in \partial B^n$, $u_\tau(x) = (x \sin(\tau \frac{\pi}{2}), a \cos(\tau \frac{\pi}{2}))$, and which minimizes the energy functionnal among the functions of $H^1(B^n, N_+)$ which agree with u_τ on ∂B^n.

It follows from theorems 1 and 2 that u_τ will be smooth harmonic and unique. Because of the uniqueness of u_τ and because of the symmetry of the problem, u_τ must be rotationnaly symmetric, i.e.

$$\begin{cases} u_\tau(x) = (\frac{x}{r} \sin\theta_\tau(r),\ a \cos\theta_\tau(r)), \quad \text{where } r = |x|. \\ \theta_\tau \in C^\infty([0,1],\ [0, \frac{\pi}{2}]) \\ \theta_\tau(0) = 0 \ ,\ \theta_\tau(1) = \tau \frac{\pi}{2} \ . \end{cases}$$

Let us calculate the equation verified by u_τ :

$$E(u_\tau) = |S^{n-1}| \int_0^1 [(\theta'_\tau)^2(\cos^2\theta_\tau + a^2 \sin^2\theta_\tau) + \frac{n-1}{r^2} \sin^2\theta_\tau] r^{n-1} dr$$

Let us introduce the diffeomorphism $\ell \in C^\infty([0, \frac{\pi}{2}], [0,\sigma])$ with

$$\begin{cases} \forall \theta \in [0, \frac{\pi}{2}] \ , \ell(\theta) = \int_s^\theta \sqrt{\cos^2 q + a^2 \sin^2 q}\, dq \\ \qquad\qquad \sigma = \int_s^{\frac{\pi}{2}} \sqrt{\cos^2 q + a^2 \sin^2 q}\, dq \ . \end{cases}$$

$\ell(\theta)$ is the arc-length of the arc $[0,\theta] \to \mathbb{R}^2$, $q \mapsto (\sin q,\ a \cos q)$.

Let us note $S \in C^\infty([0,\sigma],[0,1])$ defined by

$$\forall \theta \in [0, \frac{\pi}{2}] \ , \ S[\ell(\theta)] = \sin\theta \ .$$

(Note that if $a = 1$, S is the sinus function).

We make the change of variables $\lambda_\tau(t) = \ell[\theta_\tau(e^t)]$

$$E(u_\tau) = |S^{n-1}| \int_{-\infty}^0 [(\dot{\lambda}_\tau)^2 + (n-1) S(\lambda_\tau)^2] e^{(n-2)t}\, dt.$$

Hence, we obtain the equation which express that u_τ is harmonic

$$(3.1) \quad \ddot{\lambda}_\tau + (n-2)\dot{\lambda}_\tau - (n-1) S(\lambda_\tau) S'(\lambda_\tau) = 0$$

with the boundary conditions $\lambda_\tau(-\infty) = 0$, $\lambda_\tau(0) = \ell(\tau \frac{\pi}{2})$.

Now, let us introduce the cycloïd

$$\begin{cases} z = \frac{-(1+a)}{4} \cos 2\theta \\ y = \frac{(1-a)}{2} \theta + \frac{(1+a)}{2} \sin 2\theta \end{cases} \qquad \theta \in (0, +\infty) \ .$$

$$\begin{cases} \frac{dz}{d\theta} = \frac{1+a}{2} \sin 2\theta \\ \frac{dy}{d\theta} = \frac{1-a}{2} + \frac{1+a}{2} \cos 2\theta \ . \end{cases}$$

The sinus of the slope of the cycloid with the direction z is

$$p = \frac{\frac{dz}{d\theta}}{\sqrt{(\frac{dz}{d\theta})^2+(\frac{dy}{d\theta})^2}} =$$

$$= \frac{\frac{(1+a)}{2}\sin 2\theta}{\sqrt{[\frac{(1+a)\sin 2\theta}{2}]^2 + [\frac{(1-a)}{2}]^2 + [\frac{(1+a)\cos 2\theta}{2}]^2 + \frac{1-a^2}{2}\cos 2\theta}}$$

$$p = \frac{(1+a)\sin\theta\cos\theta}{\sqrt{\cos^2\theta+a^2\sin^2\theta}} .$$

And $S(\ell(\theta)) = \sin\theta$ implies

$$S'[\ell(\theta)]\ell'(\theta) = S'[\ell(\theta)]\sqrt{\cos^2\theta+a^2\sin^2\theta} = \cos\theta .$$

So $\quad S[\ell(\theta)]\; S'[\ell(\theta)] = \dfrac{\sin\theta\cos\theta}{\sqrt{\cos^2\theta+a^2\sin^2\theta}} = \dfrac{p}{1+a}$.

The curvilinear arc-length on the cycloïd is

$$\int_0^\theta \sqrt{\cos^2 q + a^2\sin^2 q}\, dq = \ell(\theta).$$

Now take $\beta \in \mathbb{R}$, and consider the movement of a point M on the cycloid whose curvilinear abciss is defined by

$$s(t) = \ell[\theta_\tau(t-\beta)] = \lambda_\tau(t-\beta).$$

Equation (3.1) becomes

$$(3.2) \qquad \ddot{s} + (n-2)\dot{s} - \frac{(n-1)p}{1+a} = 0 ,$$

with the boundary conditions $s(-\infty) = 0$, $s(\beta) = \ell(\tau\frac{\pi}{2})$.

This equation is the equation of the damped movement of a point which is forced to slide along the cycloid and which is subjected to a constant acceleration field $\frac{(n-1)}{1+a}\, p\, \frac{\partial}{\partial z}$.

The problem is : does this point reach the arc-length σ in a finite time ? To see this, we linearize (3.2) in a neighbourhood of $s = \sigma$ (or $\theta = \frac{\pi}{2}$), this gives

$$(3.3) \qquad \ddot{\overline{(s-\sigma)}} + (n-2)\,\dot{\overline{(s-\sigma)}} + \frac{n-1}{a^2}(s-\sigma) = 0 .$$

The characteristic equation is then

$$\mu^2 + (n-2)\,\mu + \frac{n-1}{a^2} = 0 ,$$

with discriminant $\Delta = \dfrac{(n-2)^2}{a^2}\,[a^2 - \dfrac{4(n-1)}{(n-2)^2}]$.

. If $\Delta < 0 \Leftrightarrow a^2 < \dfrac{4(n-1)}{(n-2)^2}$, we have small oscillations. So there exists $t_1 \in \mathbb{R}$ with $s(t_1) = \sigma$ and

$$\begin{cases} \forall t \in (-\infty, t_1),\ s(t_1) \in (0.\sigma). \\ \forall \tau \in (0,1),\ \exists !\ t_\tau \in (-\infty, t_1),\ s(t_\tau) = \ell(\tau\frac{\pi}{2}) \end{cases}$$

By a translation on t , we suppose that $s(0) = \sigma$. Taking $\beta = t_\tau$ i.e. $\lambda_\tau(t) = s(t+t_\tau)$, we find that λ_τ satisfies (3.1) with good boundary conditions for every $\tau \in (0,1]$. Equivalentely : $\theta_\tau(r) = \ell^{-1}[\lambda_\tau[\mathrm{Log}(e^{t}r)]]$, so that $\forall \tau \in (0,1]$, $\theta_\tau(r) = \theta_1(e^{t_\tau}r)$.

. If $\Delta \geq 0 \Leftrightarrow a^2 \geq \dfrac{4(n-1)}{(n-2)^2}$, the roots are real and non positive, the point M will never reach the abciss σ .

Indeed, let us suppose the contrary. Then there exists a real $\bar{t}$ such that

$$\begin{cases} \forall t \in (-\infty,\bar{t}),\ s(t) \in (0,\sigma),\ \dot{s}(t) > 0 \\ s(\bar{t}) = \sigma . \end{cases}$$

Let μ be a root of the characteristic equation of (3.3), μ is strictly negative, we pose

$$\forall t \in (-\infty,\bar{t}),\ x(t) = \mu(\sigma - s(t)) + \dot{s}(t).$$

Then we have

$$\dot{x} = \ddot{s} - \mu\dot{s} = -\,[(n-2)+\mu]\,\dot{s} + \frac{n-1}{1+a}\,p(s) \ .$$

$$\dot{x} = \frac{n-1}{a^2\mu}\,[\dot{s} + \frac{a^2}{1+a}\,\mu p(s)]$$

$$\dot{x} = \frac{n-1}{a^2\mu}\,x + (n-1)\left[\frac{\sin\theta\,\cos\theta}{\sqrt{\cos^2\theta + a^2\sin^2\theta}} - \int_\theta^{\frac{\pi}{2}} \frac{\sqrt{\cos^2 q + a^2\sin^2 q}}{a^2}\,dq\right] .$$

And it follows that $\dot{x} < \frac{n-1}{a^2\mu}\,x$ because if

$$H(\theta) = \frac{\sin\theta\cos\theta}{\sqrt{\cos^2\theta + a^2\sin^2\theta}} - \int_\theta^{\frac{\pi}{2}} \frac{\sqrt{\cos^2 q + a^2\sin^2 q}}{a^2}\,dq$$

H is negative on $[0, \frac{\pi}{2}]$ since $H(\frac{\pi}{2}) = 0$ and

$$H'(\theta) = \frac{[(a^2+1)\cos^2\theta + 2a^2\sin^2\theta]}{(\cos^2\theta + a^2\sin^2\theta)^{3/2}} > 0 .$$

Then since $\lim_{t\to-\infty} x(t)$ is strictly negative, we obtain that $x(\bar{t})$ is strictly negative, which is impossible because

$$x(\bar{t}) = \mu(\sigma - s(\bar{t})) + \dot{s}(\bar{t}) = \dot{s}(\bar{t}) \geq 0 .$$

Hence we conclude

$$\begin{cases} \forall t \in \mathbb{R},\ \lambda(t) \in (0,\sigma), \\ \lim_{t\to+\infty} \lambda(t) = \sigma \\ \forall_\tau\ \epsilon(0,1), \exists!\ t_\tau \in \mathbb{R},\ s(t_\tau) = \ell(\tau\frac{\Pi}{2}) \end{cases}$$

By a translation on t, we suppose that $s(0) = \ell(\frac{\Pi}{4})$.
Hence we will only obtain λ_τ for $\tau \in (0,1)$ by taking $\beta = t_\tau - t_{\frac{1}{2}}$ i.e. $\lambda_\tau(t) = s(t + t_\tau - t_{\frac{1}{2}})$, or equivalently

$$\theta_\tau(r) = \ell^{-1}\,[\lambda_\tau \mathrm{Log}(e^{t_\tau - t_{\frac{1}{2}}}\,r)]] .$$

Conclusion. Let v be in $H^1(B^n,\bar{N}_+)$ which verifies $v_{|\partial B^n} = u_{*|\partial B^n}$. It is easy to find a sequence τ_k and a sequence v_k which verifies

$$\begin{cases} \tau_k \in (0,1),\ \tau_k \to 1 \\ \forall x \in \partial B^n,\ v_k(x) = (x \sin(\tau_k \frac{\Pi}{2}),\ a\cos(\tau_k \frac{\Pi}{2})) \\ v_k \to v\ H^1(B^n,\mathbb{R}^{n+1}) \end{cases}$$

Using the following result, we have

$$(3.5) \qquad E(u_{\tau_k}) \leq E(v_k)$$

We now pass to the limit :

. If $a^2 < \frac{4(n-1)}{(n-2)^2}$, obviously, $u_{\tau_k} \to u_1\ H^1(B^n,\mathbb{R}^{n+1})$ where $u_1(x) = (\frac{x}{r}\sin\theta_1(r),\ a\cos\theta_1(r))$.

So $E(u_1) \leq E(v)$ and u_1 is a smooth and radially symmetric minimizing map.

. If $a^2 > \frac{4(n-1)}{(n-2)^2}$, obviously $u_{T_k} \rightharpoonup u_*$ $H^1(B^n, \mathbb{R}^{n+1})$, and by lower semicontinuity : $E(u_*) \leq E(v)$.

And it is not difficult to see that these functions are minimizing in $H^1(B^n, N)$ instead of $H^1(B^n, \overline{N}_+)$, because for every map $u \in H^1(B^n, N)$, if $u_+ = (u_I, |u_{II}|)$, then $E(u_+) = E(u)$, and $u_+ \in H^1(B^n, \overline{N}_+)$.

This completes the proof of theorem 3.

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