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# ON THE COHOMOLOGY OF QUANTUM ELECTRODYNAMICS

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Impressive developments have taken place in theoretical physics over the past decade or so. Gauge field theories have led to the unification of weak and electromagnetic theory while string and superstring theories show promise of unifying gravity with the other known forces. These major advances could not have been occurred were it not for developments in twentieth century mathematics which evolved independently of such applications. The language of connections and curvature on principal fiber bundles has become part of many physicists' vocabulary. More recent work on superstring theory has made significant use of generalizations of the Atiyah-Singer index theorem, Riemann surfaces and Teichmuller spaces. Mathematics as well as physics has also profited from this interplay as illustrated most dramatically by the recent results on 4 manifolds of S. Donaldson, M. Freedman and others.

In this note, I will talk about a more algebraic, or perhaps homological, link between math and physics; what A.M. Vinogradov calls the Algebraic Topology of Differential Equations [7][8]. I will illustrate this theory by presenting some partial computations for the cohomology of the Dirac and the Maxwell equations. This paper will end with a statement about a long exact sequence linking the cohomology of these theories with that of Quantum Electrodynamics. It is my hope that the methods presented here can be extended to Quantum Chromodynamics and possibly to super symmetric theories.

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This note should be considered to be a brief and incomplete survey of the methods in this field with just an indication of some of proofs and the scope of the applications. No sophisticated knowledge of physics or differential equations is assumed, however some familiarity with algebraic topology would be useful. The interested reader is encouraged to consult more detailed references for generalizations and proofs [1], [5], [6], [7].

## Section 1: Dirac Equations for Free Electrons

Over a half century ago, Dirac wrote down a pair of relativistic equations to describe a (free) electron [2]. We will introduce these equations below in a simplified context and carry them along to illustrate the variational complex.

(1) $$i\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0$$

(2) $$i\partial_{\mu}\bar{\psi}\gamma^{\mu} + m\bar{\psi} = 0$$

In this equation, $\psi$ is a map (here taken to be smooth) from Minkowski space-time, M, to $\mathbb{C}^4$, 4 dimensional complex space. The coordinates on M will be written variously as $(x^{\mu})$, $(x^0,x^1,x^2,x^3)$ and $(t,x,y,z)$. The symbol, $\partial_{\mu}$, is shorthand for $\frac{\partial}{\partial x}$, m denotes the mass of the electron, and $\gamma^{\mu}$ for $\mu=0,\ldots3$, are the 4x4 Dirac complex permutation matrices. For example



(3) $$\gamma^0 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \text{ and } \gamma^2 = \begin{bmatrix} & & & i \\ & & -i & \\ & -i & & \\ i & & & \end{bmatrix}$$

The conjugate equations involve (2) $\bar{\psi} = \psi^*\gamma^0$, a slightly modified form of the hermitian conjugate of $\psi$. The Einstein summation convention will be in force on the repeated indices $\mu = 0,\ldots,3$ so that each of the Dirac equations are 4-vector equations with 5 summands. For example the first component of equation (1) will be

(4) $$i\partial_t\psi_1 + i\,\partial_x\psi_4 - \partial_y\psi_4 + i\partial_z\psi_3 = m\psi_1$$

As these equations are first order linear PDEs, it is not difficult to find solutions. Indeed the following plane waves

(5) $$\psi(x) = \omega e^{ip.x}$$

where p is the 4 vector $(E,\vec{p})$ and $p.x = Et - \vec{p}\,\vec{x}$ can be seen to satisfy the Dirac equation and its Dirac conjugate, provided $E^2 = m^2 + |\vec{p}|^2$. The solutions with $E > 0$ are called electrons while those with $E < 0$ are called positrons. In addition the solutions have another degree of freedom which is identified with the spin of these particles [2].

Few equations in particle physics are quite as simple as these for the free electron; in general equations are non linear, of higher order, and with no elementary solutions. By an appropriate qualitative analysis of the equations, however, physicists and mathematicians have been able to find various characteristic quantities of the solutions which have important physical interpretations. For example one is interested in knowing whether solutions to the equations of motion conserve energy and momentum. Both the Hamiltonian and the Lagrangian formalism can be very useful in understanding these and other features of the equations of motion. In this note we will consider only the Lagrangian or variational approach.

## Section 2: Variational Methods

A popular method for studying equations of motion more complicated than the Dirac equations is to write down an action functional. Classically this action is the integral of the difference between the kinetic and potential energy of a physical system. Hamilton's principle of least action states that the equations of motion are given for that function which minimizes the action. In general the action integral is

(6) $$A(u) = \int_M L(x,u,\partial u)\, d^m x$$

where L is a function, called the Lagrangian density, of the independent variables, $x = (x^\mu)$, the dependent variables $u = (u^\alpha)$ and the derivatives of these dependent variables $\partial u = (\partial_\mu u^\alpha)$. The integration is carried out over the m dimensional domain, M, of the independent variables $(x^\mu)$. Standard techniques from the calculus of variations imply that under suitable smoothness conditions the critical points of the action functional are functions $u = f(x)$ which satisfy the Euler-Lagrange equations [4]:



(7) $$E_\alpha(L) = \frac{\partial L}{\partial u^\alpha} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu u^\alpha)} = 0$$

The Dirac equations are the Euler Lagrange equations for the action functional with Lagrangian density

(8) $$L(\psi,\bar{\psi}) = \bar{\psi}\,(i\gamma^\mu\partial_\mu - m)\psi$$

To see this, note that (2) follows from:

(9) $$\frac{\partial L}{\partial \psi} = -\bar{\psi}m, \qquad \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \psi)}\right) = \partial_\mu\bar{\psi}(i\gamma^\mu)$$

and (1) is given by:

(10) $$\frac{\partial L}{\partial \bar{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi, \qquad \frac{\partial L}{\partial(\partial_\mu\bar{\psi})} = 0.$$

One of the advantages of the variational methods is that symmetries of the Lagrangian density correspond to conservation laws. For example if the Lagrangian is invariant under space translations, if the physics described by these equations is the same in New York as it is in Italy, then this implies that the total momentum of the system is conserved. Such invariance of the Lagrangian density follows if $L(x,u,\partial u) = L(x+c,u,\partial u)$ or equivalently if $\frac{\partial L}{\partial x} = 0$. This obviously occurs if L does not depend explicitly on x as in (7). Similarly if the Lagrangian is invariant under time translation, if the physics is the same yesterday and today, then energy must be conserved.

The precise mathematical relationship between symmetries and conservation laws was formally established by Emmy Noether in 1918 [3] and rediscovered by countless physicists in far less general settings over the next half century. We illustrate this relation by considering the invariance of the Dirac Lagrangian under a global or rigid phase shift:

(11) $$\psi \rightarrow e^{i\theta}\psi \quad \text{and} \quad \bar{\psi} \rightarrow e^{-i\theta}\bar{\psi}$$

This transformation clearly leaves the Lagrangian density invariant and shifts the phase of the solution (5) by the same amount at each point in space-time. Noether's recipe for producing a conservation law implies that the 4 vector $(\bar{\psi}\gamma^{\mu}\psi)$ has 0 divergence:

(12) $$\partial_{\mu}(\bar{\psi}\gamma^{\mu}\psi) = \left[(\partial_{\mu}\bar{\psi})\ \gamma^{\mu}\psi + \bar{\psi}\gamma^{\mu}(\partial_{\mu}\psi)\right] + \frac{1}{i}\left[\, m\bar{\psi}\psi - m\bar{\psi}\psi \,\right] = 0 + 0$$

Note that we can rewrite this equation as

(13) $$\frac{\partial}{\partial t}\bar{\psi}\gamma^{0}\psi = -\nabla\, \bar{\psi}\vec{\gamma}\psi$$

or after integrating over a large ball B in $\mathbb{R}^3$ we get



(14) $$\frac{\partial}{\partial t}\int_B \bar{\psi}\gamma^{0}\psi\, d^3x = -\int_B \nabla(\bar{\psi}\vec{\gamma}\psi)\, d^3x = \int_{\partial B} \bar{\psi}\vec{\gamma}\psi\, d^2S \longrightarrow 0$$

provided that $\psi \to 0$ sufficiently rapidly as $|x| \to \infty$, i.e., that the electron is localized in space.

Physicists interpret this integral quantity,

(15) $$\int_{\mathbb{R}^3} \bar{\psi}\gamma^{0}\psi\, d^3x = \int_{\mathbb{R}^3} \psi^{*}\psi\, d^3x$$

as the total charge of the system and thus global phase invariance implies charge conservation [2]. This motivates the definition of a conservation law as an m component vector field with 0 divergence.

If we tried to make a different change of phase at each point in space-time, i.e. if we assume that $\theta(x)$ is a (smooth) function of x, then the Lagrangian (7) is no longer invariant under this symmetry:

(16) $$e^{-i\theta}\bar{\psi}\,(i\gamma^{\mu}\partial_{\mu} - m)\, e^{i\theta}\psi = L - i\bar{\psi}m(\partial_{\mu}\theta)\psi$$

The techniques of gauge field theory calls for adding terms to the Lagrangian so that this invariance under such a local phase shift will be restored. The resultant theory for Dirac equations, quantum electrodynamics or QED, contains Maxwell's equations. We will explore some consequences of this in Section 7 below.

## Section 3: Variational complex

Information about Lagrangian densities, Euler Lagrange equations, Noether symmetries and other features of differential equations can be encoded and studied using homological methods. Basically a system of differential equations can be translated into a (larger) system of "algebraic" equations on the infinite jet bundle over the original domain. A deRham type complex can be constructed and bigraded into a bicomplex. Various groups and differentials in this bicomplex will be related to the concepts discussed above. In particular the set of conservation laws for certain Euler Lagrange equations can be identified with a cohomology group of this bicomplex.

Although this theory has been developed in great generality, for ease of exposition we will restrict our attention to a system of differential equations whose solutions will be functions from $f: \mathbb{R}^m \to \mathbb{R}^n$ which satisfy a system $\mathcal{S}$ of kth order nonlinear differential equations. For more detailed history and development of this theory see [1], [5], [6], [7], [8].

For example a solution to the first order Dirac equations can be considered to be a map

(17) $$\psi : \mathbb{R}^4 \to \mathbb{R}^8 = \mathbb{C}^4$$

which satisfies (1) and (2) or alternatively as a section of the



(trivial) vector bundle $\mathbb{R}^4 \times \mathbb{R}^8 \to \mathbb{R}^4$.

The graph of a solution to a system $\mathcal{S}$ is a subset of $\mathbb{R}^m \times \mathbb{R}^n$. The topology of the union of all graphs of solutions tends not to be very interesting; it often is the whole space. To get more topological structure we extend or prolong the differential equations to a set of "algebraic" equations on the jet bundle associated with the projection $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$.

In our simple, local coordinate, model, we may define the kth jet bundle simply as a product of euclidean spaces and choose coordinates for the jth factor as follows:

(18) $$J^k = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{mn} \times \dots \times \mathbb{R}^{mn_k} \quad \text{for } n_k = \binom{n+k-1}{k}$$

The coordinate system for jth factor can be taken to be symbols $u_I$ where $I = (i_1, i_2, \dots, i_j)$ ranges over unordered sequences of integers from 1 to m.

If $u = f(x)$ is a smooth solution to the system $\mathcal{S}$, then the graph of this function can be extended or prolonged to the jet space using the Taylor polynomial of $f$:

(19) $$pr(f)(x) = (x, f, \partial_\mu f, \dots, \partial_I f)$$

To capture the full power of this method, however, it is necessary to extend this construction to the infinite jet bundle $J^\infty$, the projective unit of $J^k$, and to prolong or formally differentiate the original system arbitrarily often. The solutions to this prolonged equation will form a subbundle of the infinite jet bundle. It is this space that we wish to study.

## Section 4: The Variational Bicomplex

Consider first the deRham complex of $J^\infty$. Using the given local coordinate system, a basis of fundamental 1-forms can be written as $dx^\mu$, $du^\alpha$ , and $du_I^\alpha$ , where $u_I$ corresponds to the appropriate higher partial derivative of u. Coefficients of these 1-forms are smooth functions, $f(x,u,\partial_\mu u,\ldots) = f[u]$, of finitely many of the variables x, u, and $u_I$. To aid in the study of this complex, we consider the bicomplex $\Omega^{p,q}(J^\infty)$ generated by differential forms

(20) $$f[u]\, du^{\ell_1}_{I_1} \wedge du^{\ell_2}_{I_2} \wedge \cdots \wedge du^{\ell_q}_{I_q} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}.$$

The deRham differential d can be written as the sum of the differentials

(21) $$d_H : \Omega^{p,q} \to \Omega^{p+1,q} \text{ and } d_V : \Omega^{p,q} \to \Omega^{p,q+1}$$

with $d_H d_V + d_V d_H = 0$.

The total cohomology of the variational bicomplex can be computed using standard spectral sequence techniques for a bicomplex [1], [5], [7]. We will illustrate only the most elementary features of this procedure below. First the cohomology with respect to the horizontal differential will be the $E_1$ term of the spectral sequence. The



computation of $d_H$ is somewhat more complicated than that of the ordinary deRham differential. Even though we consider the variables u, and $u_I$ to be independent, the formula for $d_H$ on a p,0 chain must take into account the chain rule. For example:

(22) $$d_H(u\, u_x dy) = u_x^2\, dxdy + u\, u_{xx} dxdy$$

The chain rule also comes into play in the computation of $d_H$ on $du_I$:

(23) $$d_H\,(du_I) = du_{I,x^\mu} \wedge dx^\mu$$

In the case at hand where the base and fiber are euclidean spaces, the horizontal cohomology is almost completely trivial as would be the case with ordinary deRham cohomology. The exception occurs in the (m,0) group, in the lower right hand corner of the bicomplex. Since $\Omega^{m+1,0} = 0$ for dimension reasons, all elements of $\Omega^{m,0}$ are cocycles. However the Poincare lemma does not apply in this case where the coefficients of the basic forms are functions of the "dependent" variables as well as the independent. Indeed it is not hard to compute that the (m,0) form $L[u]d^m x$ cobounds if and only if the Euler Lagrange equations (7) on L are identically 0 [4].

This fact allows us to consider elements of $E_1^{p,q} = H^{p,q}(\Omega^{*,*};d_H)$ as equivalence classes of Lagrangian forms for a given system of Euler Lagrange equations. More precisely, two (m,0) forms, $L_1 d^m x$ and $L_2 d^m x$, yield the same Euler-Lagrange equations if and only if they differ by a coboundary, i.e, if and only if there is a form $\eta = \eta_\mu d^{m-1} x^{\hat{\mu}}$

with $d\eta = (L_1 - L_2)d^m x$. This in turn means that the m component vector field $(\eta_\mu)$ has divergence equal to $L_1 - L_2$.

The vertical differential on the Lagrangian forms is computed in far more straight forward manner:

(24) $$d_V(Ld^m x) = \sum_{\alpha, I} \frac{\partial L}{\partial(u_I^\alpha)} du_I^\alpha \, d^m x$$

For example if L depends on at most the first partial derivatives of u, then

(25) $$d_V(Ld^m x) = \left[\frac{\partial L}{\partial u^\alpha} (du^\alpha) + \frac{\partial L}{\partial u_\mu^\alpha} du_\mu^\alpha \right] d^m x$$

On the other hand, one can compute that the horizontal differential on the (m-1,1) form

(26) $$d_H\left(\frac{\partial L}{\partial u^\alpha} du^\alpha d^{m-1}\hat{x^\mu}\right) = \left[\partial_\mu \frac{\partial L_\alpha}{\partial \mu_\mu} du^\alpha + \frac{\partial L_\alpha}{\partial u_\mu} du_\mu^\alpha\right] d^m x$$

In summary the Euler Lagrange equation are given essentially by the differential $d_1$ in the $E_1$ spectral sequence:

(27) $$d_V\,(Ld^m x) = E_\alpha(L)\, du^\alpha \bmod Im(d_H)$$

A more sophisticated approach to this relation between vertical differentials and the Euler Lagrange operator can be found in [1].



## Section 5: Noether's Theorem (special case)

Now consider the subset of $J^\infty$ consisting of solutions to the prolonged Euler Lagrange $\mathfrak{S}$ equations of a fixed Lagrangian $[Ld^m x]$ in $H^{m,0}(\Omega(J^\infty); d_H)$. This means that we consider the set of solutions in $J^\infty$ to $D_I(E_\alpha(L)) = 0$ for all derivatives $D_I$. Under relatively weak regularity conditions, this solution set is a submanifold of $J^\infty$ [6] [7]. We would now like to characterize some of the groups in $\Omega^{p,q}(\mathfrak{S}) = i^*\Omega^{p,q}(J^\infty)$ In particular we would like to indicate the relationship between variational symmetries of the Lagrangian, conservation laws and $\Omega^{m-1,0}(\mathfrak{S})$.

Consider first a vector field v on J of the form $v = \Sigma\, Q_\alpha[u]\, \partial u^\alpha$. The prolongation of this vector field to $J^\infty$ is the sum

(28) $$pr(v) = \Sigma\, D_I Q_\alpha\, \partial u_I^\alpha$$

which has only finitely many non zero summands since Q[u] depends on only finitely many variables. This vector field is called a variational symmetry [4] of the Lagrangian L if

$$pr(v)(L) = 0.$$

For example if $v = \theta\psi\partial\psi - \theta\bar{\psi}\partial\bar{\psi}$ for $\theta$ constant then

(29) $$pr(v) = v + \theta\psi_\mu \partial\psi_\mu \cdots + \theta\bar{\psi}_\mu \partial\bar{\psi}_\mu \cdots$$

and $pr(v)(L) = L - L = 0$.

Equivalence classes of such symmetries can be shown to be in one to one correspondence with cohomology classes of degree $(m-1,0)$, provided the equations are totally nondegenerate in the sense that they and their prolongations are of maximal rank and that they are locally solvable [4]. This class of differential equations includes those of Cauchy Kovalevskaya type. We illustrate this correspondence for the special case where L is a function of u and its first partial derivatives only, i.e., L is a classical Lagrangian. In this case

(30) $$pr(v)(L) = Q_\alpha \frac{\partial L}{\partial u^\alpha} + \partial_\mu Q_\alpha \left(\frac{\partial L}{\partial u^\alpha_\mu}\right)$$

We claim that the cochain

(31) $$Q_\alpha \frac{\partial L}{\partial u^\alpha_\mu} d^{m-1}x^{\hat{\mu}}$$

is a $d_H$ cocycle:

(32) $$d_H \left(Q_\alpha \frac{\partial L}{\partial u^\alpha_\mu} d^{m-1}x^{\hat{\mu}}\right) = \left(\partial_\mu Q_\alpha \frac{\partial L}{\partial u^\alpha_\mu} + Q_\alpha \partial_\mu \frac{\partial L}{\partial u^\alpha_\mu}\right) d^m x$$

$$= (pr(v)(L) - Q_\alpha E_\alpha(L)) = 0$$

In the analysis above, we have interpreted elements and



differentials in the lower right corner of the bicomplex in terms of Lagrangians, Euler-Lagrange equations and conservation laws. A remarkable result of Vinogradov [7] (see also [1]) shows that for totally nondegenerate differential equations as defined above, the cohomology groups, with respect to the horizontal differential, vanish except for the last two columns, provided the base and the fiber are contractible.

## Section 6 Maxwell Equations

Consider first the classical Maxwell equations for the vector fields E and B from Minkowski space time to space.

(33) $$\nabla . B = 0 \qquad \nabla \times E - \frac{\partial B}{\partial t} = 0$$
$$\nabla . E = \rho \qquad \nabla \times B + \frac{\partial E}{\partial t} = \vec{J}$$

It has long been known that these equations can be studied in potential form using the language of connections and differential forms. Assume that

$$\alpha = A_\mu dx^\mu$$

is an arbitrary smooth 1 form on flat Minkowski space M. Alternatively we may consider the related 4 component vector field $(A_\mu): M \longrightarrow R^4$. Then the 2-form

(34) $$d\alpha = F_{\mu\nu} dx^\mu dx^\nu$$

is called the force or curvature form. Indeed if we set

(35) $$B = (F_{23}, F_{31}, F_{12})$$
$$E = (F_{01}, F_{02}, F_{03})$$

and $J = (\rho, \vec{j}) = \rho dx^0 + \vec{\jmath} d\vec{x}$

then the equations $dd\,\alpha = 0$ and $d{*}d\,\alpha = J$ are precisely the Maxwell equations above. If $J = 0$ then we say that the equations are sourceless.

A Lagrangian for the sourceless Maxwell equations $\mathcal{M}ax$ is

(36) $$L(A_\mu)\, d^m x = -1/4\, F_{\mu\nu}F^{\mu\nu} = d\,\alpha \wedge {*}d\,\alpha.$$

The Euler Lagrange equations read

(37) $$E_\nu L(\mathcal{M}ax) = \partial_\mu F_{\mu\nu} = 0$$

or $$d{*}d\alpha = 0$$

Consider now the variational bicomplex for the sourceless Maxwell equations $\Omega^{p,q}(\mathcal{M}ax)$ Then the Noether conservation law associated with the symmetry $A_\mu \longrightarrow A_\mu - (\partial_\mu \theta)$ is the 3 form

(38) $$d\theta\wedge{*}\alpha d\alpha = d(\theta{*}\alpha d\alpha) - \theta(d{*}d\alpha)$$

$$\in \mathrm{Im} d_H \subset \Omega^{3,0}(\mathcal{M}ax)$$



Our recipe produces a coboundary and thus a trivial conservation law.

Note that ${*}d\alpha$ is a cocycle in $\Omega^{2,0}(\mathcal{M}ax)$. This represents a non zero cohomology class related to the first Chern class and thus appears to contradict the two line theorem above [1]. However, the Maxwell's equations are underdetermined in the sense that there is a non trivial algebraic relationship among the derivatives of these equations

(39) $$\partial^\nu E_\mu(L) = 0$$

Indeed Noether's second theorem (see [3] [4]) implies that such a relationship exists whenever a system of differential equations admits a family of symmetries, in particular whenever a theory has a local gauge invariance. Note that the complete computation of the cohomology of these equations is still an open problem.

Section 7: QED

In one of the major successes of physics in this half century, Maxwell's equations and Dirac's equations have been combined into a remarkably successful theory called quantum electrodynamics, [2]. As noted in section 2, the Dirac Lagrangian is not invariant under local gauge transformations. The error term produced can be cancelled by including more variables in a process called minimal coupling. If we add the interaction term

(40) $$L_{\mathcal{I}nt} = \bar{\psi}\gamma^\mu A_\mu \psi$$

to the Dirac Lagrangian, then
$v = \theta\psi\,\partial_\psi - \theta\bar{\psi}\partial_{\bar{\psi}} + (\partial_\mu\theta)\,\partial_{A_\mu}$ is a variational symmetry:

(41) $$\mathrm{pr}(v)\,(L_{\mathcal{D}irac} + L_{\mathcal{I}nt}) = 0$$

and so invariance is restored. As this term is reminiscent of Maxwell's equations, we add the gauge invariant Maxwell Lagrangian to get the full QED Lagrangian:

(42) $$L_{\mathcal{QED}} = \bar{\psi}(i\gamma^\nu(\partial_\nu + A_\mu) - m)\,\psi - {}^1/_4 F_{\mu\nu}F^{\mu\nu}$$

The Euler Lagrange equations are

$$(i\gamma^\mu(\partial_\mu + A_\mu) - m)\,\psi = 0\ ,$$

its Dirac conjugate, and

(43) $$\bar{\psi}\gamma^\mu A_\mu\psi - {*d\alpha} = 0,$$

i.e. the "covariant" Dirac equations and the Maxwell equations with a source term.

The gauge symmetry vector field (41) produces a 3 cocycle

(44) $\theta\ \bar{\psi}\gamma^\mu A_\mu\psi d^3x^{\hat{\mu}} - d\theta\wedge{*d\alpha} = d(\theta{*d\alpha})$ modulo equations (43).

This indicates that the conservation law (44) is cohomologically



trivial.

This result suggests that one has a long exact sequence of cohomology theories

(45) $$H^{2,0}(\mathcal{QED}) \to \underset{[*d\alpha]}{H^{2,0}(\mathcal{M}ax)} \to \underset{[\bar{\psi}\gamma^\mu\psi]}{H^{3,0}(\mathcal{D}irac)} \to H^{3,0}(\mathcal{QED}) \to$$

In this sequence the connecting homomorphism will send the "Chern" class [*dα] to the classical Dirac current.

One could conjecture that similar long exact sequences relate the cohomology of quantum chromodynamics with the cohomology of quarks and gluons. A precise formulation of such a conjecture would involve a detailed discussion of nonabelian gauge field theories.

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