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AN INTRODUCTION TO CRITICAL POINT THEORY

(Minimax methods and periodic solutions of Hamilton systems)

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Introduction.

The purpose of these lectures is to study equation

(1) $$\varphi'(u) = 0$$

where φ is a continuously differentiable real function defined on a real Banach space X. The solutions of (1) are the critical points of φ. We shall describe a minimax approach which can be used to determine critical points of φ. The basic idea is to introduce a family $\mathcal{A}$ of subsets of X and to prove that

$$c = \inf_{A \in \mathcal{A}} \sup_{u \in A} \varphi(u)$$

is a critical value of φ. By definition, c is a critical value of φ if $\varphi^{-1}(c)$ contains at least one critical point of φ.

The construction of critical points depends on compactness conditions which correspond to a priori estimates in continuation methods. In order to avoid technicalities, applications will be made to boundary value problems for ordinary differential equations.

In § I, we describe compactness conditions and our basic tool, the deformation lemma due to Clark.

Section II contains two elementary applications of the minimax method: the Ambrosetti-Rabinowitz mountain pass theorem and the Rabinowitz saddle point theorem.

In § III, we assume that φ is invariant under the action of a discrete subgroup of X. A strong multiplicity result is obtained by using Lusternik-Schnirelmann category. This result is applied to pendulum-like forced Lagrangian systems.

In § IV, we present the dual least action principle introduced by Clarke in order to study periodic orbits of convex Hamiltonian systems.

In § V, we prove a strong multiplicity result for functionals which are invariant under the action of S^1. The basic tool is the S^1-index defined by Benci. This section contains a proof of the Ekeland-Lasry theorem on the existence of multiple periodic orbits on a convex energy surface.

Finally, in § VI we consider the non-linear eigenvalue problem

$$\varphi'(u) = \mu \chi'(u), \qquad \chi(u) = a,$$

i.e. the problem of finding critical points of φ on the manifold

$$Z = \{u : \chi(u) = a\}.$$

Applications are made to bifurcation theory and to the local results of Weinstein and Moser on periodic orbits of Hamiltonian systems near an equilibrium.

I. The deformation lemma.

We shall relate, by a deformation argument, the existence of critical values of φ to the change of topology of the sublevel sets

$$\varphi^c = \{u \in X : \varphi(u) \leq c\}.$$

Below X always denotes a real Banach space and $\varphi \in C^1(X, \mathbb{R})$. By definition φ satisfies the Palais-Smale condition (P.S.) if each sequence (u_j) such that $\varphi(u_j)$ is bounded and $\varphi'(u_j) \to 0$ contains a convergent subsequence.

Remarks. 1. (PS) is a compactness condition on φ which replaces the compactness of the manifold in the classical Lusternik-Schnirelmann theory.

2. It is, in general, easier to satisfy (PS) than to find a priori bounds for all possible solutions of $\varphi'(u) = 0$ since, in (PS), $\varphi(u_j)$ has to be bounded.

3. It follows immediately from (PS) that, for each $c \in \mathbb{R}$, the set

$$K_c = \{u \in X : \varphi'(u) = 0 \text{ and } \varphi(u) = c\}$$

is compact.

The basic deformation result is the following:

Lemma 1. If $\varphi \in C^1(X,\mathbb{R})$ satisfies (PS) and if U is an open neighborhood of K_c, then, for every $\bar\varepsilon > 0$, there exists $\varepsilon \in]0,\bar\varepsilon[$ and $\eta \in C([0,1]\times X, X)$ such that

(a) $\eta(0,u) = u$ for all $u \in X$.

(b) $\eta(1, \varphi^{c+\varepsilon} \setminus U) \subset \varphi^{c-\varepsilon}$.

(c) $\eta(t,u) = u$ if $\varphi(u) \notin [c-\bar\varepsilon, c+\bar\varepsilon]$.

Proof. We consider the special case of X Hilbert space and $\varphi \in C^2(X,\mathbb{R})$.

If $\bar\varepsilon > 0$ is given, there is a $\varepsilon \in]0, \bar\varepsilon/2[$ such that, if $u \in \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap (\complement U)_{2\sqrt{\varepsilon}}$, then

(2) $$\|\nabla\varphi(u)\| \geq 2\sqrt{\varepsilon}.$$

Indeed, if it is not the case, there is a sequence (u_k) such that

$$u_k \in (\complement U)_{2/\sqrt{k}}, \quad c - \frac{2}{k} \leq \varphi(u_k) \leq c + \frac{2}{k}, \quad \|\nabla\varphi(u_k)\| < \frac{4}{\sqrt{k}}$$

and, by (PS), we can assume, going if necessary to a subsequence, that $u_k \to u$. But then

$$u \in \complement U, \quad \varphi(u) = c, \quad \nabla\varphi(u) = 0,$$

a contradiction.

Let us define

$$A = \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap (\complement U)_{2\sqrt{\varepsilon}},$$

$$B = \varphi^{-1}([c-\varepsilon, c+\varepsilon]) \cap (\complement U)_{\sqrt{\varepsilon}},$$

$$\psi(u) = \frac{\operatorname{dist}(u, \complement A)}{\operatorname{dist}(u, \complement A) + \operatorname{dist}(u, B)}$$

so that $0 \leq \psi(u) \leq 1$, $\psi(u) = 1$ in B and $\psi(u) = 0$ in $\complement A$.

Define the locally Lipschitz vector field f on X by

$$f(u) = -\psi(u)\nabla\varphi(u)/\|\nabla\varphi(u)\| \quad , \text{if } u \in A,$$
$$= 0 \qquad\qquad\qquad\qquad , \text{if } u \notin A.$$

As f is bounded on X, the Cauchy problem

$$\dot\sigma = f(\sigma)$$
$$\sigma(0) = u$$

has, for each $u \in X$, a unique solution $\sigma(\cdot, u)$ defined on $[0, +\infty[$, with $\sigma(\cdot,\cdot)$ continuous. Let us define $\eta \in C([0,1]\times X, X)$ by

$$\eta(t,u) = \sigma(\sqrt{\varepsilon}\, t, u).$$

For $t \geq 0$, we have

$$\|\sigma(t,u) - u\| = \left\|\int_0^t f(\sigma(\tau,u))\,d\tau\right\| \leq \int_0^t \|f(\sigma(\tau,u))\|\,d\tau \leq t,$$

so that $\sigma(t, \beta U) \subset (\beta U)_{\sqrt{\varepsilon}}$ for $t \in [0, \sqrt{\varepsilon}]$.

We have for all $u \in X$ and $t \geq 0$

$$\frac{d}{dt}\varphi(\sigma(t,u)) = \langle \nabla\varphi(\sigma(t,u)), f(\sigma(t,u))\rangle$$

$$= -\psi(\sigma(t,u))\,\|\nabla\varphi(\sigma(t,u))\| \leq 0.$$

Let $u \in \varphi^{c+\varepsilon} \setminus U$. If $\varphi(\sigma(t,u)) < c - \varepsilon$ for some $t \in [0, \sqrt{\varepsilon}[$, then $\varphi(\sigma(\sqrt{\varepsilon}, u)) < c - \varepsilon$ and $\eta(1,u) \in \varphi^{c-\varepsilon}$. If not, then $\sigma(t,u) \in B$ for all $t \in [0, \sqrt{\varepsilon}]$. By the definition of f and (2), we obtain

$$\varphi(\sigma(\sqrt{\varepsilon}, u)) = \varphi(u) + \int_0^{\sqrt{\varepsilon}} \frac{d}{dt}\varphi(\sigma(t,u))\,dt$$

$$= \varphi(u) + \int_0^{\sqrt{\varepsilon}} (\nabla\varphi(\sigma(t,u)), f(\sigma(t,u)))\,dt$$

$$= \varphi(u) - \int_0^{\sqrt{\varepsilon}} \|\nabla\varphi(\sigma(t,u))\|\,dt$$

$$\leq c + \varepsilon - 2\varepsilon = c - \varepsilon.$$

Thus $\eta(1,u) \in \varphi^{c-\varepsilon}$ and hence $\eta(1, \varphi^{c+\varepsilon}\setminus U) \subset \varphi^{c-\varepsilon}$. Clearly (a) and (b) are also satisfied. □

Remark. In the general case, the proof of lemma 1 depends on the construction of a "pseudo-gradient" vector field (see [4], [5]), introduced by Palais.

Let us give a simple but fundamental application of lemma 1.

Theorem 1. Assume that $\varphi \in \mathcal{C}^1(X,\mathbb{R})$ satisfies (PS) and is bounded from below. Then $c = \inf_X \varphi$ is a critical value of φ.

Proof. If $K_c = \emptyset$, then lemma 1 applied to $U = \emptyset$ and $\bar\varepsilon = 1$ implies the existence of $\varepsilon \in]0,1[$ and $\eta \in \mathcal{C}([0,1]\times X, X)$ such that $\eta(1, \varphi^{c+\varepsilon}) \subset \varphi^{c-\varepsilon}$. But $\varphi^{c-\varepsilon} = \emptyset$ by the definition of c. This contradiction gives the result. □

We shall apply theorem 1 to a variational problem with periodic boundary conditions.

Let $\mathcal{C}^\infty_T$ be the space of indefinitely differentiable T-periodic functions from $\mathbb{R}$ into $\mathbb{R}^N$.

Fundamental lemma [ ]. Let $u, v \in L^1(0,T;\mathbb{R}^N)$. If, for every $f \in \mathcal{C}^\infty_T$,

(3) $$\int_0^T (u(t), f'(t))\,dt = -\int_0^T (v(t), f(t))\,dt,$$

then

(4) $$\int_0^T v(s)\,ds = 0$$

and there exists $c \in \mathbb{R}^N$ such that, a.e. on $[0,T]$,

$$u(t) = \int_0^t v(s)\,ds + c.$$

Remarks. 1) A function v satisfying (1) is called a weak derivative of u. By a Fourier series argument, the weak derivative, if it exists, is unique. The weak derivative of u will be denoted by $\dot u$.

2) By the fundamental lemma

$$u(t) = \int_0^t \dot u(s)\, ds + c,$$

a. e. on $[0,T]$. As usual, we shall identify the equivalence class u and its continuous representant

$$\tilde u(t) = \int_0^t \dot u(s)\, ds + c.$$

In particular, by (4), $u(0) = u(T) = c$, and

$$u(t) - u(\tau) = \int_\tau^t \dot u(s)\, ds$$

for $t, \tau \in [0,T]$.

3) If $\dot u$ is continuous, then $\dot u$ is the classical derivative of u everywhere on $[0,T]$.

4) By a classical result of Lebesgue theory, $\dot u$ is the classical derivative of u a. e. on $[0,T]$.

Let $1 < p < \infty$. The Sobolev space $W_T^{1,p}$ is the space of functions

$$u \in L^p(0,T;\mathbb{R}^N)$$

having a weak derivative

$$\dot u \in L^p(0,T;\mathbb{R}^N).$$

Let us recall that, if $u \in W_T^{1,p}$, then

$$u(t) = \int_0^t \dot u(s)\, ds + c$$

and $u(0) = u(T)$. The norm over $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}} = \left(\int_0^T |u(t)|^p\, dt + \int_0^T |\dot u(t)|^p\, dt \right)^{1/p}.$$

It is easy to verify that $W^{1,p}_T$ is a reflexive Banach space and that $C^\infty_T \subset W^{1,p}_T$.

We shall denote by H^1_T the Hilbert space $W^{1,2}_T$ with the inner product

$$((u, v)) = \int_0^T [(u(t), v(t)) + (\dot{u}(t), \dot{v}(t))]\, dt$$

and the corresponding norm $\|u\| = \|u\|_{W^{1,2}_T}$.

Let us recall that

$$\|u\|_{L^p} = \left(\int_0^T |u(t)|^p\, dt \right)^{1/p} \text{ and } \|u\|_\infty = \max_{t \in [0,T]} |u(t)|.$$

The following result is an easy consequence of the mean value theorem and Hölder inequality or Parseval equality.

Proposition 1. There exists $c > 0$ such that, if $u \in W^{1,p}_T$, then

$$\|u\|_\infty \le c \|u\|_{W^{1,p}_T}.$$

Moreover, if $\int_0^T u(t)\, dt = 0$, then

$$\|u\|_\infty \le c \|\dot{u}\|_{L^p}.$$

If $u \in H^1_T$ and $\int_0^T u(t)\, dt = 0$, then

$$\int_0^T |u(t)|^2\, dt \le \frac{T^2}{4\pi^2} \int_0^T |\dot{u}(t)|^2\, dt,$$ (Wirtinger's inequality)

and

$$\|u\|_\infty^2 \le \frac{T}{12} \int_0^T |\dot{u}(t)|^2\, dt,$$ (Sobolev inequality).

Proposition 2. If the sequence (u_k) converges weakly to u in $W_T^{1,p}$ then (u_k) converges uniformly to u on $[0,T]$.

Proof. By proposition 1, the injection of $W_T^{1,p}$ into $\mathcal{C}([0,T];\mathbb{R}^N)$ with its natural norm $\|\cdot\|_\infty$ is continuous. Since $u_k \rightharpoonup u$ in $W_T^{1,p}$, it follows that $u_k \rightharpoonup u$ in $\mathcal{C}([0,T];\mathbb{R}^N)$. By the Banach-Steinhaus theorem, (u_k) is bounded in $W_T^{1,p}$ and, hence, in in $\mathcal{C}([0,T];\mathbb{R}^N)$. Moreover the sequence (u_k) is equi-uniformly continuous since, for $0 \le s \le t \le T$, we have

$$|u_k(t) - u_k(s)| \le \int_s^t |\dot u_k(\tau)|\, d\tau$$

$$\le (t-s)^{1/q} \left(\int_s^t |\dot u_k(\tau)|^p d\tau\right)^{1/p}$$

$$\le (t-s)^{1/q} \|u_k\|_{W_T^{1,p}}$$

$$\le C(t-s)^{1/q}.$$

By Ascoli-Arzelà theorem, (u_k) is relatively compact in $\mathcal{C}([0,T];\mathbb{R}^N)$. By the uniqueness of the weak limit in $\mathcal{C}([0,T];\mathbb{R}^N)$, every uniformly convergent subsequence of (u_k) converges to u. Thus u_k converges uniformly on $[0,T]$ to u. $\square$

As examples of applications to differential equations, let us consider the following problems:

a) Lagrangian systems.

i) Classical formulation:

$$\begin{cases} \dfrac{d}{dt} D_y L(t, u(t), \dot{u}(t)) = D_x L(t, u(t), \dot{u}(t)), \\ u(0) = u(T). \end{cases}$$

when $L(t,x,y)$, $D_x L(t,x,y)$ and $D_y L(t,x,y)$ are continuous on $[0,T] \times \mathbb{R}^N \times \mathbb{R}^N$.

ii) Integral formulation:

$$\begin{cases} D_y L(t, u(t), \dot{u}(t)) = \int_0^t D_x L(s, u(s), \dot{u}(s))\, ds + c, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. on } [0,T], \\ u(0) = u(T). \end{cases}$$

iii) Weak formulation:

Find $u \in W_T^{1,p}$ such that, for every $f \in \mathcal{C}_T^\infty$,

$$\int_0^T (D_y L(t, u(t), \dot{u}(t)), \dot{f}(t))\, dt = -\int_0^T (D_x L(t, u(t), \dot{u}(t)), f(t))\, dt.$$

iv) Variational formulation:

Find a critical point $u \in W_T^{1,p}$ of the functional φ defined on $W_T^{1,p}$ by

$$\varphi(u) = \int_0^T L(t, u(t), \dot{u}(t))\, dt.$$

b) Second order systems.

If the lagrangian L is given by
$$L(t,x,y) = \frac{|y|^2}{2} - V(t,x),$$
the corresponding classical formulation is
$$\begin{cases} \ddot{u}(t) + D_x V(t,u(t)) = 0, \\ u(0) = u(T), \ \dot{u}(0) = \dot{u}(T). \end{cases}$$

c) Hamiltonian systems.

Let
$$J = \begin{bmatrix} 0 & id_{\mathbb{R}^N} \\ -id_{\mathbb{R}^N} & \end{bmatrix}$$
be the standard symplectic matrix. If the lagrangian is given by
$$L(t,x,y) = (Jy, x) + H(t,x),$$
the corresponding classical formulation is
$$\begin{cases} J\dot{u}(t) + D_x H(t,u(t)) = 0, \\ u(0) = u(T). \end{cases}$$

In order to apply critical point theory to the variational problem, we need a sufficient condition for the differentiability of φ on $W_T^{1,p}$.

Proposition 3. Let L, D_xL, D_yL be continuous on $[0,T]\times\mathbb{R}^N\times\mathbb{R}^N$. If there exists $a \in \mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$ and $1<p<\infty$ such that
$$|L(t,x,y)| \le a(|x|)(1+|y|^p),$$
$$|D_xL(t,x,y)| \le a(|x|)(1+|y|^p),$$
$$|D_yL(t,x,y)| \le a(|x|)(1+|y|^{p-1}),$$
then φ is continuously differentiable on $W_T^{1,p}$ and

5) $$\langle \varphi'(u), v\rangle = \int_0^T [(D_xL(t,u(t),\dot{u}(t)), v(t)) + (D_yL(t,u(t),\dot{u}(t)), \dot{v}(t))]\,dt.$$

Sketch of proof. It suffices to prove that φ has at every point u a directional derivative $\varphi'(u) \in (W^{1,p}_T)^*$ given by (5) and that the mapping

$$\varphi' : W^{1,p}_T \to (W^{1,p}_T)^* : u \mapsto \varphi'(u)$$

is continuous. The first part follows from Leibnitz formula of differentiation under integral sign and the second from a theorem of Krasnosel'skii. See [1] or [4] for a complete proof. □

Let us now consider the problem

$$(6) \qquad \begin{cases} \ddot{u}(t) + D_x V(t, u(t)) = 0 \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases}$$

in the case when $D_x V$ is bounded on $[0,T] \times \mathbb{R}^N$. In general the problem has no solution. But it suffices to require a coercivity condition for the average of V with respect to t in order to obtain a solution by minimization.

Theorem 2. Assume that V and $D_x V$ are continuous on $[0,T] \times \mathbb{R}^N$ and that there exists a constant $a > 0$ such that

$$|D_x V(t,x)| \leq a$$

on $[0,T] \times \mathbb{R}^N$. If

$$(7) \qquad \int_0^T V(t,x)\, dt \to -\infty, \text{ as } |x| \to \infty,$$

the problem (6) has at least one solution which minimizes φ on H^1_T.

<u>Proof.</u> We shall apply Theorem 1.

1) <u>φ is bounded from below.</u>

Setting

$$u = \bar{u} + \tilde{u} \quad , \quad \bar{u} = \frac{1}{T}\int_0^T u(t)\,dt$$

we have, for $u \in H^1_T$, using Sobolev inequality,

$$(8) \quad \varphi(u) = \int_0^T \left[\frac{|\dot{u}(t)|^2}{2} - V(t,u(t))\right] dt$$

$$= \|\dot{u}\|^2_{L^2}/2 - \int_0^T V(t,\bar{u})\,dt - \int_0^T\int_0^1 (D_2V(t,\bar{u}+s\tilde{u}(t)),\tilde{u}(t))\,ds\,dt$$

$$\geq \|\dot{u}\|^2_{L^2}/2 - \int_0^T V(t,\bar{u})\,dt - T\,a\,\|\tilde{u}\|_\infty$$

$$\geq \|\dot{u}\|^2_{L^2}/2 - C\,\|\dot{u}\|_{L^2} - \int_0^T V(t,\bar{u})\,dt\,.$$

Since, by assumption $\int_0^T V(t,\bar{u})\,dt$ is bounded from above, φ is bounded from below.

2) <u>φ satisfies the (PS) condition.</u>

Let u_k be a sequence in H^1_T such that $\varphi(u_k)$ is bounded and $\varphi'(u_k) \to 0$ as $k \to \infty$. Because of (8) and assumption (7), the sequence (u_k) is bounded in H^1_T. Going if necessary to a subsequence, we can assume that $u_k \rightharpoonup u$ in H^1_T and $u_k \to u$ uniformly on $[0,T]$. But then

$$(9) \quad \langle \varphi'(u_k) - \varphi'(u), u_k - u\rangle \to 0 \text{ as } k \to \infty.$$

Using proposition 3, we have

$$\langle \varphi'(u_k) - \varphi'(u), u_k - u\rangle = \int_0^T |\dot{u}_k(t) - \dot{u}(t)|^2\,dt + \int_0^T (D_2V(t,u_k(t)) - D_2V(t,u(t)), u_k(t)-u(t))\,dt$$

It is then easy, using (9), to verify that $\|\dot{u}_k - \dot{u}\|_{L^2} \to 0$, and hence that $u_k \to u$ in H^1_T.

3) <u>End of the proof.</u>

By theorem 1, $c = \inf \varphi$ is a critical value of φ, i.e. there exists $u \in H^1_T$ such that $\varphi'(u) = 0$, $\varphi(u) = \inf \varphi$.

By proposition 3, we have

$$0 = \langle \varphi'(u), v\rangle = -\int_0^T (D_x V(t, u(t)), v(t))\,dt + \int_0^T (\dot u(t), \dot v(t))\,dt$$

for all $v \in H^1_T$ and hence for all $v \in \mathcal{C}^\infty_T$. In particular $D_x V(t, u(t))$ is the weak derivation of $\dot u$. But $D_x V(t, u(t))$ is continuous, so that we obtain

$$\ddot u(t) = D_x V(t, u(t))$$

on $[0,T]$. Since $u \in H^1_T$, $u(0) = u(T)$. Since $\dot u$ has a weak derivation, $\dot u(0) = \dot u(T)$. $\blacksquare$

<u>Remarks</u>. 1. It is interesting to note that $u(0) = u(T)$ is an <u>essential boundary condition</u> and $\dot u(0) = \dot u(T)$ is a <u>natural boundary condition</u>.

2. If

$$\int_0^T V(t,x)\,dt \to +\infty \quad \text{as } |x| \to \infty,$$

then φ is neither bounded from below, nor from above. We shall consider this interesting situation in the next section.

3. The existence of critical point by minimizations follows also from Ekeland variational principle or from the direct method of the calculus of variations (see [1], [3], [4], [6]).

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