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A NEW APPROACH TO THE MORSE-CONLEY THEORY

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Introduction

In this note we present a new approach to the Morse theory which is based on a generalization of the Conley index to non locally compact spaces. The variant of the Morse theory which we obtain seems suitable for applications to nonlinear functional analysis. We refer to a paper in preparation [B] for some of such applications.

Since we are not well acquainted with the very extensive literature on Morse theory, we did not attempt to provide a listing even of the main papers on this subject. We apologize for this to the readers and to the authors.

§ 1 - The homotopic index.

Let M be a metric space on which a semiflow η is defined i.e. a (continuous) map

$$\eta : \mathbb{R}^+ \times M \longrightarrow M$$

such that $\eta(0,x) = x$ and $\eta(t_1,\eta(t_2,x)) = \eta(t_1+t_2,x)$; $(t_1,t_2 \in \mathbb{R}^+,\ x \in M)$.

When no ambiguity is possible we will write $x \cdot t$ instead of $\eta(t,x)$.

A semiflow which is defined for every $t \in \mathbb{R}$ is called a flow. If X is any subset of M and T a positive constant we set

$$G^T(X) = G^T(X,\eta) = \{x \in M \mid x\cdot[0,T] \subset M\} \cap \{\bigcup_{t \geq 0} \eta(t,X)\}$$

If η is a flow, clearly we have

$$G^T(X) = \{x \in M \mid x\cdot[-T,T] \subset X\} = \bigcap_{t \in [-T,T]} \eta(t,X).$$

Also we set

$$\Sigma = \Sigma(\eta) = \{X \subset M \mid X \text{ is closed and } \exists\, T > 0 \text{ s.t. } G^T(X,\eta) \subset \mathring{X}\}$$

where $\mathring{X}$ denotes the interior of X.

<u>Def.1.1</u> A pair of closed subset of X, (N,N_0) with $N_0 \subset N$ is called <u>index pair</u> if

(i) $G^T(\overline{N-N_0}) \subset N-N_0$ ~~[illegible] ($\bar{A}$ denotes the closure of A)~~

(ii) N_0 is positively invariant with respect to N (i.e. $x \in N_0$ and $x\cdot[0,t] \subset N \Rightarrow x\cdot[0,t] \subset N_0$)

(iii) N_0 is an <u>exit</u> set for N (i.e. $x \in N$ and $x\cdot[0,t] \not\subset N \Rightarrow \exists\, t^* \in [0,t]$ such that $x\cdot t^* \in N_0$)

We say that (N,N_0) is an index pair for $X \in \Sigma$ if

(iv) $\overline{N-N_0} \subset X$ and there exists $T > 0$ such that $G^T(X) \subset \overline{N-N_0}$.

Now it is necessary to recall some concepts from the homotopy theory.
If X is a topological space and A is a closed subset then X/A denotes the spaces obtained by X identifing all the points of A.
Two spaces X/A and Y/B are called homotopic equivalent if there are maps $\phi : X/A \to Y/B$ and $\psi : Y/B \to X/A$ such that $\phi([A]) = [B]$; $\psi([B]) = [A]$ and such that $\phi \circ \psi$ and $\psi \circ \phi$ are homotopic to the identity by homotopies which leave the points $[A]$ and $[B]$ fixed respectively.
The class of all spaces homotopically equivalent to X/A is called homotopy type of X/A and denoted by $[X/A]$.
The homotopy type of X/X is denoted by $\underline{0}$; if X is a contractible space, the homotopy type of X/ϕ is denoted by $\underline{1}$. Morever, by convenction, we set $\phi/\phi = \underline{0}$.

<u>Def. 1.2</u> For $X \in \Sigma$, the homotopy index of X is the homotopy type of an index pair (N,N_0) relative to X; in formula we write

$$h(X) = h(X,\eta) = [N/N_0]$$

The definition 1.2 makes sense if we prove that

(1-1)
(a) $\forall\, X \in \Sigma$ there exists an index pair (N,N_0) for X

(b) if (N,N_0) and $(\tilde{N},\tilde{N}_0)$ are two index pairs relative to X, then $[N/N_0] = [\tilde{N}/\tilde{N}_0]$

In order to prove(1-1) some work is necessary. First, we need an other notation; for $T > 0$ we set

(1-2) $\Gamma^T(X) = \Gamma^T(X,\eta) = \{ x \in G^T(X,\eta) \mid x\cdot[0,T] \cap \partial X \neq \emptyset \}$

We need now a technical lemma:

Lemma 1.3 Suppose that $X, Y \in \Sigma$; then

(i) $X \subset Y \to G^T(X) \subset G^T(Y)$ for every $T > 0$

(ii) $T_1 > T > 0 \quad G^{T_1}(X) \subset G^T(X)$

(iii) $G^{T_1+T_2}(X) = G^{T_1}(G^{T_2}(X))$

(iv) if $G^T(X) \subset \mathring{X}$ then $G^{2T}(X) \subset \text{int}\,[G^T(X)]$

(v) $G^T(X)$ is closed

(vi) if $X \in \Sigma$ then $G^T(X)$ and $\eta(t,X) \in \Sigma$

(vii) $\Gamma^T(X)$ is closed

(viii) $\Gamma^T(X) \subset \partial G^T(X)$

Proof (i),(ii) and (iii) follow easily from the definition of $G^T(X)$.

(iv) First of all observe that if

(1-3) $x \in G^T(X)$, then $x\cdot t$ is defined for $t \in [-T,T]$ i.e. we can go back in time up to the point $-T$; and this by the definition of $G^T(X)$.

In order to prove (iv) we arge indirectly and we suppose that there exists $y \in G^{2T}(X) \cap \partial G^T(X)$. Then there exists a sequence $y_n \to y$ such that $y_n\cdot[-T,T] \not\subset X$. This implies that there exist times $t_n \in [-T,T]$ such that $y_n\cdot t_n \notin X$; we can extract a sequence t_n' such that $t_n' \to \bar{t}$, so we have that $y_n\cdot t_n' \to y\cdot\bar{t} \notin \partial X$. Since $y \in G^{2T}(X)$, $y\cdot[-2T,2T] \subset X$ and so $y\cdot\bar{t} \in G^T(X)$ (since $|\bar{t}| \le T$).

And this contradicts our assumption that $G^T(X) \cap \partial X = \emptyset$.

(v) We have $G^T(X) = \{ \bigcap_{t\in[0,T]} \eta(t,X)\} \cap \{x \in X \mid x\cdot[0,T] \subset X\}$

The first set of above formula is closed since for every $t > 0$ $\eta(t,X)$ is closed. If we set $A_t = \eta(t,\cdot)$, then $\{ x \in X \mid x\cdot[0,T] \subset X\} = \bigcap_{t\in[0,T]} A_t^{-1}(X)$.

So also this set is closed. Therefore $G^T(X)$ is closed.

(vi) $G^T(X) \in \Sigma$ by (iv). $\eta(t,X) \in \Sigma$ by the continuity of η.

(vii) Let $\{x_n\} \subset \Gamma^T(X)$ with $x_n \to \bar{x}$. Then there exists $t_n \in [0,T]$ such that $x_n\cdot t_n \in \partial X$. Let t_n' be a subsequence of t_n converging to some $\bar{t} \in [0,T]$; then $x_n\cdot t_n' \to \bar{x}\cdot\bar{t} \in \partial X$. Therefore $\bar{x} \in \Gamma^T(X)$.

(viii) Let $x \in \Gamma^T(X)$; then $\exists\, t \in [0,T]$ such that $x\cdot t \in \partial X$; thus there exists a sequence $y_n \in X^c$ (X^c denotes the complement of X in M) converging to $x\cdot t$. This implies that $y_n(-t) \to x$. But $y_n(-t) \in G^T(X)$, therefore $x \in \partial G^T(X)$.

Now we can prove (1.1) a).

Theorem 1.4 (Existence of index pair). Let $X \in \Sigma$ and let T be large enough that $G^T(X) \subset \mathring{X}$. Then

$$(G^T(X), \Gamma^T(X))$$

is an index pair for X.

Proof By lemma 1.3 (vi),(vii), $G^T(X)$ and $\Gamma^T(X)$ are closed. We have to check points (i),(ii) and (iii) of Def.1.2.

(i) By lemma 1.2 (viii), $\overline{G^T(X) - \Gamma^T(X)} = G^T(X)$; so by lemma 1.2 (v), the conclusion follows.

(ii) Let $x \in \Gamma^T(X)$ and suppose that

(1.4) $x\,[0,t] \subset G^T(X)$

we want to prove that $x\cdot[0,t] \subset \Gamma^T(X)$. suppose that this fact is not true; then there exists $\bar{t} \in [0,t]$ such that $x\cdot\bar{t} \notin \Gamma^T(X)$.

Now set

$$t^* = \inf\{\tau \in [0,t] \mid x\cdot\tau \notin \Gamma^T(X)\}$$

Clearly $t^* \in [0,t)$ and

(1-5) (a) $x\cdot t^* \in \Gamma^T(X)$ since $\Gamma^T(X)$ is closed by lemma 1.2 (vii) ;

(b) $x\cdot(t^*+\varepsilon_n) \notin \Gamma^T(X)$ (with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$).

If we set $y = x\cdot t^*$, by (1.5) and the definition of $\Gamma^T(X)$ we have

$$y\cdot[0,T] \cap \partial X \neq \phi$$

$$y\cdot[\varepsilon_n,T] \cap \partial X = \phi$$

From the above formulas we have that

(1-6) $y \in \partial X$.

On the other hand, by (1-4), $y \in G^T(X)$ and by our assumptions $y \in \mathring{X}$; this fact contradicts (1.6).

(iii) It is trivial

<u>Theorem 1.5</u> (Equivalence of index pairs). Let (N,N_o) and $(\tilde N,\tilde N_o)$ be two index pairs such that exists $T > 0$ such that

$$G^T(\overline{N - N_o}) \subset \overline{\tilde N - \tilde N_o} \quad \text{and}$$

$$G^T(\overline{\tilde N - \tilde N_o}) \subset \overline{N - N_o}$$

Then $[N/N_o] = [\tilde N/\tilde N_o]$.

<u>Remark</u> . The proof of theorem 1.5 is essentially contained in Salamon [S]. He gave a short and elegant proof of Conley's theorem of equivalence of index pairs (in the compact contest). Salamon's proof can be adepted to our case.

<u>Sketch of the proof of th.1.5</u> . We can suppose that $G^T(\overline{N - N_o}) \subset \text{int}\, \tilde N - \tilde N_o$ and that $G^T(\overline{\tilde N - \tilde N_o}) \subset \text{int}(N - N_o)$ (if not it is enough to replace T by 2T and use lemma 1.2(iv). Now let $f : N_1/N_o \to \tilde N/\tilde N_o$ be defined as follows

$$f([x]) = \begin{cases} [x\cdot 3T] & \text{if } x\cdot[0,2T] \subset N_1 - N_o \text{ or } x[T,3T] \subset \tilde N_1 - \tilde N_o \\ [N_o] & \text{otherwise} \end{cases}$$

the function f is continuous (for detais of the proof see [S] lemma 4.7). In an analogous way we can define a map $\tilde f : \tilde N/\tilde N_o \times [T,\infty) \to N/N_o$. We have to prove that $\tilde f \circ f$ and $f \circ \tilde f$ are homotopic to the identity in N/N_o and $\tilde N/\tilde N_o$ respectively.

For $t \in [0,T]$ define the map $h : [0,T] \times N/N_o \to N/N_o$ as follows

$$h(t,[x]) = \begin{cases} [x\cdot 6t] & \text{if } x[0,6t] \subset N_1 - N_o \\ [N_o] & \text{otherwise.} \end{cases}$$

It is easy to show that h is continuous and that

$$h(T,[x]) = \tilde f \circ f \quad \text{and } h(0,[x]) = Id_{N/N_o} .$$

In the some way it is possible to construct a homotopy $\tilde h : [0,T] \times \tilde N/\tilde N_o \to \tilde N/\tilde N_o$.//

<u>Corollary 1.6</u>. If (N,N_o) and $(\tilde N, \tilde N_o)$ are two index pairs for X, then $[N,N_o] = [\tilde N,\tilde N_o]$. in particular (1-1) (b) holds.

<u>Proof</u>. If (N,N_o) and $(\tilde N,\tilde N_o)$ are two index pairs for X, we have

$$G^T(\overline{N - N_o}) \subset G^T(X) \subset \tilde N - \tilde N_o \quad \text{by definition 1.1}$$

and

$$G^T(\overline{\tilde N - \tilde N_o}) \subset G^T(X) \subset N - N_o \; .$$

The conclusion follows from theorem 1.5. //

So at this point h(X) is well defined. An other consequence of theorem 1.5 is the following Corollary

<u>Corollary 1.7</u> . Let $X, Y \in \Sigma$ and suppose that $\exists\, T \geq 0$ such that

(1-7) $\quad G^T(X) \subset Y \quad$ and $\quad G^T(Y) \subset X$

Then $h(X) = h(Y)$.

Proof. Let (N, N_o) and $(\tilde{N},\tilde{N}_o)$ be two index pairs for X and Y respectively. Then

(1-9) $\quad G^T(N-N_o) \subset G^T(X) \subset Y$ by definition 1.1 and (1-7).

Since $(\tilde{N},\tilde{N}_o)$ is an index pair for Y, $\exists\ T_1 > 0$ such that

$$G^{T_1}(Y) \subset \mathrm{int}(\tilde{N}_1 - \tilde{N}_o)$$

Therefore by the above formula, (1-9) and lemma 1.2 (iii), we have that

$$G^{T+T_1}\ \overline{(N-N_o)} \subset \tilde{N} - \tilde{N}_o$$

For the some reason there exists $T_2 > 0$ such that

$$G^{T+T_2}\ \overline{(\tilde{N}-\tilde{N}_o)} \subset N - N_o$$

Thus by theorem 1.5 (replacing T with $T + \max(T_1,T_2)$) the conclusion follows.//

Corollary 1.8. For every $T > 0$ $\quad h(G^T(X)) = h(X)$.

Proof. Trivial. /

Corollary 1.9. If there is $T > 0$ such that $G^T(X) = \Phi$, then $h(X) = \underline{0}$.

Notice that Corollary 1.9 cannot be inverted as the following example shows.

Example 1.10. Take

$$M = \mathbb{R} \quad ; \quad \eta(t,x) = x-t \quad ; \quad X = [0,+\infty)$$

Then $h(X) = 1$ but $G^T(X) \neq \Phi$ for every $T > 0$.

However there is a good test to see if the index of a set is $\underline{0}$.

Theorem 1.11. Suppose that $X \in \Sigma$ and that

(1-10) for every $x \in X$, there is $t > 0$ such that $x \cdot t \notin X$. Then $h(X) = \underline{0}$.

We need some lemmas to prove theorem 1.11.

Lemma 1.12. Suppose that (N,N_o) is an index pair and that τ is a positive constant such that

(1-11) $\quad x \cdot [0,\tau] \subset N - N_o$

Then there exists an open neighborood V of x such that for every $y \in V \cap N$,

$$y \cdot [0,\tau] \subset N - N_o \ .$$

Proof. We argue indirectly and suppose that the conclusion of the lemma is not true. Then exists a sequence $x_n \to X$ ($x_n \in N - N_o$) and a sequence $t_n \in [0,\tau]$ such that

$$x_n \cdot t_n \notin N - N_o \ .$$

We set

$$\tilde{t}_n = \sup \{t \in [0,t_n] \text{ such that } x \cdot [0,t] \subset N\}$$

$\tilde{t}_n$ is a bounded sequence; so we can suppose that it is convergent to some $\bar{t} \in [0,\tau]$. By our construction, $x_n \cdot \tilde{t}_n \in N_o$; so $x \cdot \bar{t} \in N_o$ since N_o is closed. This last statement contradicts (1-11); so the lemma is proved. //

Lemma 1.13. Let $(N, N_o) = (G^T(X),\Gamma^T(X))$ be an index pair for X (cf.Th.1.4).

We set

$$U = \{x \in N \mid \exists\ t \in [0, 2T] \text{ such that } x \cdot t \in N^c\}$$

where N^c denotes $M - N$.

Then U satisfies the following properties

(i) $\quad$ U is relatively open in N ;

(ii) given two positive constants $t_1 < t_2$ such that

$$x \cdot t_i \in U \text{ and } x \cdot [0,t_i] \subset N \quad (i=1,2)$$

then for every $t \in [t_1,t_2]$, $x \cdot t \in U$;

(iii) $N_o \subset U$;

(iv) $(\bar{U},N_o)$ is an index pair and $[\bar{U}/N_o] = \underline{0}$.

Proof. (i) and (ii) are easy to check.

In order to prove (iii) we argue indirectly and suppose that there is $x \in N_o$ such that $x \cdot [0,2T] \subset N$. Since N_o is positively invariant with respect to N, $x \cdot [0,2T] \subset N_o$. Then if we set $y = x \cdot T$, it follows that $y \in N_o$ and $y \in G^T(N)$. Since $G^T(N) \subset \dot{N}$ by lemma 1.3 (iii) and (iv) and $N_o \subset \partial N$, by lemma 1.3 (viii), we have obtained a contradiction.

Now let us prove (iv). First observe that $N_o \subset \bar{U}$ by (iii). (i) of Def. 1.1 is satisfied since $\overline{\bar{U} - N_o} = \bar{U}$ and $G^{2T}(\bar{U}) = \Phi \subset \text{int}(\bar{U})$.

To check (ii), it is enough to observe that $\bar{U}$ N. (iii) follows directly by the definition of $\bar{U}$. So $(\bar{U},N_o)$ is an index pair.

$[\bar{U}/N_o] = h(U) = \underline{0}$ by Corollary (1-9). //

Proof of Th. 1.11. Let N, N_o and U as in lemma 1.13. For every $x \in N$, we choose a $t(x) > 0$ such that

$$x \cdot [0,t(x)] \subset N \qquad \text{and} \qquad x \cdot t(x) \in U .$$

This is possible by (1-10) and lemma 1.13 (iii). If $x \in U$ we choose $t(x) = 0$. Also if $x \notin U$, we can choose $t(x)$ such that $t(x) \notin N_o$.

Now for $x \in N - N_o$, let V_x be an open neighborood of N (open in the topology of N) such that

(1.12) for every $y \in V_x$, $y \cdot [0,t(x)] \subset N$ and $y \cdot t(X) \in U$.

This is possible by our choice of $t(x)$, lemma 1.12 and lemma 1.13 (i).

For $x \in N_o$, set $V_x = U$. Thus $\{V_x\}_{x \in N}$ is an open cover of N (open in the relative topology of N).

Let $\{V_i\}_{i \in I}$ be a locally finte rifiniment of $\{V_x\}_{x \in N}$ which exists since N is a metric space.

Observe that, by our construction, for every $i \in I$, there exists $t_i \geq 0$ such that

(1-13) $\quad \eta(t_i,V_i) \subset U \quad$ and $\quad \eta([0,t_i],V_i) \subset N$.

Now let $\{\beta_i(x)\}_{i\ I}$ be a partition of the unity relative to $\{V_i\}_{i \in I}$ i.e. a set of function $\beta_i : N \to \mathbb{R}$ whose support is $\bar{V}_i$ and $\sum_{i \in I} \beta_i(x) = 1$ for every $x \in N$. Such partition exists since N is a metric space.

Now set

$$\tau(x) = \sum_{i \in I} \beta_i(x) t_i .$$

Clearly $\tau(x)$ is a continuous function. We claim that

(1-14) $\quad x \cdot \tau(x) \in U$.

In order to see this, fix $\bar{x} \in N$ and set

$$t_1(\bar{x}) = \min\{t_i \mid \bar{x} \in V_i\} \quad ; \quad t_2(\bar{x}) = \max\{t_i \mid \bar{x} \in V_i\} .$$

By (1-13), $\eta(t_i, \bar{x}) \in U$ (i=1,2) and $\eta([0,t_i], \bar{x}) \subset N$.

Therefore (1.14) follows from lemma 1.13 (ii).

Moreover observe that by our construction

(1-15) $\quad \tau(x) = 0 \quad$ for every $x \in N_o$.

Now consider the map $h : [0,1] \times N \to U$ defined by

$$h(s,x) = \eta(s \cdot t(x),x)$$

h is an homotopy equivalence between N and $\bar{U}$, and by (1-15) it is also an homotopy equivalence between N/N_o and $\bar{U}/N_o$.

Therefore, by lemma 1.13 (iv)

$$h(X) = [N/N_o] = [\bar{U}/N_o] = \underline{0} . \qquad //$$

Remark 1.14. Now, few words to compare the Conley index with our generalization.

A closed set X is called by Conley [C] an isolating neighborood if $I(X) \subset \mathring{X}$ where $I(X) = \{x \in X : x\mathbb{R} \subset X\}$ or, using our notation, $I(X) = \bigcap_{t\geq 0} G^T(X)$.

Let $\tilde{\Sigma}$ be the family of isolating neighboroods in M; then if M is compact $\Sigma = \tilde{\Sigma}$. If M is not compact, in general, $\Sigma \subset \tilde{\Sigma}$. So, in our approch, it was necessary to restrict the class of sets X for which to define index pairs (and introduce the operator $G^T(\cdot)$).

Now, observe that the relationship (1-7) gives an equivalence relation on Σ (which we will denote by $\approx$).

Corollary 2.4 states that the index is constant on each equivalence class of $\approx$. If M is compact, then $X \approx Y$ if and only if $I(X) = I(Y)$ (the easy proof of this left to the reader).

So, when M is compact, h depends only on the maximal invariant set I(X) contained in X; therefore it is an <u>index of isolated invariant sets</u>. Example 1.10 shows that this is not the case when the compactness is not assumed (in fact $h(X) = \underline{1}$ but $I(X) = \phi$). Concluding, the Conley index is an index of isolated invariant sets; our generalization is an index of a class of closed set Σ which has been closen in order that the main properties of the Conley theory can be preserved.

Example 1.15. Let M = E be an Hilbert space and let L be a bounded normal invertible operator.

We consider the flow η defined by the differential equation

(1-16) $$\dot{x} = -Lx$$

We want to compute $h(X,\eta)$ where X is a bounded closed neighborood of O.

By our assumption E can be splitted as follows

(1-17) $$E = E^+ \oplus E^-$$

where E^+ and E^- are two mutually ortogonal subspaces such that exists a constant

$\alpha > 0$

(1-18) $$\langle Lx,x\rangle \geq \alpha\|x\|^2 \qquad \forall\ x \in E^+$$
$$\langle Lx,x\rangle \leq -\alpha\|x\|^2 \qquad \forall\ x \in E^-$$

According to the splitting (1-17),(1-16) can be written as follows

$$\dot{x}^+ = -L^+x^+$$
$$\dot{x}^- = -L^-x^-$$

where $x = x^+ + x^-$ with $x^\pm \in E^\pm$ and $L^\pm = L|_{E^\pm}$.

Now, if Y is any other bounded closed neighborood of O, by 1-18, it is easy to check that $X,Y \in \Sigma(\eta)$ and that (1-7) is satisfied. Then $h(X) = h(Y)$.

In particular we can take

$$Y = (B_R \cap E^+) \times (B_R \cap E^-)$$

where B_R is the ball of Radious R.

It is easy to check that

$$(Y,\ (B_R \cap E^+) \times \partial(B_R \cap E^-))$$

is an index pair and that it is homotopically equivalent to

$$(B_R \cap E^-,\ \partial(B\ \ \cap E^-)) .$$

Also we have

$$\left[\frac{B_R \cap E^-}{\partial(B_R \cap E^-)}\right] = \begin{cases} [S^N,*] & \text{if } \dim E^- = N \\ [S^\infty,*] = \underline{0} & \text{if } E^- \text{ is infinite dimensional} . \end{cases}$$

So concluding we have

$$h(X) = h(Y) = [S^N,\ *]$$

where N is dim E^- and remembering that $[S^\infty,\ *] = [*,\ *] = \underline{0}$.

2 - Stability and homotopy invariance of the generalized Conley index.

We need new notation :

$$\mathcal{F}(M) = \{X \subset M \mid X \text{ is closed}\}$$

$\mathcal{F}(M)$ can be equipped with the extended Hausdorff metric :

$$d_H(X,Y) = \sup_{x\in X} d(x,Y) + \sup_{y\in Y} d(y,X)$$

We need also the following notation

$$X \overset{\delta}{\sqsubset} Y \overset{\text{def}}{\Longleftrightarrow} X \subset Y \text{ and } \operatorname{dist}(X,\partial Y) \geq \delta$$

$$X \sqsubset Y \overset{\text{def}}{\Longleftrightarrow} \exists \delta > 0 \text{ s.t. } X \overset{\delta}{\sqsubset} Y.$$

We set

$$\Sigma_o = \Sigma_o(\eta) = \{X \in \mathcal{F}(M) \mid \exists T, \delta > 0 \text{ s.t. } G^T(N_\delta(X)) \sqsubset X, \text{ and } \eta(\cdot,T) \text{ is uniforlmly continuous }\}.$$

Clearly $\Sigma_o \subset \Sigma$.

Theorem 2.1. Let $X \in \Sigma_o(\eta)$ and let $\tilde{\eta}$ be a flow such that

$$d(\eta(t,x), \tilde{\eta}(t,x)) \leq \varepsilon \qquad \forall\, t \in [-T,T], \forall\, x \in X$$

where ε and t are suitable positive constants, which depend on X and η. Then

(i) $X \in \Sigma_o(\tilde{\eta})$

(ii) $h(X,\tilde{\eta}) = h(X,\eta)$

Before proving theorem 2.1 we will see two important consequences of this theorem.

Corollary 2.2. Let $X,\eta,\tilde{\eta}$ be as in theorem 2.1 and let $\tilde{X}$ be a closed set such that

$$(2\text{-}1) \qquad d_H(X,\tilde{X}) < \varepsilon_1$$

where $d_H(\cdot,\cdot)$ is the Hausdorff distance and ε_1 is a positive distance depending on $X,\eta,\tilde{\eta}$ but not on $\tilde{X}$.

Then

(i) $X \in \Sigma_o(\tilde{\eta})$

(ii) $h(\tilde{X},\tilde{\eta}) = h(X,\eta)$.

Proof. By Th. 2.1 (i), there exists $T,\delta_1,\delta_2 > 0$ such that

$$G^T(N_{\delta_1}(X),\tilde{\eta}) \overset{\delta_2}{\sqsubset} X$$

Now choose $\varepsilon_1 > 0$ smaller then $\min(\delta_1/2,\delta_2/2)$. Then by (2-1),

$$(2\text{-}2) \qquad G^T(N_{\delta_1}(X),\tilde{\eta}) \overset{\varepsilon_1}{\sqsubset} \tilde{X} \cap X$$

Moreover, by the choice of ε_1, we have

$$\tilde{X} \subset N_{\varepsilon_1}(X) \subset N_{\delta_1/2}(X) \quad \text{and} \quad N_{\varepsilon_1}(\tilde{X}) \subset N_{\delta_1}(X) .$$

By the above formula and (2-2) we get

$$(2\text{-}3) \qquad G^T(N_{\varepsilon_1}(\tilde{X}),\tilde{\eta}) \subset G^T(N_{\delta_1}(X),\tilde{\eta}) \overset{\varepsilon_1}{\sqsubset} X \cap \tilde{X} .$$

The above formula proves (i).

Moreover by (2-2) and (2-3), we have that

$$G^T(X,\tilde{\eta}) \subset \tilde{X} \qquad \text{and} \qquad G^T(\tilde{X},\tilde{\eta}) \subset X .$$

Then by corollary 1.7 we have

$$h(X,\tilde{\eta}) = h(\tilde{X},\tilde{\eta}).$$

The conclusion follows by (ii) of Theorem 2.1. //

Corollary 2.3. Let $\eta_\lambda, \lambda \in [0,1]$, be a family of flow depending continuously on λ with respect to the topology of the uniform convergence on $X \times [-T,T]$ for every $T > 0$ where $X \subset M$.

Suppose that X_λ is a family of sets contained in X and depending uniformly on λ with respect to the Hausdorff topology.

Finally suppose that $X_\lambda \in \Sigma_o(\eta_\lambda)$ for every $\lambda \in [0,1]$.

Then $h(X_\lambda, \eta_\lambda)$ does not depend on λ.

Proof. By Corollary 2.2, for every $\bar\lambda \in [0,1]$, there exists a neighborood of $\bar\lambda$, $I_{\bar\lambda}$ such that

$$h(X_\lambda, \eta_\lambda) \quad \text{is constant for } \lambda \in I_{\bar\lambda}.$$

Then the conclusion follows strightforward. //

The proof of theorem 2.1 is involved and relies on several lemmas.

Lemma 2.4. Take $X \in \Sigma$ and T large enough such that

(2-4) $$G^{T/2}(X) \subset \mathring{X}$$

Set $\phi_1 : \mathrm{int}(G^{T/2}(X) \to G^T(X)/\Gamma^T(X)$ be defined as follows

$$\phi_1(x) = \begin{cases} [x \cdot T] & \text{if } x\cdot T \notin G^T(X) \\ [\Gamma^T(X)] & \text{if } x\cdot T \notin G^T(X) \end{cases} .$$

Then ϕ_1 is continuous.

Proof. It is obvious that $\phi_1(x)$ is continuous if $x\cdot T \in \mathrm{int}(G^T(x))$ or $x\cdot T \notin G^T(X)$.

So we have to consider only the case $x\cdot T \in \partial G^T(X)$. First notice that

(2-5) $$x \in \mathrm{int}(G^{T/2}(X)) \Longrightarrow x\cdot[0,T/2] \subset \mathring{X}.$$

Moreover

$$x\cdot T \in G^T(X) \Longrightarrow x\cdot\left[\frac{1}{2}T, \frac{3}{2}T\right] \subset G^{T/2}(X)$$

Thus by (2-4) and the above formula $x\left[\frac{1}{2}T, \frac{3}{2}T\right] \subset \mathring{X}$ and by (2-5) it follows that

(2-6) $$x\left[0, \frac{3}{2}T\right] \subset \mathring{X}.$$

We clain that

(2-7) $$x\cdot T \in \partial G^T(X) \Longrightarrow x\cdot T \in \Gamma^T(X).$$

In fact if $x\cdot T \in \partial G^T(X)$ there exists $t \in [0,2T]$ such that $x\cdot t \in \partial X$.

By (2-6) we have that $t \geq \frac{3}{2}T \geq T$. Then by the definition of $\Gamma^T(X)$, (2-6) follows. So we have that

$$x\cdot T \in \partial G^T(X) \Longrightarrow \phi_1(x) = [\Gamma^T(X)]$$

and by the above formula the continuity of ϕ_1 at x follows easily. //

Lemma 2.5. The function $\phi_2 : G^T(X)/\Gamma^T(X) \to G^T(X)/\Gamma^T(X)$ defined as follows

$$\phi_2([x]) = \begin{cases} [x\cdot T] & \text{if } x\cdot T \in G^T(X) \\ [\Gamma^T(X)] & \text{otherwise} \end{cases}$$

is continuous.

Proof. The proof of this lemma is contained in the proof of Th. 1.5 when it is shown that $h(t,[x])$ is continuous. //

Lemma 2.6. Let $X \in \Sigma_o(\eta)$ and let $T > 0$ be large enough that

(2-8) $$G^{T/2}(X) \overset{\delta}{\subset} X \qquad \text{for some } \delta > 0$$

Then there exists $\delta_1 = \delta_1(\eta, X)$ such that

$$x \in N_{\delta_1}(\Gamma^T(X)) \Longrightarrow x\cdot\left[0, \frac{3}{2}T\right] \cap \partial X \neq \phi .$$

Proof. Choose δ_1 small enough that

(2-9) $d(x_1, x_2) < \delta_1 \Longrightarrow d(x_1\cdot T, x_2\cdot T) \le \delta/2 \qquad \forall x_1, x_2 \in X$.

This is possible by the uniform continuity of $\eta(T,\cdot)$.

So we have

$$
\begin{aligned}
& x \in N_{\delta_2}(\Gamma^T(X)) \Longrightarrow \\
& \exists\, \bar{x} \in \Gamma^T(X) : d(x,\bar{x}) \le \delta_1 \Longrightarrow && [\text{by } (2\text{-}9)] \\
& d(x\cdot T, \bar{x}\cdot T) < \delta/2 \Longrightarrow && [\text{since } \bar{x}\cdot T \in \partial X] \\
& d(x\cdot T, \partial X) < \delta/2 \Longrightarrow && [\text{by } (2\text{-}8)] \\
& x\cdot T \notin G^{T/2}(X) \Longrightarrow && [\text{by the definition of } G^{T/2}(X)] \\
& x\cdot[\tfrac{1}{2}T, \tfrac{3}{2}T] \cap \partial X \ne \phi \,. && //
\end{aligned}
$$

In the following lemmas we shall write $\eta_t(x)$ instead of $\eta(t,x)$ to simplify the notation.

Lemma 2.7. Take $X \in \Sigma_o(\eta)$ and choose T large enough that

(2-10) $$G^T(X) \overset{\delta}{\sqsubset} G^{T/2}(X) \overset{\delta}{\sqsubset} \overset{\circ}{X} \,.$$

Let $\tilde{\eta}$ be a flow such that

(2-11) $$d(\tilde{\eta}_t(x), \eta_t(x)) \le \frac{\delta_1}{2} \qquad \forall x \in X \quad \forall t \in [-T,T]$$

where $\delta_1 = \delta_1(\eta,X) < \delta$ is defined in lemma 2.6.

Let $h : [0,1] \times G^T(X)/\Gamma^T(X) \to G^T(X)/\Gamma^T(X)$ be defined as follows

$$
h(\lambda, [x]) = \begin{cases} [\eta_{2T} \circ \tilde{\eta}_{-\lambda T} \circ \eta_{\lambda T}(x)] & \text{if } \eta_{[0,2T]}(x) \subset G^T(X) \\ [\Gamma^T(X)] & \text{otherwise} \end{cases}
$$

Then h is continuous.

Proof. By (2-11) taking $t = -\lambda T$ and replacing x with $\eta_{-\lambda T}(x)$ we get

(2-12) $$d(\tilde{\eta}_{-\lambda T} \circ \eta_{\lambda T}(x), x) \le \delta_1/2 \;(\le\delta/2) \quad \forall \lambda \in [0,1] \quad \forall x \in G^T(X) \,.$$

Then by (2-10), the function

$$x \to \tilde{\eta}_{-\lambda T} \circ \eta_{\lambda T}(x)$$

maps $G^T(X)$ into $\mathrm{int}(G^{T/2}(X))$ for every $\lambda \in [0,1]$.

Now consider the function $g : [0,1] \times G^T(X) \to G^T(X)/\Gamma^T(X)$ defined as follows

$$
g(\lambda,x) = \begin{cases} [\eta_{2T} \circ \eta_{-\lambda T} \circ \eta_{\lambda T}(x)] & \text{if } \eta_{[0,2T]} \subset G^T(X) \\ [\Gamma^T(X)] & \text{otherwise} \,. \end{cases}
$$

Then we have $g(\lambda,x) = \phi_2 \circ \phi_1 \circ (\tilde{\eta}_{-\lambda T} \circ \eta_{\lambda T})$ where

$$
\begin{aligned}
\tilde{\eta}_{-\lambda T} \circ \eta_{\lambda T} &: [0,1] \times G^T(X) \to G^{T/4}(X) \\
\phi_1 &: G^{T/4}(X) \to G^T(X)/\Gamma^T(X) && \text{is defined by lemma 2.4} \\
\phi_2 &: G^T(X)/\Gamma^T(X) \to G^T(X)/\ ^T(X) && \text{is defined by lemma 2.5 .}
\end{aligned}
$$

Since all the above maps are continuous also g is continuous.

It remains to prove that

$$h(\lambda, [x]) = g(\lambda, x) \,.$$

So we have to prove that if $x \in \Gamma^T(X)$ then $g(t,x)$ is constant, so that the above equality makes sense.

By (2-12) we have

$$x \in \Gamma^T(X) \Longrightarrow \tilde{\eta}_{-\lambda T} \circ \eta_{\lambda T}(x) \in N_{\delta_1}(\Gamma^T(X)) \,.$$

By the above formula and lemma 2.6 we have that

$$x \in \Gamma^T(X) \Longrightarrow \eta_{[0,2T]} \circ \eta_{\lambda T} \circ \eta_{-\lambda T}(x) \cap \partial X \ne \phi \,.$$

Therefore $g(\lambda,x) = [\Gamma^T(X)] \quad \forall\ x \in \Gamma^T(X)$. //

<u>Lemma 2.8</u>. Take T large enough that

$$G^T(X) \overset{\delta}{\sqsubset} G^{T/2}(X) \overset{\delta}{\sqsubset} X$$

and take $\delta_1 = \delta_1(X,\eta) < \delta/4$.

Now take $\tilde{\eta}$ close enough to η such that

$$\text{(i)} \quad d(\tilde{\eta}_t(x),\ \eta_t(x)) \le \delta_1 \quad \text{for every } x \quad X \text{ and } t \quad [-2T,2T]$$

$$\text{(2-13)} \qquad \text{(ii)} \quad N_{\delta_1}(G^T(X)) \subset \tilde{G}^{T/2}(X)$$

$$\text{(iii)} \quad \tilde{G}^T(X) \overset{2\delta_1}{\sqsubset} X$$

Then the function $f : G^T(X)/\Gamma^T(X) \to \tilde{G}^T(X)/\tilde{\Gamma}^T(X)$ defined as follows

$$f([x]) = \begin{cases} [\tilde{\eta}_{2T} \circ \eta_{-T}(x)] & \text{if } \tilde{\eta}_{2T} \circ \eta_T(x) \in \tilde{G}^T(x) \\ [\tilde{\Gamma}^T(x)] & \text{otherwise} \end{cases}$$

is continuous (we have used the notation $\tilde{G}^T(X) = G^T(X,\tilde{\eta})$ and $\tilde{\Gamma}^T(X) = \Gamma^T(X,\tilde{\eta})$).

<u>Proof</u>. By (2-13) (i) we set

$$\text{(2-14)} \qquad d(\tilde{\eta}_{-t} \circ \eta_t(x),\ x) \le \delta_1 \quad \forall\ x \in G^T(X)\ \forall\ t \in [0,T] .$$

Then, by (2-13) (ii), the function $\tilde{\eta}_T \circ \eta_T$ maps $G^T(X)$ into int $G^{T/2}(X)$.

Now define $g : G^T(X) \to \tilde{G}^T(X)/\tilde{\Gamma}^T(X)$ as follows

$$g(x) = \begin{cases} \tilde{\eta}_{2T} \circ \eta_T(x) & \text{if } \tilde{\eta}_{2T} \circ \eta_T(x) \in \tilde{G}^T(X) \\ [\tilde{\Gamma}^T(x)] & \text{otherwise .} \end{cases}$$

Notice that

$$g(x) = \tilde{\phi}_1 \circ (\tilde{\eta}_T \circ \eta_{-T})$$

where $\tilde{\phi}_1 :$ int $\tilde{G}^{T/2}(X) \to \tilde{G}^T(X)/\tilde{\Gamma}^T(X)$

is the map of lemma 2.4 with $G^T(X)$, $\Gamma^T(X)$ and η_t replaced by $\tilde{G}^T(X)$, $\tilde{\Gamma}^T(X)$, and $\tilde{\eta}_t$ respectively.

Therefore g is continuous.

It remains to prove that

$$f([x]) = g(x) .$$

So we have to prove that

$$x \in \Gamma^T(X) \implies g(x) \quad \text{is constant}$$

or more exactly $g(x) = [\tilde{\Gamma}(x)]$.

Use (2-13) (i) with $t = 2T$ and x replaced by $\eta_{-T}(x)$ with $x \in \Gamma^T(X)$; then we have

$$d(\tilde{\eta}_{2T} \circ \eta_T(x),\ \eta_{2T} \circ \eta_{-T}(x)) \le \delta_1$$

or

$$d(\tilde{\eta}_{2T} \circ \eta_T(x),\ \eta_T(x)) \le \delta_1$$

Since x $\Gamma^T(X)$, we have that $\eta_T(x)$ ∂X, and by the above formula

$$d(\tilde{\eta}_{2T} \circ \eta_T(x),\ \partial X) \le \delta_1 .$$

Thus $\tilde{\eta}_{2T} \circ \eta_T(x) \notin \tilde{G}^T(X)$. So we have proved that

$$x \in \Gamma^T(X) \implies g(x) = \tilde{\Gamma}^T(X)$$

and this completes the proof of the lemma . //

<u>Proof of Theorem 2.1</u>. Take T and ε such that (2-11) and (2-13) are satisfied with $\delta_1 < 2\varepsilon$.

Moreover, if ε is small enough, we have also

$$(2\text{-}14)\quad \begin{array}{ll}(a) & N_{\delta_1}(\tilde{G}^T(X) \subset G^{T/2}(X)) \\ (b) & \tilde{G}^T(X) \overset{2\delta_1}{\sqsubset} X\end{array}$$

Now let $f : G^T(X)/\Gamma^T(X) \to \tilde{G}^T(X)/\tilde{\Gamma}^T(X)$ be the function defined in lemma 2.8.
We have to prove that f is an homotopy equivalence.
We clain that $\tilde{f} : \tilde{G}^T(X)/\tilde{\Gamma}^T(X) \to G^T(X)/\Gamma^T(X)$ is the homotopy inverse of f ($\tilde{f}$ is defined as f replacing $G^T(X)$ with $\tilde{G}^T(X)$, etc...).
f and $\tilde{f}$ are continuous by virtue of lemma 2.8 and (2-14).
Moreover $\tilde{f} \circ f = h(1,\cdot)$ where h is defined in lemma 2.7.
Lemma 2.7 shows that $\tilde{f} \circ f \sim h(0,\cdot)$ (where "$\sim$" means homotopy equivalence).
Moreover it is strightforward to show that $h(0,\cdot) \sim Id$. Thus $\tilde{f} \circ f$ Id. Analogously we can show that $f \circ \tilde{f} \sim Id$ and this proves Theorem 2.1.

Example 2.9. Let η be the flow defined on M by the differential equation

$$\dot{x} = F(x) \ .$$

We suppose that M is an Hilbert space E (or an Hilbert manifold). Let $\bar{x}$ a nondegenerate critical point for F i.e. $F(\bar{x}) = 0$ and $F'(\bar{x}) : T_{\bar{x}} M \to T_{\bar{x}} M$ (where $T_{x_0} M$ denotes the tangent space at x_0) is defined (as Frechèt derivative) and it is an invertible norma operator.
Since $F'(x)$ is a normal operator, we have (cf. Ex.1.15)

$$T_{\bar{x}} M = E^+ \oplus E^-$$

where E^+ is the stable manifold of η and E^- the unstable manifold.
Now let η_0 be the flow defined by the following equation

$$\dot{x} = \bar{x} + F'(x_0)\cdot x \ .$$

By Theorem 2.1 it follows that

$$h(U,\eta) = h(U,\eta_0)$$

where U is a neighborood of $\bar{x}$ sufficiently small.
Therefore by the example 1.15, it follows that

$$(2\text{-}15)\qquad h(U,\eta) = (S^{m(x)}, *)$$

where $m(x) = \dim E^-$.

§ 3 - The generalised Conley index and compactness.

For $X \in \Sigma(\eta)$ we set

$$I(X) = \bigcap_{T>0} G^T(X) = \{x \in X \mid \eta(t,x) \in X \text{ for every } t \in \mathbb{R} \text{ such that } \eta(t,x) \text{ is defined}\}.$$

The following compactness assumption is very important for our theory:

Def.3.1. Let $X \in \Sigma$. We say that X satisfies the property (C) if for every neighborhood U of I(X) there exists T > 0 such that

$$G^T(X) \subset U .$$

Observe that the property (C) is hereditary, i.e. if X satisfies the property (C) and $Y \subset X$ $(Y \in \Sigma)$, then Y satisfies the property (C).

Prop.3.2. Suppose that $X, Y \in \Sigma$ and that satisfy the property (C). Then

$$I(X) = I(Y) \Longrightarrow h(X) = h(Y).$$

Proof. Let $S = I(X) = I(Y)$. $U = X \cap Y$ is a neighborood of S. Then, since X and Y satisfy the property (C) there exists T > 0 such that

$$G^T(X) \subset U \subset Y \quad \text{and} \quad G^T(Y) \subset U \subset X .$$

The conclusion follows by Corollary 1.7. //

Def.3.3. We say that $S \subset X$ is a (C)-invariant set if

(i) S is an invariant set

(ii) S has a neighborood U which satisfies the property (C) and such that $I(U) = S$.

By the remarks before Prop.3.3. and by the Prop. 3.3., it follows that any neighborood sufficiently small of S has the same homotopy index.

Therefore it is natural to define the index of a (C)-invariant set S as follows:

(3-1) $h(S) = h(U)$ where $U \in \Sigma$ neighborood of S sufficiently small.

The following proposition gives a criterium to check if a set $U \in \Sigma$ satisfies the property (C).

Prop.3.4. Let $U \in \Sigma$ and suppose that

(3-2) given a sequence $x_n \in U$ and a sequence $t_n \to +\infty$ such that $x_n \cdot [0,t_n] \subset U$, then the sequence $x_n \cdot t_n$ has a limit point.

Then U satisfies the property (C).

Proof. We argue indirectly and suppose that there exists a neighborood V of I(U) such that for every T > 0

$$G^T(U) \not\subset V .$$

Then there exists a sequence $y_n \in U$ and a sequence $t_n \to +\infty$ such that

$$y_n \in G^{t_n}(X) - V .$$

If we set $x_n = y_n(-t_n)$, then $x_n \cdot [0,t_n] \subset U$. Then by (3-2) $x_n \cdot t_n$ is convergent to some $\bar{y}$ (may be considering a subsequence). By its construction $\bar{y} \cdot \mathbb{R} \subset U$, therefore $\bar{y} \in S$.

However, since $\bar{y} = \lim_{n\to+\infty} y_n$ we have that $\bar{y} \notin \mathring{V}$. And this is a contradiction since V is a neighborood of S. //

Corollary 3.5. Let M be a locally compact space. Then any compact invariant isolated set $S \subset M$ is a (C)-invariant set.

Therefore, the index (3-1) is defined for such S.

Proof. Clearly every compact neighborood of S satisfies (3-2). //

Remark 3.6. When M is locally compact we get the "classical" Conley theory (of Remark 1.14).

The property (3.2) (which was introduced by Rybakowsky [R]) can replace the local compactness of M in such a way that the main properties of the "original" Conley index are preserved (in particular it is possible to define the index of an isolated invariant set).
Our theory has been developed without any request of compactness, replacing the index of an invariant set with the index of a set $X \in \Sigma$.
A compactness property, as the property (C), is required only to define the index of an invariant set as in the original Conley theory.

Prop. 3.7. Let U satisfy the property (C) and suppose that I(U) is compact. Then $U \in \Sigma_Q$.
Proof. Let

$$\varepsilon = d(\partial U, I(U)).$$

Since I(U) is compact then $\varepsilon > 0$. Then setting $V = N_{\varepsilon/2}(I(U))$, we have that $V \in \Sigma$ and that, for T large enough

$$G^T(U) \subset V \quad \text{(since U satisfies the property (C))}.$$

Thus $V \overset{\varepsilon/2}{\sqsubset} U$ as we wanted to prove. //

Example 3.8. Let $\bar{x}$ be as in example 2.9. Then $\bar{x}$ is a (C)-invariant set and

$$h(\bar{x}) = (S^{m(\bar{x})}, *) .$$

§ 4 - The generalized Morse index.

Let $H^*(\cdot,\cdot)$ denote the Alexander-Spanier cohomology with coefficients in some field F (cf. [Sp]).
We recall that the Alexander-Spanier cohomology satisfies the following property which is not shared by the singular cohomology theory.

Th. 4.1. Let (X,A) and (Y,B) two pairs of topological spaces. We suppose that X and Y are paracompact Hausdorff spaces and that A and B are closed in X and Y respectively. Moreover suppose that X - A and Y - B are homeomorphic. Then

$$H^*(X,A) \cong H^*(Y,B) .$$

Proof. See [Sp], Th.5, pag 318. //

Now for every pairs of closed spaces (X,A) we set

$$p(X,A) = p_t(X,A) = \sum_{q=0}^{\infty} \left[\dim H^q(X,A)\right] t^q$$

p(X,A) is a formal series whose coefficients are cardinal numbers; these numbers are known as Betti numbers.
If X is a compact manifold with boundary A, then p(X,A) reduces to a polynomium, called Poincaré or Betti polynomium.
p(X,A) is a topological invariant which carries part of the information contained in the cohomology algebra $H^*(X,A)$.
When $A = \emptyset$ we shall write p(X) instead of $p(X,\emptyset)$.
Will shall denote by S the set of formal series with cardinal coefficients.
The following properties of p(X,A) will be used to study the generalized Morse index.

Lemma 4.2. Let (X,A) and (Y,B) be couples of closed subspaces of a metric space. Then

(i) $p(X,A) = p(X/A,[A])$

(ii) if $X \cap Y = \Phi$ then

$$p(X \cup Y,\ A \cup B) = p(X,A) + p(Y,B)$$

(iii) $p((X,A) \times (Y,B)) = p(X,A) \cdot p(Y,B)$

where $(X,A) \times (Y,B) = (X \times Y,\ X \times B \cup Y \times A)$

(iv) if $B \subset A \subset X$ then there exists $Q(t) \in S$ s.t.

$$p_t(X,A) + p_t(A,B) = p_t(X,B) + (1+t)Q(t).$$

Proof. (i) Let $\pi : X \to X/A$ be the projection map. Then $\pi_{|X-A}$ is a homeomorphism between X-A and $X/A - [A]$. Thus the conclusion follows from Th. 4.1.

(ii) trivial.

(iii) Since (X,A) and (Y,B) are closed pairs, there is an exact Mayer-Vietoris sequence for the $\bar{H}^*$ cohomology (cf. [Sp] pag.291).

But every closed pairs of Hausdorff-paracompact spaces is a tout pair for the Alexander-Spanier cohomology (cf. [Sp] pag.315).

Therefore $\bar{H}^* = H^*$ on such pairs. Therefore the Künneth formula can be applied to such pairs (cf. [Sp] pag.249) and we get

$$H^*((X,A) \times (Y,B)) = H^*(X,A) \otimes H^*(Y,B).$$

From the above formula the conclusion follows.

(iv) Let us consider the exact sequence relative to the triple $B \subset A \subset X$:

$$\text{(4-1)} \quad \cdots \xrightarrow{\delta^*_{q-1}} H^q(X,A) \xrightarrow{i^*_q} H^q(X,B) \xrightarrow{j^*_q} H^q(A,B) \xrightarrow{\delta^*_q} \cdots$$

and set

$$a_q = \dim (\ker i^*_q)$$

$$b_q = \dim (\ker j^*_q)$$

$$c_q = \dim (\ker \delta^*_q)$$

By the exactness of (4-1) we get

$\dim H^q(X,A) = c_{q-1} + a_q$ (with the convenction that $c_{-1} = 0$)

$\dim H^q(X,B) = a_q + b_q$

$\dim H^q(A,B) = b_q + c_q$

Then we have

$$p(X,A) = \sum_{q=0}^{\infty} (c_{q-1} + a_q) t^q$$

$$p(X,B) = \sum_{q=0}^{\infty} (a_q + b_q) t^q$$

$$p(A,B) = \sum_{q=0}^{\infty} (b_q + c_q) t^q$$

Then

$$p(X,A) + p(A,B) = p(X,B) + \sum_{q=0}^{\infty} (c_{q-1} + c_q) t^q = p(X,B) + (1+t) \sum_{q=0}^{\infty} c_q t^q.$$

The conclusion follows setting $Q(t) = \sum_{q=0}^{\infty} c_q t^q$.

Notice that the formula (iv) holds even if some of the coefficients are infinite cardinal numbers. //

We can now define the generalized Morse index :

Def.4.3. The generalized Morse index (GIM) is a map

$$i : \Sigma(\eta) \to S$$

defined by

$$i_t(X,\eta) = p_t(N,N_0)$$

where (N,N_0) is an index pair for X.

When no ambiguity is possible we shall write $i(X)$ instead of $i_t(X,\eta)$.

Using Th.1.4. we could define the GIM in the following (formally) simpler way

$$i_t(X,\eta) = \lim_{T \to +\infty} p_t(G^T(X), \Gamma^T(X))$$

Example 4.4. Let η, $\bar{x}$, U be as in the Example 2.9. Then

$$i(U) = \sum_{q=0}^{\infty} \dim H^q(S^{m(\bar{x})}, *)t^q = t^{m(\bar{x})} \qquad [\text{by } 3.9]$$

since we have
$$H^q(S^k, *) = \begin{cases} \{0\} & \text{if } q \neq k \\ F & \text{if } q = k\ . \end{cases}$$

Remark 4.4'. By lemma 4.2 (i), $p(N,N_o) = p(N/N_o, [N_o])$; so the generalized Morse index depends only on h(X); thus it is well defined by (1-1)(a) and (b). The above remark implies that the GIM carries less information than the homotopic index. Nevertheless is more usefull since it is much easier to deal with. The following theorem illustrates the first properties of the generalized Morse index :

Theorem 4.5. The GIM satisfies the following properties

(i) if $X \in \Sigma$ and for every $x \in X$, there is t>0 such that $x \cdot t \notin X$, then $i(X)=0$;

(ii) if $X \in \Sigma$ is contractible and positively invariant, then $i(X) = 1$;

(iii) if $X,Y \in \Sigma$ and $X \cap Y = \phi$ then $i(X \cup Y) = i(X) + i(Y)$;

(iv) if η_i is a semiflow on M_i (i=1,2), then a semiflow $\eta_1 x \eta_2$ is defined on $M_1 x M_2$ as follows

$$(\eta_1 x \eta_2)(t,(x_1, x_2)) = (\eta_1(t,x_1), \eta_2(t,x_2)) \quad ;$$

then if $X_i \in \Sigma(\eta_i)$ (i=1,2), we have that $X_1 x X_2 \in \Sigma(M_1 x M_2, \eta_1 x \eta_2)$ and

$$i(X_1 x X_2, \eta_1 x \eta_2) = i(X_1, \eta_1) \cdot i(X_2, \eta_2) \ .$$

Proof. (i) follows from theorem 1.11; (ii) follows by the fact that

$$H^q(X) = 1 \quad \text{if and only if } q = 0.$$

(iii) and (iv) follow by lemma 4.3 (ii) and (iii) respectively. //

Next we are going to prove a property of the GIM which is a generalization of

the classical Morse inequalities.

Def.4.6. Take $X_1, X_2 \in \Sigma$ with $\mathring{X}_1 \cap \mathring{X}_2 = \phi$. We say that X_2 is over X_1 if there exists T > 0 such that $X_1 \cap G^T(X_1 \cup X_2)$ is positively invariant with respect to $G^T(X_1 \cup X_2)$.

If X_2 is over X_1 or X_1 is over X_2 then we say that X_1 and X_2 are η-connected. Otherwise we say that they are η-disconnected.

Example 4.7. I : If $X_1 \cap X_2 = \phi$, then X_1 and X_2 are η-disconnected.

II : Let f be a Liapunov function for (M,η) and let c be a constant which is a regular value for f(i.e. $f(x) = c \Longrightarrow f'(x) \neq 0$). We set

$$X_1 = \{x \in M \mid f(x) \le c\} \ ; \ X_2 = \{x \in M \mid f(x) \ge c\} \ .$$

Then $X_1, X_2 \in \Sigma$ and X_2 is over X_1.

Def.4.8. Let $X \in \Sigma$. A family of sets $\{X_k\}_{k \le N}$ is called a Morse decomposition of X if

(i) $X = \bigcup_{k=1}^{N} X_k$

(ii) $X_k \in \Sigma$ for k=1,...,N

(iii) $\mathring{X}_k \cap \mathring{X}_h = \phi$ for $k \neq h$

(iv) X_{h+1} is over $\bigcup_{k=1}^{h} X_k$ for h=1,...,N-1.

Example 4.9. Let f be a Liapunov function for (M,η) and let $c_1 < c_2 < \ldots < c_{N-1}$ be a sequence of regular values for f. Let $c_o = -\infty$ and $c_N = +\infty$ and

$$X_k = \{x \in X \mid c_{k-1} \le f(x) \le c_k\}$$

then $\{X_k\}$ is a Morse decomposition of X.

The next theorem states one for the most important properties of the index (as far as the applications are concerned).

Theorem 4.10. If X_k is a Morse decomposition of X, then there exists $Q \in S$ such that

$$\sum_{h=1}^{N} i(X_k) = i(X) + (1+t)Q(t) \qquad Q \in S.$$

In order to prove Theorem 4.9. some lemmas are necessary.

Lemma 4.10'. Let $X = X_1 \cup X_2$ and suppose that X_2 is over X_1. Then there exist closed spaces $N_o \subset N_1 \subset N_2$ such that (N_2,N_o), (N_2,N_1), (N_1,N_2) are index pairs for X, X_2 and X_1 respectively.

Proof. Take T big enough in order that

(a) $X_1 \cap G^T(X)$ is positively invariant with respect to $G^T(X)$.

(4-4) (b) $(G^T(X), \Gamma^T(X))$ is an index pair for X.

(c) $G^T(X_1) \subset \mathring{X}_1$.

We set

$$N_o = \Gamma^T(X)$$
$$N_1 = (X_1 \cap G^T(X)) \cup \Gamma^T(X)$$
$$N_2 = G^T(X).$$

We want to prove that N_o,N_1,N_2 satisfy the required properties. We now prove that (N_1,N_o) is an index pair for X. Let us check (i) of Def. 1.1. Since $\overline{N-N_o} = X_1 \cap G^T(X)$

$$G^T(\overline{N-N_o}) = G^T(X_1 \cap G^T(X)) \subset G^T(X_1) \subset \mathring{X}_1 \quad \text{by (4-4)(c).} \tag{4-5}$$

Also by lemma 1.3 (i),(iii) and (iv)

$$G^T(\overline{N-N_o}) \subset G^T(G^T(X)) = G^{2T}(X) \subset \text{int}\left[G^T(X)\right] \tag{4-6}$$

Then by (4-5) and (4-6)

$$G^T(\overline{N-N_o}) \subset \text{int}(N-N_o).$$

(iii) of Definition 4.8. holds since $(X_1 \cap G^T(X))$ is positively invariant in $G^T(X)$ by Definition and $\Gamma^T(X)$ is positively invariant in $G^T(X)$ by Th. 1.4. Now let us check (iii) of Def.1.1. If $x \in N_1$, and it leaves N_1 at some times, it has to leave $G^T(X)$ also, since N_1 is positively invariant in $G^T(X)$. Thus there exists t^* such that $x \cdot t^* \in \Gamma^T(X)$ since $\Gamma^T(X)$ is an exit set for $G^T(X)$. Finally since $G^T(X_1) \subset N_1 - N_o$, (iv) of Def. 1.1. holds.
Let us check that (N_2,N_1) is an index pair for X_2.

$$\overline{N_2 - N_1} = \overline{G^T(X) - X_1} = G^T(X) \cap X_2.$$

Then arguing as we have done for $G^T(X) \cap X_1$, it follows that $\overline{N_2 - N_1} \in \Sigma$.

(ii) of Def. 1.1. holds since N_1 is positively invariant in N_2 and (iii) holds since $N_1 \supset \Gamma^T(X)$ and $\Gamma^T(X)$ is an exit set for N_2.

(iv) follows by the fact that $G^T(X_2) \subset \overline{N_2 - N_1}$. //

Corollary 4.11. If $X = X_2 \cup X_1$ and X_2 is over X_1, then there exists Q such that

$$i(X_1) + i(X_2) = i(X) + (1+t)Q(t).$$

Proof. By lemma 4.2. (iv) applied to the triple N_o,N_1,N_2 defined in lemma 4.10. we have

$$p(N_2,N_1) + p(N_1,N_o) = p(N_2,N_o) + (1+t)Q(t).$$

The conclusion follows by lemma 4.10. and the definition of the cohomological index. //

Remark 4.12. It is easy to check that if X_1 and X_2 are η-disconnected, then, for T large enough

$$G^T(X_1 \cup X_2) = G^T(X_1) \cup G^T(X_2) \quad \text{and} \quad G^T(X_1) \cap G^T(X_2) = \Phi .$$

Then

$$i(X) = i(G^T(X_1 \cup X_2)) \qquad \text{by Corollary 1.8.}$$

$$= i(G^T(X_1)) + i(G^T(X_2)) \qquad \text{by Th. 4.5 (iii)}$$

$$= i(X_1) + i(X_2) \quad .$$

Comparing this result with Corollary 4.11. we deduce that $Q(t) \neq 0$ implies that X_1 and X_2 are η-connected.

Proof of Th. 4.9. We argue by induction. For $N = 2$ it is true since it is nothing else but Corollary 2.11.

We can suppose that it is true for N-1; so there exists $Q_1 \in \Sigma$ such that

$$\sum_{k=1}^{N-1} i(X_k) = i(\bigcup_{k=1}^{N-1} X_k) + (1+t)Q_1(t).$$

Now, since X_N is over $\bigcup_{k=1}^{N-1} X_k$, applying Corollary 4.11. an other time, we get

$$i(X_N) + i(\bigcup_{k=1}^{N-1} X_k) = i(X) + (1+t)Q_2(t) \qquad \text{with } Q_2(t) \in \Sigma.$$

Then the conclusion follows with $Q(t) = Q_1(t) + Q_2(t)$. //

If we have enough compactness we can define the Morse index of an isolated invariant set as follows (cf. also (3-1)).

Def. 4.13. Let S be a (C)-invariant set (cf.Def.3.1.), then we set

$i(S) = i(U)$ where $U \in \Sigma$ is a sufficiently small neighborood of S.

From the above definition and theorem 4.10 we get

Corollary 4.14. Let X and X_k be as in theorem 4.10.

Moreover suppose that X_k satisfy the property (C) $(k=1,\ldots,N)$ and set $S_k = I(X_k)$.

Then we have

$$\sum_{k=1}^{N} i(S_k) = i(X) + (1+t)Q(t) \qquad Q \in S .$$

Observe that in Corollary 4.14. the property (C) for X is not required.

Example 4.15. Let η be a flow as in Example 2.9. Suppose that X and the X_k's satisfy the assumptions of lemma 4.14.Moreover suppose that each X_k contains only one nondegenerate critical point x_k.

Therefore, by the Example 4.4. and Corollary 4.14,, we get

$$\text{(4-7)} \qquad \sum_{k=1}^{N} t^{m(x_k)} = i(X) + (1+t)Q(t) \qquad Q \in \Sigma .$$

More in particular, if $F(X) = D f(x)$, then $m(x)$ reduces to the classical Morse index and (4-7) reduces to the classical Morse inequalities.

§ 5 - Variational systems.

Let $f \in C^1(M, \mathbb{R})$ and suppose that f' is bounded on bounded sets. If η is a semiflow on M we denote by $Df(x)$ the Dini derivative of f at x i.e.

$$Df(x) = \max \lim_{t \to 0^+} \frac{f(x \cdot t) - f(x)}{t}$$

$K(X)$ will denote the set of the critical points of f in X, i.e.

$$K(X) = \{ x \in X : f'(x) = 0 \} .$$

We need also an other notation

$$\Delta_a^b = f^{-1}([a,b]) .$$

Def. 5.1. A variational system relative to f is a couple

$$\{\eta , \Sigma (\eta) \} \quad \text{such that}$$

(i) if $X \in \Sigma$ and $f|_X$ is bounded, then $\forall \varepsilon > 0 \; \exists \; \delta > 0$ such that

$$Df(x) \leq - \delta \qquad \forall x \in X - N_\varepsilon (K(X))$$

(in particular $Df(x) \leq 0 \quad \forall x \in M$).

(ii) $d(x, \eta(t,x)) \leq \alpha(|t|)$ where α is a monotone function such that $\alpha(0) = 0$.

(iii) $K(\Delta_a^b)$ is compact.

We recall the condition (c) of Palais and Smale [P. S] which is essential in costructing variational systems.

Def.5.2. Let $f \in C^1(M)$. We say that f satisfies (P.S.) if any sequence $\{x_n\}$ such that $f(x_n)$ is bounded and $f'(x_n) \to 0$ is precompact.

If f satisfies P.S. and we assume that

(5-2)
$$f \in C^2(M)$$
$$\|f'(x)\| \leq M_1$$

then the equation

(5-3) $\qquad \dot{x} = -f'(x)$

has a unique solution for every $x \in M$ and $t \in \mathbb{R}$.
If η is the flow relative to (5-3) then it is not difficult to check that $\{\eta,\Sigma\}$ is a variational system for f.
However it is not necessary to assume (5-2) in order to construct a variational system relative to f.

Prop. 5.2. If f satisfies P.S. then there exists a variational system $\{\eta,\Sigma\}$ relative to f.
Proof. In [P] Palais has proved that f admits a pseudogradient vector field i.e. a map $F : M \to T M$ such that

(i) the equation $\dot{x} = F(x)$ has a unique solution for every initial point $x \in M$

(5-4) (ii) $< F(x), f'(x) > \geq \alpha(\|f'\|)$ where α is a strictly monotone function with $\alpha(0) = 0$

(iii) $\|F\|$ is bounded .

Now, if η is the flow relative to the equation (5-4) (i), it is not difficult to prove that $\{\eta, M\}$ is a variational system relative to f. //

We now need a technical lemma :
Lemma 5.3. Let $\{\eta,\Gamma\}$ be a variational system relative to f. Then we have

(i) $\exists\ T > 0$ such that $G^T(\Delta_a^b)$ is bounded

(ii) let α and β $(a < \alpha \le \beta < b)$ be two constants such that $K(\Delta_a^\alpha \cup \Delta_\beta^b) = \phi$. Then $\exists\ T > 0$ such that $G^T(\Delta_a^b) \subset \Delta_\alpha^\beta$

(iii) if a and b are regular values for f, then $\Delta_a^b \in \Sigma_o$ (Σ_o is defined in § 2)

(iv) if c is the only critical value of $f|_X$ $(X \in \Sigma,\ X \subseteq \Delta_a^b)$, then $I(X) = K(X)$ and X satisfies the property (C).

Proof. (i) Take r big enough in order that $K(\Delta_a^b) \subset B_r$ (this is possible by (iii) of Def. 5.1.), and set

$$\delta = \inf\{\|Df(x)\| : x \in \Delta_a^b - B_r\}.$$

By (i) of Def. 5.1., we have $\delta > 0$.

Now take

$$T = \frac{b-a}{\delta}$$

We claim that

(5-5) $$x \in G^T(\Delta_a^b) \implies \exists\ \bar{t} \in [-T,T] \text{ such that } x \in B_r \cap \Delta_a^b.$$

In fact if $x\cdot[-T,T] \subset \Delta_a^b - B_r$, then $Df(x\cdot t) \le -\delta$ for every $t \in [-T,T]$.

Therefore $b - a \ge f(x(-T)) - f(x(T)) \ge \int_{T}^{-T} Df(x\cdot t)dt \ge 2T\delta \ge 2(b-a)$.

The contradiction above implies (5-5).

Now if $x \in \Delta_a^b - B_r$ with $R > r + \alpha(T)$, by (ii) of Def. 5.1. we have that

$$x\cdot[-T,T] \cap B_r = \phi.$$

By the above formula and (5-5) the conclusion follows.

(ii) It follows easly by (i) and (iii) of Def.5.1.

(iii) By (i) and (ii) it follows that there exists $\varepsilon, R > 0$ such that

$$G^T(\Delta_{a-\varepsilon}^{b+\varepsilon}) \subset \Delta_{a+\varepsilon}^{b-\varepsilon} \cap B_R.$$

We claim that

$$N_\varepsilon(\Delta_{a+\varepsilon}^{b-\varepsilon} \cap B_R) \subset \Delta_a^b$$

Suppose that the above formula does not hold. Then there exist sequences $\phi_n \notin \Delta_a^b$ and $z_n \in \Delta_{a+\varepsilon}^{b-\varepsilon} \cap B_R$ such that $d(\phi_n, z_n) \to 0$.

Then, by the mean value theorem we have :

$$\varepsilon < |f(y_n) - f(x_n)| \le \|f'(\xi_n)\|\, d(y_n, x_n)$$

The above formula is absurd since we have supposed that f' is bounded on bounded sets.

(iv). Let U be any closed neighborood of K(X). We have to prove that $\exists\ T > 0$ such that $G^T(X) \subset U$.

Now let $V \subset U$ be any other closed neighborood of K(X) with

(5-6) $$d(V, \partial U) > 0$$

and set

(5-7) $$\delta = \inf\{\|Df(x)\| : x \in X - V\}.$$

By (i) of Def.1.5. we have $\delta > 0$.

Now suppose that $x \in X - U$ and that $x\cdot t \in V$. By (5-6) and (iii) of Def. 5.1., there exists $\tau > 0$ such that

$$|t| > \tau.$$

Then by (5-7)

(5-8) $$|f(x) - f(x\cdot t)| \ge \delta\tau.$$

Now we set $\varepsilon = \delta\tau$ and $\delta_1 = \inf\{\|Df(x)\| : x \in (\Delta_{c+\varepsilon}^b \cup \Delta_a^{c-\varepsilon}) \cap X\}$.

Since c is the only critical value of f in X, then $\delta_1 > 0$.

Now we set $\quad T = \dfrac{b-a}{\delta_1} + \tau$. We claim that $x \notin U \Longrightarrow x \notin G^T(X)$.

In order to prove the claim above we suppose $f(x) \leq c$ (if $f(x) \geq c$ we argue in the same way).

By (5-8) we have that $f(x\cdot\tau) \leq c-\varepsilon$.

Now arguing indirectly we suppose that

$$x\cdot[\tau,T] \subset X \ .$$

Then we have

$$b-a > f(x\cdot\tau) - f(x\cdot T) \geq \int_\tau^T Df(x\ t)\ dt \geq (T-\tau)\delta_1 = b-a \ .$$

The contradiction above implies the conclusion. //

Now set

(5-9) $\mathcal{K}_0 = \{K \subset K(M) \mid d(K, K(X)-K) > 0$ and K consists of a finite number of connected components $\}$.

Theorem 5.4. Let $\{M,\eta\}$ be a variational system relative to f. Then

(i) if $K \in \mathcal{K}_0$, then it is a (C)-invariant set; in particular $i(K)$ is well defined.

(ii) if $\{\tilde\eta,\tilde\Sigma\}$ is an other variational system, and $K \in \mathcal{K}_0$, then

$$i(K,\eta) = i(K,\tilde\eta).$$

This means that $i(K)$ depends only on f and not on η.

(iii) if $K_1, K_2 \in \mathcal{K}_0$ and $K_1 \cap K_2 = \phi$, then $i(K_1 \cup K_2) = i(K_1) + i(K_2)$.

(iv) if $X \in \Sigma$, $f|_X$ is bounded below $K(X) \in \mathcal{K}_0$, then

$$i(K(X)) = i(X) + (1+t)\ Q(t) \qquad Q \in \text{[illegible]}.$$

Proof. (i) is a trivial consequence of lemma 5.3.(iv).

(ii) To semplify the proof from unessential technicalities we suppose that M is a (may be infinite dimensional) manifold and that η and $\tilde\eta$ respectively are the flows relative to the following equations

$$\dot x = F(x) \qquad\qquad \dot x = \tilde F(x) \ .$$

Now let η_λ be the flow relative to the following equation

$$\dot x = (1-\lambda)F(x) + \lambda F(x) \qquad\qquad \lambda \in [0,1] \ .$$

Clearly for every $\lambda \in [0,1]$, (M,η_λ) is a variational system relative to F and K is a (C)-invariant set for η_λ.

Take $\bar\lambda \in [0,1]$ and let $U_{\bar\lambda}$ be a neighborood of K which satisfies the property (C); it exists by (i).

By proposition 3.7., $U_{\bar\lambda} \in \Sigma$.

Then by the theorem 2.1., $i(U_{\bar\lambda},\eta_\lambda)$ is constant for $\lambda \in \Gamma_{\bar\lambda}$ where $\Gamma_{\bar\lambda}$ is a suitable neighborood of $\bar\lambda$.

This implies that $i(K,\eta_\lambda)$ is constant for $\lambda \in \Gamma_{\bar\lambda}$ for every $\bar\lambda \in [0,1]$.

Thus it follows that

$$i(\eta,K) = i(\eta_0,K) = i(\eta_1,K) = i(\tilde\eta,K) \ .$$

(iii) It follows by Theorem 4.5.(iii).

(iv) Since $K(X)$ has a finite number of connected components, $f|_X$ has only a finite number of critical values $c_1,\dots,c_N$.

Set

$a_0 = \inf f(X)$

a_ℓ any number in $(c_\ell, c_{\ell+1})$ for $\ell = 1,\dots,N$ and $a_{N+1} = +\infty$.

Now set

$$X_\ell = \Delta_{a_\ell}^{a_\ell+1} \cap X \qquad \text{for } \ell = 0,\ldots,N \ .$$

Then $\{X_\ell\}$ is a Morse decomposition of X (cf.Ex.4.9.). Then by theorem 4.9. we have

(5-10) $$\sum_{\ell=0}^{N} i(X_\ell) = i(X) + (1+t)\, Q(t) \qquad t \in S \ .$$

By lemma (5.3.) (iv), $i(X_\ell) = i(K(X_\ell))$. Using proposition 5.4.(iii), we have

$$i(K(X)) = i\Big(\bigcup_{\ell=0}^{N} K(X_\ell)\Big) = \sum_{\ell=0}^{N} i(K(X_\ell)) = \sum_{\ell=0}^{N} i(X_\ell) \ .$$

By the above formula and (5-10) the conclusion follows. //

Now we suppose that M is Hilbert manifold modelled on a space E (i.e. $T_x M \cong E \quad \forall x \in M$) and that η is the flow relative to the differential equation

$$\dot{x} = F(x) \ .$$

Suppose that x is a critical point of F such that $f''(x) : T_x M \to T_x M$ is defined. If

(5-11) $\qquad f''(x) : T_x M \to T_x M \qquad$ has a discrete spectrum , we set

(5-12)
$$\begin{cases} m(x) = & \text{dimension of the space spamed by the eigenvector of } f''(x) \\ & \text{corresponding to negative eigenvalues} \\ m^*(x) = m(x) + \dim\,[\ker f''(x_o)] \end{cases}$$

We recall that a critical point x is called non-degenerate if $\ker f''(x) = \{o\}$. In this case $m(x) = m^*(x)$.

<u>Theorem 5.5</u>. If x_o is a nondegenerate critical point of f, then $\{x_o\} \in \mathcal{K}_o$ and

$$i(x_o) = t^{m(x_o)} \ .$$

<u>Remark</u>. Observe that in theorem 5.4. we do not assume that $f''(x)$ is defined in a neighborood of x_o; it is sufficient that it is defined in x_o. A similar result has been obtained in the contest of the classical Morse theory by Mercuri and Palmieri [Me.P.] .

<u>Proof</u>. Since x_o is nondegenerate, it is isolated; thus $\{x_o\} \in \mathcal{K}_o$. Now let $\tilde{\eta}$ be the flow relative to the differential equation

$$\dot{x} = -f''(x_o)\cdot(x-x_o) \ .$$

If U is a small enough neighborood of x_o, then $\{\tilde{\eta},U\}$ is a variational system relative to f.

Then by Proposition 5.4. (ii)

$$i(\{x_o\},\eta) = i(x_o,\tilde{\eta}) .$$

But we know by Example 4.4. that $i(x_o,\tilde{\eta}) = t^{m(x)}$. //

The next Corollary follows straightforwardly :

<u>Corollary 5.7</u>. Suppose that $X \in \Sigma$ and that X contains only nondegenerate critical points of f, $x_1,\ldots,x_N$. Then

$$\sum_{k=1}^{N} i(x_k) = i(X) + (1+t)\, Q(t) \qquad t \in S$$

<u>Remark 5.8</u>. Notice that in the Corollary above we require only that $f''(x)$ is defined only when x is a critical point of f. This situation occurs quite often when we apply the Morse theory to P.D.E's.

We now need some other notation. If K is a set of critical points of f, then we set

$$(5\text{-}13)\qquad m(K) = \inf_{x\in K} m(x) \qquad m^*(K) = \sup_{x\in K} m^*(x)$$

The following theorem is quite useful in applications :

Theorem 5.9. Suppose that $U \in \Sigma_o$, that $f|_U$ is bounded and that $f \in C^2(U)$. Then

$$i(U) = \sum_{\ell=m(K)}^{m^*(K)} a_\ell t^\ell \qquad a_\ell \geq 0$$

where $K = K(U)$.

The proof of theorem 5.9. is based on some results of Marino and Prodi [M.P.] which can be summarised in the following lemma :

Lemma 5.10. Let f, U and K as in theorem 5.9. Then for every $\varepsilon > 0$ there exists a function $g_\varepsilon \in C^2(U)$ such that

(i) $|f - g_\varepsilon|_{C^2(U)} \leq \varepsilon$

(ii) g_ε has only a finite number of critical points in U and they are not degenerate

(iii) all the critical points of g_ε in U are contained in $N_\varepsilon(U)$.

Proof of Th. 5.9. Let $\lambda_1(x) \leq \lambda_2(x) \leq \ldots \leq \lambda_k(x) \leq \ldots$ be the eigenvalues of $f''(x)$. They are continuous functions of x in U since $f \in C^2(U)$.

Now let $s = m(K)$ and $r = m^*(K) + 1$.

By the definition of $m(K)$ and $m^*(K)$ we have that

$$(5.14)\qquad \lambda_s(x) < 0 < \lambda_r(x) \qquad \text{for every } x \in K.$$

Now take ε_1 small enough in order that (5-14) holds for every $x \in N_{\varepsilon_1}(K)$. This is possible since the $\lambda_k(x)$ are continuous in x and K is compact. Now let

$$\lambda_1^\varepsilon(x) \leq \lambda_2^\varepsilon(x) \leq \ldots \leq \lambda_k^\varepsilon(x) \leq \ldots$$

be the eigenvalue of the operator $g_\varepsilon''(x)$ where g_ε is a function as in lemma 5.10.

Now chose $\varepsilon < \varepsilon_1$ small enough that

$$(5\text{-}15)\qquad \lambda_s^\varepsilon(x) < 0 < \lambda_r^\varepsilon(x) \qquad \forall x \in N_{\varepsilon_1}(K).$$

Thus we have that all the critical points $x_1, \ldots, x_N$ of g_ε are nondegenerate, contained in N_{ε_1}, and by (5-15)

$$(5\text{-}16)\qquad s \leq m(x_k) \leq r - 1 \qquad k=1,\ldots,N$$

where $m(x_k)$ is the Morse index of x_k for g_ε.

Now if η_g is the flow relative to the equation

$$\dot{x} = -g_\varepsilon'(x)$$

and if ε has been chosen small enough, $U \in \Sigma_o(\eta_g)$ and

$$(5\text{-}17)\qquad i(U,\eta) = i(U,\eta_g)$$

by virtue of theorem 2.1.

By Corollary (5.7)

$$(5\text{-}18)\qquad \sum_{k=1}^{N} i(x_k) = i(U,\eta_g) + (1+t)\, Q(t) \qquad Q \in S$$

where the x_k s are the critical values of g_ε in U.

By theorem 5.5. and 5.13. we have

$$\sum_{k=1}^{N} i(x_k) = \sum_{k=1}^{N} t^{m(x_k)} = \sum_{\ell=s}^{r-1} a_\ell t^\ell .$$

By (5-17), 5-18) and the above formula we have

$$\sum_{\ell=s}^{r-1} a_\ell t^\ell = i(U,\eta) + (1+t)\, Q(t) .$$

From the definition of s and r the conclusion follows. //

<u>Corollary 5.11</u>. If a and b are not critical values for $f' \in C^2(\Delta_a^b)$

$$i(\Delta_a^b) = \sum_{\ell=m(K)}^{m^*(K)} a_\ell t^\ell$$

where $K = K(\Delta_a^b)$.

<u>Proof</u>. Use Theorem 5.9. and Lemma 5.3. (iii). //

REFERENCES

[B] BENCI, V., Some applications of the generalized Morse-Conley index, to appear in "Conferenze del Seminario di Matematica dell'Università di Bari"

[C] CONLEY, C., Isolated invariant sets and the Morse index, Regional Conference Series in Mathematics, 38, (1976)

[C.Z.] CONLEY, C., and ZHENDER, E., Morse type index theory for flows and periodic solutions for Hamiltonian equations, Comm. Pure Appl. Math. <u>37</u>, (1984), 207-253

[M.P.] MARINO ,A., and PRODI G., Metodi perturbativi nella teoria di Morse, Boll. U.M.I., <u>11</u>, Suppl. Fasc.3, (1975), 115-132

[Me.P.] MERCURI F., and PALMIERI, G., Morse theory with low differentiability, preprint

[P] PALAIS, R.S., Ljusternik Schnirelman theory on Banach manifolds, Topology, <u>5</u>, (1966), 115-132

[P.S.] PALAIS, R.S., and SMALE, S., A generalized Morse Theory, Bull. Am. Math. Soc., <u>70</u>, (1964), 165-172

[R] RYBAKOWSKI, K.P., On the homotopy index for infinite dimentional semiflows, Trans. Am. Math. Soc., <u>296</u>, (1982), 351-381

[S] SALAMON, D., Connected simple systems and the Conley index of isolated invariant sets, to appear in Trans. Am. Math. Soc.

[Sp] SPANIER, E.H., Algebraic topology, Mc Graw-Hill, (1966)

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VIERI BENCI

SOME APPLICATIONS OF THE GENERALIZED MORSE-CONLEY INDEX

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Introduction

The generalized Morse-Conley Index was recently generalized to infinite dimensional spaces by the author [B1]. The purpose of this paper is to apply the generalized Morse-Conley index to problems in differential equations.

Some results are known and they are presented here just to give different proofs using the generalized Morse-Conley theory. However, there are some new results as Theorems 3.10, 3.14, 4.7 and related results. It seems to us that these results are not easy to obtain using other methods.

This paper is organized as follows. In sections 1 and 2 we recall basic definitions and review results on the generalized Morse-Conley theory as presented in [B1]. In section 3 we give some abstract critical point theorems and apply them to obtain existence results for elliptic pde's. In section 4 we present some critical point theorems for symmetric functional and apply them to prove the existence of periodic solutions of a second order system of ode's.

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1. Definition and main properties of the generalized Morse-Conley Index.

Let M be a metric space (with distance $d(\cdot\ ,\ \cdot)$) on which a semiflow η is defined i.e. a continuous map

$$\eta : R^+ \times M \to M$$

such that $\eta(0, x) = x$ and $\eta(t_1, \eta(t_2, x)) = \eta(t_1 + t_2, x)$ $(t_1, t_2 \in R^+, x \in M)$. When no ambiguity is possible we use the shorthand notation $x \cdot t$ instead of $\eta(t, x)$. If a semiflow is defined also for negative t, it is called a flow. If X is an open subset of M and T a positive constant we set

$$G^T(X) = G^T(X, \eta) = \{x \in M \mid x \cdot [-T, T] \text{ is defined and } x \cdot [-T, T] \subset \overline{X}\}$$

where $\overline{X}$ denotes the closure of X.

$$\Gamma^T(X) = \{x \in G^T(x) \mid x \cdot [0, T] \cap \partial X \neq \phi\}$$

where ∂X denotes the boundary of X.

Also we set

$$\Sigma = \Sigma(\eta) = \{X \subset M \mid X \text{ is open and } \exists T > 0 \text{ such that } G^T(X) \subset X\}.$$

$\mathcal{S}$ will denote the set of formal power series in t with nonnegative coefficients (or to be more precise with coefficients which are cardinal numbers).

The generalized Morse-Conley Index (GIM) is a map

$$i(\cdot\ , \eta) : \Sigma \to \mathcal{S}$$

defined as follows

$$i_t(X, \eta) = \lim_{T \to +\infty} \sum_{k=0}^{\infty} \dim[\overline{H}^k(G^T(X, \eta), \Gamma^T(X, \eta))]t^k \tag{1.1}$$

where $\overline{H}^*(\cdot\ , \cdot)$ denotes the Alexander-Spanier [Sp] cohomology with coefficients in some field K.

The limit in (1.1) exists in a trivial sense; in fact in [B1], it is proved that, for T large enough, $H^*(G^T(X), \Gamma^T(X))$ does not depend on T. When no ambiguity is possible we shall write $i(X)$ instead of $i_t(X, \eta)$.

Now we shall list some of the properties of the GIM which have been proved in [B1].

THEOREM 1.1. The GIM satisfies the following properties

(i) if $X \in \Sigma$ then $G^T(X) \in \Sigma$ and $i(G^T(X)) = i(X)$ $\forall T > 0$

(ii) if $X \in \Sigma$ then $\eta(T, X) \in \Sigma$ and $i(\eta(T, X)) = i(X)$ $\forall T > 0$

(iii) if $X,\ Y \in \Sigma$ and $\exists T > 0$ such that $G^T(X) \subset Y$ and $G^T(Y) \subset X$, then $i(X) = i(Y)$

(iv) if $x \in X$ and for every $x \in X$ $\exists t > 0$ such that $x \cdot t \notin X$, then $i(X) = 0$

(v) if $X \in \Sigma$ is contractible and positively invariant, then $i(X) = 1$

(vi) if $X,\ Y \in \Sigma$ and $\overline{X} \cap \overline{Y} = \phi$, then $i(X \cup Y) = i(X) + i(Y)$

(vii) if η_i is a semiflow on M_i $(i = 1, 2)$, then a semiflow $\eta_1 \times \eta_2$ is defined on $M_1 \times M_2$; in this case if $X_i \in \Sigma(\eta_i)$ $(i = 1, 2)$, then $X_1 \times X_2 \in \Sigma(\eta_1 \times \eta_2)$ and

$$i(X_1 \times X_2) = i(X_1, \eta_1) \cdot i(X_2, \eta_2).$$

We need now some other definitions:

DEF. 1.2. Let $X_1,\ X_2 \in \Sigma$ with $X_1 \cap X_2 = \phi$. We say that X_2 is *over* X_1 is there exists $T > 0$ such that $X_1 \cap G^T(X_1 \cup X_2)$ is positively invariant with respect to $G^T(X_1 \cup X_2)$.

DEF. 1.3. Let $X \in \Sigma$. A family of sets $\{X_k\}_{k \in N}$ is called a Morse decomposition of X if

(i) $$\overline{X} = \bigcup_{k=1}^{N} \overline{X}_k$$

(ii) $$X_k \in \Sigma \text{ for } k = 1, \ldots, N$$

(iii) $$X_k \cap X_h = \phi \text{ for } k \neq h$$

(iv) $$X_{h+1} \text{ is over } \bigcup_{j=1}^{h} X_j \text{ for } h = 1, \ldots, N-1.$$

Example. Let f be a Liapounov function for (M, η) (i.e. a function strictly decreasing on non-stationary trajectories), and let $c_1 < c_2 < \ldots < c_{N-1}$ be a sequence of regular values for f (i.e. $f(x) = c_i \Rightarrow f'(x) \neq 0, i = 1, \ldots, N-1$). Now set $c_0 = -\infty$ and $c_N = +\infty$ and

$$X_k = \{x \in X \mid c_{k-1} < f(x) < c_k\} \quad k = 1, \ldots, N; \; X \in \Sigma.$$

It is easy to check that $\{X_k\}$ is a Morse decomposition of X.

THEOREM 1.4. If $\{X_k\}_{k \leq N}$ is a Morse decomposition of X, then there exists $Q \in \mathscr{S}$ such that

$$\sum_{k=1}^{N} i(X_k) = i(X) + (1+t)Q(t).$$

Now let $\Sigma_0(\eta) \subset \Sigma(\eta)$ be a family of sets which satisfy the following properties

$$(1.2) \quad \begin{cases} \text{(i)} \quad \exists T, \delta > 0 \text{ such that } N_\delta[G^T(N_\delta(X))] \subset X \\ \text{(ii)} \quad \exists \delta > 0 \text{ such that } \eta(\cdot, T)|_{N_\delta(x)} \text{ is uniformly continuous in } X. \end{cases}$$

If $X \in \Sigma_0(\eta)$, then $i(X, \eta)$ does not change for "small" perturbation of X and η.

THEOREM 1.5. Let $X \in \Sigma_0(\eta)$. Then there are constants ϵ and T depending on X and η with the following property.

If $\tilde{X}$ and $\tilde{\eta}$ satisfy the following inequalities

(i) $d_H(X, \tilde{X}) < \epsilon$ where $d_H(\cdot, \cdot)$ denotes the Hausdorf distance

(ii) $d(\eta(t, x), \tilde{\eta}(t, x)) < \epsilon \quad \forall x \in X, \quad \forall t \in [-T, T]$

then

(i) $\tilde{X} \in \Sigma_0(\tilde{\eta})$

(ii) $i(\tilde{X}, \tilde{\eta}) = i(X, \eta)$.

From theorem 1.5 it is not difficult to prove the homotopy invariance of the GIM:

THEOREM 1.6. Let $X_\lambda \subset Z$ ($Z \subset M$; $\lambda \in [0, 1]$) be a family of sets depending continuously on λ with respect to the Hausdorff metric and let η_λ be a family of flows uniformly continuous in $[-T, T] \times Z$, depending continuously on λ with respect to the topology of the uniform convergence in $[-T, T] \times X$ for every $T > 0$.

Then, if $X_\lambda \in \Sigma_0(\eta_\lambda)$, the index

$$i(X_\lambda, \eta_\lambda)$$

does not depend on λ.

We end this section defining the index of an invariant set (when possible).

DEF. 1.7. We say that $S \subset X$ is a (C)-invariant set if

(i) S is an invariant set

(ii) S has a neighborhood U such that for any other neighborhood V of S there exists $T > 0$ such that $G^T(U) \subset V$.

Definition 2.6 is justified by the following

THEOREM 1.8. If U and V are sufficiently small open neighborhood of a (C)-invariant set S, then $U, V \in \Sigma$ and

$$i(U) = i(V).$$

The following definition follows is a natural way from theorem 1.8.

DEF. 1.9. If S is a (C)-invariant set then we set

$$i(S) = i(U)$$

where V is a sufficiently small open neighborhood of S.

A study of flows where all the invariant sets are (C)-invariant has been done in [RY].

2. The Generalized Morse Index and Variational Systems

We now suppose that M is a Hilbert manifold and we take $f \in C^1(M, R)$, moreover we suppose that f' is bounded on bounded sets. We shall use the following notation.

$$f_a^b := \{x \in M \mid a < f < b\}$$

$$f^b := f_{-\infty}^b; \; f_a := f_a^{+\infty}$$

$$K(A) = K(A, f) = \{x \in A \mid f'(x) = 0\}.$$

DEF. 2.1. A variational system relative to f is a couple η, Γ with $\Gamma \subset \Sigma(\eta)$ such that

(0) $f|_\Gamma$ is bounded

(i) $\forall X \in \Gamma,\ \forall \varepsilon > 0\ \ \exists \delta > 0$

$$Df(x) \leqslant -\delta \qquad \forall x \in X - N_\varepsilon(K(x))$$

where

$$Df(x) = \max \lim_{t \to 0^+} \frac{f(x \cdot t) - f(x)}{t} .$$

In particular $t \to f(x \cdot t)$ is strictly decreasing unless $x \in K(X)$.

(ii) $d(x, \eta(t, x)) \leqslant \alpha(|t|)$ where α is a monotone function with $\alpha(0) = 0$

(iii) $K(X)$ is compact for every $X \in \Gamma$.

The interest of variational systems as defined above relies on the following theorem.

THEOREM 2.2. Let $f \in C^1(M, R)$ satisfy $(P \cdot S)$ in $[a, b]$, i.e.

$$(2.1) \qquad \begin{cases} \text{every sequence } x_n \text{ such that } f(x_n) \to c \in [a, b] \text{ and } f'(x_n) \to 0 \\ \text{has a converging subsequence.} \end{cases}$$

Then there exists a variational system $\{\Gamma, \eta\}$ relative to f, where $\Gamma = \Sigma(\eta) \cap \mathscr{P}(f_b^a)$. In particular, if f satisfies $(P \cdot S)$ in $(-\infty, +\infty)$ (i.e. for every $c \in R$) then $\Gamma = \{x \in \Sigma(\eta) \mid f|_X$ is bounded$\}$. The proof of this theorem is essentially contained in [B1].

We set

$$\mathscr{K} = \{K = K(X) \mid X \in \Gamma \text{ and } K \text{ has a finite number of connected components}\}.$$

Notice that the definition of K depends only on f but not on the particular variational system we have chosen.

PROPOSITION 2.3. Let $\{\eta, \Gamma\}$ be a variational system relative to f. Then we have the following

(i) if a and b are regular values of f and $f_a^b \in \Gamma$, then $f_a^b \in \Sigma_0$.

(ii) for every $X \in \Gamma$, $\exists T > 0$ such that $G^T(X)$ is bounded.

(iii) if $K \in \mathscr{K}$, then it is a (C) - invariant set; in particular $i(K)$ is well defined

(iv) if $\{\eta, \Gamma\}$ and $\{\tilde{\eta}, \tilde{\Gamma}\}$ are two variational systems relative to f then

$$i(K, \eta) = i(K, \tilde{\eta})$$

(v) suppose that $x \in \Gamma$, and that x_0 is the only critical point of f in X.

Then $\{x_0\} \in \mathscr{K}$ and

$$i(X) = i(x_0)$$

(vi) if $f|_X$ is bounded below and $K(X) = \phi$, then $i(X) = 0$

(vii) if a and b are regular values of f, then $i(f_a^b) = \Sigma_n \dim\,[H_n(f^b, f^a)]t^n$ where H_n denotes the singular homology.

Proof. The proofs of (i), ... , (iv) can be found in [B1]. In order to prove (v), it is sufficient to observe that for every neighborhood U of x with $U \subset X$, there exist $T > 0$ such that $G^T(X) \subset U$. The conclusion follows from Th. 1.1 (iii).

(vi) If X does not have critical points, by (i) of definition 2.1, every trajectory $x \cdot t$ (with $x \in X$) exist from the set X. Then by Theorem 1.1 $i(X) = 0$.

(vii) Since a and b are regular values of f the Alexander-Spanier cohomology of the pair (f^b, f^a) coincides with the singular cohomology. Moreover since our coefficients are in a field,

$$\dim H^n(f^b, f^a) = \dim H_n(f^b, f^a).$$

Then the conclusion follows from the fact that (f^b, f^a) is an index pair for f_a^b (cf. e.g. [B1]). □

The following definition is very important in our generalization of the Morse theory.

DEF. 2.4. Let $X \in \Gamma$ and $K = K(X)$.
A finite family of sets $\{U_j\}_{j \in N}$ is called an ε-Morse covering of K if

(i) $\overline{U}_j$ is connected for $j = 1, \dots, N$

(ii) $K \subset \bigcup_{j=1}^{N} U_j \subset N_\epsilon(K)$

(iii) $U_j \in \Gamma$ and $\sum_{j=1}^{N} i(U_j) = i(X) + (1+t)\, Q(t) \qquad Q \in \mathcal{S}$

The above definition is justified by the following theorem

THEOREM 2.4. If $X \in \Gamma$, then for every $\epsilon > 0$ there exists an ϵ-Morse covering of $K(X)$.

Proof. For every $c \in \overline{f(X)}$ there exists $\delta(c) > 0$ such that $f_{c-\delta(c)}^{c+\delta(c)} \in \Gamma$ and that

(2.1) $$G^{T(c)}(f_{c-\delta(c)}^{c+\delta(c)}) \subset N_{\epsilon/2}(K).$$

Since $\overline{f(X)}$ is compact there exists a finite covering $\{(c_i - \delta(c_i),\ c_i + \delta(c_i))\}_{i \le N}$ of $\overline{f(X)}$.

Now let $b_0, \dots, b_n$ an increasing sequence of regular values of f such that $b_0 = \inf_X f$; $b_n = \sup_X f$ and for every $\ell = 1, \dots, n-1$.

(2.2) $$c_i + \delta(c_i) \le b_{\ell-1} < b_\ell \le c_i + \delta(c_i) \qquad \text{for some } i = 1, \dots, N.$$

Now we set

(2.3) $$A_\ell = X \cap f_{b_{\ell-1}}^{b_\ell} \qquad \ell = 1, \dots, n.$$

By our construction we have that A_ℓ is a Morse decomposition of Γ (cf. Def. 1.3) and by (2.1), (2.2) and (2.3) we get

(2.4) $$G^T(A_\ell) \subset N_{\epsilon/2}(K) \qquad \text{for } T \text{ large enough.}$$

By theorem 1.4 we have

(2.5) $$\sum_{\ell=1}^{n} i(A_\ell) = i(X) + (1+t)\, Q(t) \qquad Q \in \mathcal{S}$$

Setting

$$\sigma = \{\ell \mid A_\ell \cap K \neq \Phi\}$$

by (2.5) and Th. 1.1 (iv) we have

(2.6) $$\sum_{\ell \in \sigma} i(A_\ell) = i(X) + (1+t)\, Q(t) \qquad Q \in \mathcal{S}$$

Now set

(2.7) $$U_\ell = A_\ell \cap N_\epsilon(K)$$

Then by (2.4) and the fact that $A_\ell \in \Gamma$ we have that

$$G^T(A_\ell) \subset U_\ell \qquad \text{and } G^T(U_\ell) \subset A_\ell$$

Then, by Th. 1.1 (iii), we have

$$i(A_\ell) = i(U_\ell)$$

Using the above formula and (2.6) we get

(2.8) $$\sum_{\ell \in \sigma} i(U_\ell) = i(X) + (1+t)\, Q(t) \qquad Q \in \mathcal{S}$$

Now for $\ell \in \sigma$, let $\{\overline{U}_{\ell,k}\}_{k < n_\ell}$ be the family of connected components of $\overline{U}_\ell$.

We claim that $\{U_{\ell,k}\}_{\ell \in \sigma; k \le n_\ell}$ is a ϵ-Morse covering of $K(X)$. (i) and (ii) of the Def. 2.4 are trivially satisfied.

$U_{\ell,k} \in \Gamma$ since $U_\ell \in \Gamma$. Moreover, since

$$\overline{U}_{\ell,k} \cap \overline{U}_{\ell,h} = \Phi \qquad \text{for } k \neq \ell.$$

By theorem 1.1 (vi), we have

$$\sum_{k=1}^{n_\ell} i(U_{k,\ell}) = i(U_\ell).$$

By the above formula and (2.8) we get

$$\sum_{\substack{\ell \in \sigma \\ k \le n_\ell}} i(U_{k,\ell}) = i(X) + (1+t)\, Q(t).$$

So we have proved the theorem. □

COROLLARY 2.5. Suppose that the assumptions of theorem 2.4 are satisfied. Moreover suppose that $K(X)$ consists of a finite number of connected components $K_1, \ldots, K_n$. Then

$$\sum_{s=1}^{N} i(K_s) = i(X) + (1+t)\,Q(t).$$

Proof. It follows from Th. 2.4 (a) and Proposizion 2.3 (iii). □

COROLLARY 2.6. Suppose that f satisfies (P.S.) in $[c, +\infty)$ and suppose that $K(f_c)$ contains only a finite number of connected components $K_1, \ldots, K_N$. Then

$$\sum_{s=1}^{N} i(K_s) = i(f_c) + (1+t)\,Q \qquad Q \in \mathcal{S}$$

Proof. Let $c_1 = \max f|_{K(x)} + 1$. Then $f_c^{c_1} \in \Gamma$ and $f_{c_1} \in \Sigma$. By theorem 2.4, we get

$$i(f_c^{c_1}) + i(f_{c_1}) = i(f_c) + (1+t)\,Q_1 \qquad Q_1 \in \mathcal{S}$$

By theorem (1.1) (iv) and the definition of c_1,

(2.9) $$i(f_{c_1}) = 0.$$

By corollary 2.5

(2.10) $$i(f_c^{c_1}) = \sum_{s=1}^{N} i(K_s) - (1+t)Q_2.$$

The conclusion follows by (2.9) and (2.10). □

Now we are going to relate the Generalized Morse Index with the differential structure of (M, f). Suppose that x is a critical point of f such that

$$f''(x) : T_x M \to T_x M$$

is defined.

For the rest of this section we shall suppose that the nonpositive part of the spectrum of f consists of isolated eigenvalues of finite multiplicity and that f satisfies P.S. Now we set

$m(x)$ = dimension of the space spanned by the eigenvectors of $f''(x)$ corresponding to negative eigenvalues

$m^*(x) = m(x) + \dim\,[\ker f''(x)]$.

We shall call $m(x)$ the numerical Morse index of x.

We recall that a critical point x is called nondegenerate, if $f''(x)$ exists and it is invertible. In this case we have $m(x) = m^*(x)$. If $f|_X$ has only nondegenerate critical points then it is called a Morse function (on X).

We recall a theorem of Marino and Prodi [MP].

THEOREM 2.7. If f, Γ is a variational system, then for every $X \in \Gamma$ and for every $\varepsilon \in (0, \bar\varepsilon]$ (where $\bar\varepsilon = \bar\varepsilon(X)$) there exists a Morse function on X such that $\|f - f_\varepsilon\|_{C^1(X)} \leq \varepsilon$ and f_ε satisfies P.S. in X.

The following theorem characterizes the index of nondegenerate critical points.

THEOREM 2.8. If x_0 is a nondegenerate critical point of f, then $\{x_0\} \in \mathcal{K}$ and

$$i(x_0) = t^{m(x_0)}.$$

Proof. See [B1] Th. 5.5. □

Notice that in Th. 2.8 the fact that $f''(x)$ is defined in a neighborhood of x_0 is not needed. A similar result has also been obtain by Mercuri and Palmieri [MP]. Theorem 2.8 suggests the following definition:

DEF. 2.9. A critical point x is called topologically nondegenerate if $\{x\} \in \mathcal{K}$ and $i_1(x) = 1$ (i.e. if $i(x) = t^m$ for some $m \in N$). As a consequence of Th. 2.8, we can write the "classical" Morse relations:

COROLLARY 2.10. Suppose that $X \in \Gamma$ contains only topologically nondegenerate critical points of f. Let $\alpha(m)$ denote the number of critical points having Morse index m. Then

$$\sum_{m=0}^{\infty} \alpha(m) t^m = i(X) + (1+t)\,Q(t) \qquad Q \in \mathcal{S}$$

Proof. It follows from Corollary 2.5 and theorem 2.7. □

From Theorem 2.7 and Corollary 2.10, we get the following results:

COROLLARY 2.11. If $X \in \Gamma$, then $i(X)$ is finite (i.e. $i_1(X) < +\infty$).

Proof. By theorem 2.7 and theorem 1.5 we can find a Morse function f_ε such that $i(X, f) = i(X, f_\varepsilon)$. Since f_ε satisfies P.S. and $f_\varepsilon|_X$ is bounded, then f_ε has only a finite number of critical points. Then the conclusion follows from Corollary 2.10. □

Theorem 2.7 suggests the following definition.

DEF. 2.12. If x is a critical point of f, the number $i_1(x)$ will be called the multiplicity of f.

Notice that the definition 2.12 (as well as definition 2.9) can be extended also to critical sets $K \in \mathcal{K}$. Using this definition we have:

COROLLARY 2.13. If $X \in \Gamma$, then $f|_X$ has at least $i_1(X)$ critical points if counted with their multiplicity. *Proof.* Obvious.

Notice that Corollary 2.13 does not need the function f to be of class C^2.

We end this section with a result which is useful in some applications. If K is a set of critical points of f we set

$$(2.11) \qquad \begin{cases} m(K) = \inf\limits_{x \in K} m(x) \\ \\ m^*(K) = \sup\limits_{x \in K} m^*(x) \end{cases}$$

THEOREM 2.14. Suppose that $U \in \Gamma \cap \Sigma_0$, and that $f \in C^2(U)$. Then

$$i(U) = \sum_{\ell = m(K)}^{m^*(K)} a_\ell t^\ell$$

where $K = K(U)$.

Proof. See [B1] Th. 5.9. □

3. Some existence theorem.

Let $f \in C^1(E, \mathbb{R})$ and suppose that

(f_1) $f = \frac{1}{2}(Lx, x) - \psi(x)$ where L is an invertible bounded selfadjoint operator.

(f_2) $\lim\limits_{\|x\| \to +\infty} \psi'(x) = 0$, where ψ' is a compact operator

It is easy to check that (f_1) and (f_2) imply that f satisfies (P.S.). Therefore, there is a flow η such that $\{\eta, \Gamma\}$ is a variational system relative to f. Such a flow can be chosen of the form

$$(3.1) \qquad \dot{x} = -Lx + \tilde{\psi}(x)$$

where $\tilde{\psi}$ is a Lipschitz continuous compact operator (cf. e.g. [B3]).

LEMMA 3.1. There is $R > 0$ such that $B_R = : \{x \in E \mid \|x\| \leq R\} \in \Sigma(\eta)$ and $i(B_r) = t^{m(\infty)}$ where $m(\infty) = \#\{$negative eigenvalues of L counted with their multiplicity$\}$.

Proof. Let P^- be the projectors on the span of the negative eigenvalues of L and let $P^+ = (P^-)^\perp$. Let L^+, L^- and x^+, x^- be the corresponding decomposition of L and x respectively.

Now consider the family of flows η_τ relative to the family of equations:

$$(3.2) \qquad \begin{cases} \dot{x}^+ = L^+x^+ - \tau P^+ \tilde{\psi}(n) \\ \\ \dot{x}^- = L^-x^- - \tau P^- \tilde{\psi}(n) \end{cases} \qquad \tau \in [0, 1].$$

For $\tau = 1$ equation (3.2) is equal to equation (3.1).

Now it is not difficult to check that for R large enough, by virtue of (f_2), $\exists T$ such that $G^T(B_{R+1}) \subset B_{R-1}$ and such R and T are independent of τ. Therefore, $B_R \in \Sigma_0(\eta_\tau)$ for every $\tau \in [0, 1]$. By Theorem 1.6, this implies that $i(B_R, \eta_\tau)$ is independent of τ. Therefore $i(B_R) = i(B_R, \eta_1) = i(B_R, \eta_0)$. But $i(B_R, \eta_0) = t^{m(\infty)}$ by Proposition 2.3 (v) and theorem 2.7. □

THEOREM 3.2. Suppose that (f_1) and (f_2) hold, then f has at least one critical point.

Proof. It follows by Lemma 3.1 and Proposition 2.3 (iv).

THEOREM 3.3. Suppose (f_1) and (f_2) and that

$$f'(0)=0 \text{ and } i(0)=t^{m(0)} \text{ with } m(0)\neq m(\infty).$$

Then f has at least two other critical points $x_1, x_2 \neq 0$ (which may coincide if they are degenerate). Moreover, if they are not degenerate we have

$$i(x_1)=t^{m(\infty)};\quad i(x_2)=t^{m(0)\pm 1}.$$

Proof. By theorem 2.4 we have that, for ϵ sufficiently small

$$i(N_\epsilon(0))+\sum_{j=1}^{N} i(U_j^\epsilon)=i(x)=(1+t)\,Q(t)\quad Q\in\mathscr{S}$$

By lemma 3.1 and by the fact that $i(0)=t^{m(0)}$ we get

$$t^{m(0)}+\sum_{j=1}^{N} i(U_j^\epsilon)=t^{m(\infty)}+(1+t)\,Q(t)\qquad Q\in\mathscr{S}$$

Now, since $m(\infty)\neq m(0)$, then the above formula is satisfied if Q contains the term $t^{m(0)}$ or $t^{m(0)-1}$. Let us consider the first case. In this case $\sum_{j=1}^{N} i(U_j^\epsilon)$ must contain the terms $t^{m(0)+1}$ and $t^{m(\infty)}$. Now there are two possibilities. First possibility: there exists ϵ sufficiently small

$$\exists U_{j_1}^\epsilon \text{ and } U_{j_2}^\epsilon \text{ with}$$

and

$$i(U_{j_1}^\epsilon)=t^{m(0)}\quad+\text{ other possible terms}$$
$$i(U_{j_2}^\epsilon)=t^{m(\infty)}\quad+\text{ other possible terms.}$$

Then there are at least two critical points x_1 and x_2, and if they are nondegenerate

$$i(x_1)=t^{m(0)+1} \text{ and } i(x_2)=t^{m(\infty)}.$$

The other possibility is that for every ϵ, $\exists U_j^\epsilon$ such that

$$i(U_j^\epsilon)=t^{m(0)}+t^{m(\infty)+1}+\text{ other possible terms.}$$

In this case U_j^ϵ contains a degenerate point with multiplicity at least 2. If $Q(t)$ contains the term $t^{m(0)-1}$ we argues is a similar way. □

Example 1. Consider the problem

$$(3.3)\qquad \begin{cases} u\in H_0^1(\Omega) & \Omega\subset R^n \text{ smooth and bounded}\\ \Delta u+g(u)=0. \end{cases}$$

Suppose that

$$(3.4)\qquad \lim_{x\to\pm\infty} g'(x)=\ell \begin{cases}\text{with } \ell\notin\sigma(-\Delta) \text{ where } \sigma(-\Delta) \text{ denotes the}\\ \text{spectrum of } -\Delta \text{ in } L^2(\Omega).\end{cases}$$

We define L as follows:

$$(Lu,v)_{H_0^1(\Omega)}=\int_\Omega [(\nabla u,\nabla v)+\ell u\cdot v]\,dx\qquad u,v\in H_0^1(\Omega)$$

and ψ as follows

$$\psi(u)=\int_\Omega [G(u)+\ell u^2]\,dx \text{ where } G(s)=\int_0^s g(t)\,dt.$$

Then we have

$$f(u)=\frac{1}{2}(Lu,u)-\psi(u)=\int_\Omega\left[\frac{1}{2}|\nabla u|^2-G(u)\right]dx.$$

(3.3) are the Euler-Lagrange equation for the functional f. Thus the critical points of f are the solutions of (3.3). Clearly the assumptions of theorem 3.1 apply. Thus we have

THEOREM 3.4. If (3.4) holds then (3.3) has at least one solution.

Now suppose that

$$(3.5)\qquad g(0)=0 \text{ and } g'(0)\notin\sigma(-\Delta).$$

Then 0 is a critical point of f and

$$i(0)=m(0)$$

where

(3.5′) $m(0) = \#\{\text{eigenvalues of} -\Delta \text{ in } L^2(\Omega) \text{ less or equal to } \ell\}$.

Thus if

(3.6) the interval $[g'(0), \ell]$ (resp. $[\ell, g'(0)]$) contains at least one eigenvalue of $-\Delta$

we have that $m(0) \neq m(\infty)$. Thus the assumptions of Theorem 3.3 are satisfied. So we get:

THEOREM 3.5. If (3.4), (3.5) and (3.6) hold, then the equation (3.3) has two nontrivial solutions which may coincide if they are degenerate.

REMARK 3.6. While theorem 3.4 can be easily proved using the Leray-Schauder topological degree, this is not the case for theorem 3.5 when $m(0)$ and $m(\infty)$ are both odd or even. However, since the operator in (3.3) is asymptotically quadratic, theorem (3.5) can be proved using a finite dimensional reduction argument and the standard Morse-Conley theory (cf. e.g. [AZ]).

The next theorem will describe a situation in which the finite reduction method cannot be applied. We consider a case in which ψ has a superquadratic growth i.e.

(f_3) $\exists p > 2 \ \exists \rho_0 > 0$ such that $\langle \psi'(\rho x), \rho x\rangle > p\psi(\rho x) > 0 \ \ \forall \rho > \rho_0 \ \forall x \in S$

where $S = \{x \in E \mid \|x\| = 1\}$.

It is well known that (f_1) and (f_3) imply (P.S.) in $(-\infty, +\infty)$. Therefore, by theorem 2.2 there exists a variational system $\{\eta, \Gamma\}$ relative to f.

LEMMA 3.7. If f satisfies (f_1) and (f_3), then there exists $c \in R$ such that

$$i(f_c) = 0.$$

Proof. Notice that (f_3) implies that

$$\exists \nu > 0 \ \text{ such that } \ \psi(\rho x) > \nu \rho^p \ \text{ for } x \in S \text{ and } \rho \text{ large.}$$

Thus, for ρ large and $x \in S$, we have:

$$\frac{d}{d\rho} f(\rho x) = \rho \langle Lx, x\rangle - \langle \psi'(\rho x), x\rangle \leq \rho \| L_0 \| - \frac{1}{\rho}\psi(x) \leq$$

$$\leq \rho(\| L_0 \| - \nu\rho^{p-1}).$$

Thus $\exists \rho_0$ such that $d/d\rho \, f(\rho x) < 0 \ \ \forall x \in S$ and $\rho \geq \rho_0$. Thus it is possible to choose

$c \ll 0$ such that for every $x \in S$ there exists a unique $\rho = \rho(x) > 0$ such that

$$f(\rho(x)x) = c.$$

This allows us to define a function

$$\varphi : (B, S) \to (f_c, \partial f_c) \quad B = \{x \in E \mid \|x\| \leq 1\}$$

as follows $\varphi(tx) = t\rho(x)x$ ($x \in S$ and $t \in [0, 1]$).

It is easy to check that φ is a homeomorphism. Then we have

$$i(f_c) = \sum_{q=0}^{\infty} \dim [\bar{H}^q(f_c, \partial f_c)] t^q = \sum_{q=0}^{\infty} \dim [\bar{H}^q(B, S)] t^q = 0$$

since the infinite dimensional sphere S is contractible. □

THEOREM 3.8. Suppose that (f_1) and (f_3) hold. Moreover suppose that 0 is a nondegenerate critical point of f. Then f has at least one other critical point.

Proof. Let $i(0) = t^{m(0)}$. Then if no other critical points exists, by Corollary 2.6 we get

$$t^{m(0)} = 0 + (1 + t)\, Q(t)$$

and this is a contradiction. □

Example II. We study again equation (3.3), but this time we suppose that g satisfies the following asymptotic condition:

(3.7)
$$\begin{cases} \text{(a)} \ \ \exists \vartheta > 2, \ \exists R > 0, \ \text{such that } g(t) \cdot t > \vartheta \, G(t) > 0 \ \text{ if } |t| > R \\ \text{(b)} \ \ \exists K_1, K_2 > 0 \ \text{such that } |g(t)| \leq K_1 + K_2 |t|^\alpha \\ \qquad \text{where } \alpha < \dfrac{n+2}{n-2} \ ; \ n = \dim \Omega. \end{cases}$$

We set

(3.8)
$$\psi(x) = \int_\Omega G(x)\, dx.$$

LEMMA 3.9. If (3.7) (a) holds, the functional (3.8) satisfies (f_3) for every $p \in (2, \vartheta)$ in the space $E = H_0^1(\Omega)$.

Proof. Take $u \in H_0^1(\Omega)$, with $\| u \| = 1$ then

$$(3.9) \qquad \langle \psi'(\rho u), \rho u \rangle = \int_\Omega g(\rho u) \cdot \rho u \, dx = \int_{\Omega_1} g(\rho u) \rho u + \int_{\Omega_2} g(\rho u) \rho u \, dx$$

where $\Omega_1 = \{ x \mid \rho \mid u(x) \mid > R \}$ and $\Omega_2 = \Omega - \Omega_1$. Thus using (3.7) we have

$$\langle \psi'(\rho u), \rho u \rangle \geq \int_{\Omega_1} \vartheta G(\rho u) \, dx - M_1 \quad \left[\text{with } M_1 = \int_{\Omega_1} g(\rho u) \rho u \, dx\right]$$

$$= \int_\Omega \vartheta G(\rho u) \, dx - M_2 \quad \left[\text{where } M_2 = \int_{\Omega_2} \vartheta G(\rho u) \, dx + M_1\right]$$

$$= p \int_\Omega G(\rho \cdot u) \, dx + (\vartheta - p) \int_\Omega G(\rho u) \, dx - M_2 >$$

$$> p \, \psi(\rho u) + (\vartheta - p) \rho^\vartheta K - M_2$$

[since G grows more than $K\rho^\vartheta$ for some positive K]. □

THEOREM 3.10. Suppose that g satisfies (3.7) and (3.5). Then the equation 3.3 has at least a nontrivial solution.

Proof. Consider the functional $f(u) = 1/2 \int_\Omega |\nabla u|^2 dx - \psi(u)$. By (3.7) (b), f is a functional of class C^1. By (3.5), 0 is a critical point of f and $f''(0)$ is defined. Then by Th. 2.8 $i(0) = t^{m(0)}$ where $m(0)$ is defined by (3.6) Thus all the assumptions of theorem 3.8 are satisfied. □

Now we are going to use the generalized Morse theory to prove a well known theorem of Ambrosetti and Rabinowitz (see e.g. [AR] or [R1]) with an additional information on the Morse index of the critical points.

THEOREM 3.12: *Mountain Pass Theorem.* Suppose that $f \in C^1(E)$ satisfies (P.S.) and that there is a set S in E which splits $E - S$ in two connected components. Moreover, suppose that there exist $a, b \in R$ $(a < b)$ and $\epsilon > 0$ such that

(i) $f(x) > a + \epsilon \quad \forall x \in S$

(ii) $f(x_i) < a \quad i = 1, 2$ where x_1 and x_2 are two points belonging to different connected components of $E - S$

(iii) $f(x) < b - \epsilon \quad \forall x \in Q$ where Q is a curve joining x_1 and x_2.

Then $K = K(f_a^b) \neq \phi$ and if all the critical points of f in K are topologically nondegenerate there exists a point $\bar{x}$ such that

$$i(\bar{x}) = t.$$

Moreover, if $f \in C^2(N_\epsilon(K))$ there exist two points (which might coincide) such that

$$m(x_1) \leq 1 \leq m^*(x_2).$$

Proof. We suppose that a and b are regular value (otherwise replace a with $a + \epsilon_1$ and b with $b - \epsilon_2$ with $\epsilon_1, \epsilon_2 < \epsilon$). Since f^a has at least two connected components; thus $H_0(f^a)$ has at least two generators $[x_1]$ and $[x_2]$.

Now consider the map $i_0 : H_0(f^a) \to H_0(f^b)$ induced by the natural embedding. Since x_1 and x_2 belong to the same connected component of f^b, then $i_0([x_2] - [x_1]) = 0$. Then, by the exactness of the sequence,

$$\to H_1(f^b, f^a) \xrightarrow{j_1} H_0(f^a) \xrightarrow{i_0} f_0(f^a).$$

it follows that $[x_1] - [x_2] \in \operatorname{Im} j_1$. Therefore $H_1(f^b, f^a) \neq 0$. Then, by Proposition (2.3) (vii),

$$i(f_a^b) = t + \text{other possible terms.}$$

Thus the first two statements of the theorem follow by Prop. (2.3) (vi) and Corollary 2.5 respectively. In order to obtain the second part of the theorem use theorem 2.14 with $U = f_a^b$. Then we have

$$\sum_{q = m(K)}^{m^*(K)} a_q t^q = t + \text{other possible terms}$$

and therefore

$$m(K) \leq 1 \leq m^*(K).$$

The conclusion follows from the definition $m(K)$ and $m^*(K)$ (see (2.11)). □

REMARK 3.12. The interest of the above theorem is in the fact that we give some information about the generalized Morse index of the critical points. A similar result for C^2-functionals has been obtained by Hofer [H] and Solimini [So].

The generalization of the Mountain Pass Theorem is the "linking theorem" (see e.g. [R1] or [BBF]).

DEF 3.13. Let $Q \subset E$ be a manifold homeomorphic to B^n $(n > 1)$ and let S be a smooth manifold in E of codimension n. We say that S and ∂Q link if, for every isotopy $h : Q \to E$ such that $h|_{\partial Q} = Id$, we have $h(Q) \cap S \neq \phi$. We say that S and ∂Q link traversally if they link and

(3.7) $\qquad S \cap \partial Q = x_0$ and $\partial(N_\varepsilon(S)) \cap Q$ is diffeomorphic to the n-ball B^n.

THEOREM 3.14. Linking theorem. Suppose that f satisfies (P.S) and that

(i) there exists two manifold ∂Q and S which link transversally (with dim Q = codim $S = n$)

(ii) $\exists a, b \in R$ and $\varepsilon > 0$ such that $f|_S > a + \varepsilon; f|_{\partial Q} < a$ and $f|_Q < b$.

Then $K = K(f_a^b) \neq \phi$ and if all the critical points of f in K are topologically nondegenerate there is a point $\bar{x}$ such that

$$i(\bar{x}) = t^n.$$

Moreover if $f \in C^2(N_\varepsilon(K))$ there exist two points $x_1, x_2 \in K$ (which might coincide) such that

$$m(x_1) \leq n \leq m^*(x_2).$$

Notice that the Mountain Pass Theorem is a particular case of the Linking Theorem when $n = 1$.

LEMMA 3.15. Suppose that S and ∂Q link transversally, with dim Q = codim $S = n$.

Then $H_n(E, E - S) \simeq K$ and $[\partial Q]$ is the generator.

Proof. Let N be a neighborhood of S. If N is chosen in a suitable way then, $(N, \partial N)$ has the structure of fiber bundle on S with fiber (B^n, S^{n-1}). Then by Thom isomorphism theorem $H_{q+n}(N, \partial N) \cong H_q(S)$ and in particular $H_n(N, \partial N) \cong H_0(S) \cong K$ and by (3.7), the generator will be $[\alpha] = [\partial N \cap Q]$. Also $H_*(N, \partial N) \simeq H_*(N, N - S)$ and by excision $H_*(N, N - S) \simeq H_*(E, E - S)$ and $[\alpha]$ is also a generator of $H_n(E, E - S)$. Since α is homologous to ∂Q, ∂Q is a generator in $(E, E - S)$. □

Proof of Theorem 3.14. Consider the map

$$H_n(f^b, f^a) \xrightarrow{i_n} H_n(E, E - S)$$

where i is the natural embedding. Then, since $i_n([\partial Q]) \neq 0$ by Lemma 3.15, then

$$H_n(f^b, f^a) \neq 0.$$

Then by proposition 2.3 (vii) we have that

$$i(f_a^b) = t^n + \text{other possible terms}.$$

Thus the first two statements of the theorem follows from Prop. 2.3 (vi) and Corollary 2.5 respectively.

In order to obtain the second part of the theorem, use theorem 2.14 with $U = f_a^b$. Then we have that

$$\sum_{k=m(K)}^{m^*(K)} a_k t^k = t^n + \text{other possible terms}$$

and therefore $m(K) \leq n \leq m^*(K)$. □

REMARK 3.16. As in the case of Th. 3.12 the interest of Th. 3.14 does not rely on the existence result which can be obtained in an easier way with minimax methods (see e.g. [R1]). The interest lies in the information about the Morse index of the critical points which is relevant in some class of problems (cf. e.g. [BF1]).

COROLLARY 3.17. (Saddle point theorem). Suppose that $f \in C^1(E)$ satisfies P.S. and let $E = E_n \oplus \tilde{E}$ where E_n is a n-dimensional space. Moreover suppose that $\exists e > 0$

(i) $f(x) \geq a + e \quad \forall x \in \tilde{E}$

(ii) $\exists R > 0 : f(x) \leq a \qquad \forall x \in E_n \cap \partial B_R$

(iii) $f(x) \leq b - e \qquad \forall x \in B_R$.

Then the same conclusion of Theorem 3.14 holds.

Proof. Take $S = \tilde{E}$ and $Q = B_R$. Then it is well known that S and ∂Q link (see e.g. [R1] or [BBF]), and it is immediate to see that they link transversally. □

REMARK 3.18. A somewhat weaker version of Corollary 3.17 has been obtained also by Laser and Solimini [LS] when $f \in C^2(E)$.

COROLLARY 3.19. Suppose that $f \in C^1(E)$ satisfies (P.S.) and let $E = E_{n-1} \oplus \tilde{E}$ where E_{n-1} is an $(n-1)$-dimensional space $(n \geq 2)$. Moreover suppose that there exists constants $\rho, R_1, R_2, e_1 > 0$ (and $R_1 > \rho$) such that

(i) $f(x) \geq a + e \qquad \forall x \in \tilde{E} \cap \partial B_\rho$

(ii) $f(x) \leq a \quad \forall x \in \partial Q$

(iii) $f(x) \leq b - e \quad \forall x \in Q$

where $Q = \{y + tz \mid y \in E_{n-1}, \|y\| \leq R_1$ and $t \in [0, R_1]\}$ and $z \in \tilde{E}$ with $\|\tilde{z}\| = 1$. Then the same conclusion of Theorem 3.14 holds.

Proof. Take $S = \tilde{E} \cap \partial B_\rho$. Then it is well known that S and ∂Q link (see e.g. [R1] or [BBF]) and it is immediate to check that they link transversally. Then the conclusion follows from theorem 3.14. □

4. Some existence theorems for invariant functionals

Now we consider how to use the index in a symmetric situation. We suppose that a compact Lie group G acts on E, i.e. that there exists a map

$$\varphi : G \times E \to E$$

such that $\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1 \cdot g_2, x)$. As usual we shall write gx instead of $\varphi(g, x)$. We recall some definition. If $x \in E$, the subgroup of G defined by

$$G_x = \{g \in E \mid gx = x\}$$

is called the isotropy group of x.

We say that G acts freely on $A \subset E$ if $G_x = Id$ for every $x \in A$. A point x is called a fix point if $G_x = G$. The set of all fix points of G will be denoted Fix(G). The set $O_x = \{g \mid \exists g \in G : g = gx\}$ is called the orbit of G passing through x. A set $A \subset E$ such that $gx \in A$ for every $g \in G$ and every $x \in A$ is called *G-invariant*. A functional $f : E \to R$ is called *G-invariant* if

$$f(gx) = f(x) \qquad \forall g \in G.$$

If the function $f \in C^1(E)$ satisfies P.S., then it is possible to construct a variational system

$\{\eta, \Gamma\}$ where η is a *G-invariant* flow, i.e.:

$$\eta(t, gx) = g\eta(t, x) \quad \forall t \in R \text{ and } \forall g \in G.$$

We shall call the triple $\{\eta, \Gamma, G\}$ an equivariant variational system (notice that we do not require that the sets in Γ are G-invariant).

If x is a critical point for a G-invariant function f, then all the points of the orbit are critical points. Such an orbit is called "critical orbit". We have the following proposition:

PROPOSITION 4.1. Let $\{\eta, \Gamma, G\}$ be an equivariant variational system relative to f where G is a group of finite order, and let O_x be an isolated critical orbit of f. Then there exist a polynomial $P(t) \in \mathcal{S}$ such that

$$i(t, O_x) = \gamma \cdot P(t) \text{ where } \gamma = \frac{\text{Ord } G}{\text{Ord } G_x} \tag{4.1}$$

(notice that γ is the number of points of the critical orbit).

Proof. Let $x = x_1, \ldots, x_\gamma$ be the points of the orbit O_x. Then by Theorem 1.1 (vi) we have

$$i(t, O_x) = i(t, N_\varepsilon(O_x)) = \sum_{i=1}^{\gamma} i(t, N_\varepsilon(x_i)) \tag{4.2}$$

where ϵ is small enough that $N_\epsilon(x_{\ell_1}) \cap N_\epsilon(x_{\ell_2}) = \phi$ for $\ell_1 \neq \ell_2$. Now since the index is a local property, $i(N_\epsilon(x_i)) = i(N_\epsilon(x_j))$. Then the conclusion follows immediately by (4.2). □

PROPOSITION 4.2. Let $\{\eta, \Gamma, G\}$ be an equivariant system relative to f where G is a group of finite order p. Take $A \in \Gamma$ such that

(i) A is G-invariant

(ii) G acts freely on $K(A)$

(iii) f is of class C^2 is a neighborhood of $K(A)$.

Then there exists $m \in N$ such that

$$i(p-1, A) = p \cdot m.$$

Proof. Since the set of points on which G acts freely is an open set, it is possible to choose $\epsilon > 0$ such that G acts freely on $N_\epsilon(K)$ where $K := K(A)$. Also we can take ϵ small enough that $f \in C^2(N_\epsilon(K))$.

Now by theorem (2.4) (b) we get

(4.3) $$i(N_\epsilon(K)) = i(A) + (1+t)Q_1 \qquad Q_1 \in \mathscr{S}$$

By (iii) and the fact that G acts freely on $N_\epsilon(K)$ it is possible to choose a Morse function f_ϵ arbitrarily close to f (apply the theorem 2.7 at the function $f \circ \pi^{-1}$ where π : $N_\epsilon(K)/G$ is the natural projection). Then by theorem 1.5 we have

(4.4) $$i(t, N_\epsilon(K), f_\epsilon) = i(t, N_\epsilon(K), f)$$

and by Corollary 2.10, we have

(4.5) $$\sum_{m=0}^{N} \varphi(m) t^m = i(t, N_\epsilon(K), f) + (1+t) Q_2 \qquad Q_2 \in \mathscr{S}$$

Notice that all the $\varphi(m)$'s are multiple of p since the action is free. Then by (4.3), (4.4) and (4.5) we have

$$\sum_{m=0}^{N} \varphi(m) t^m = i(A) + (1+t)(Q_1 + Q_2).$$

Since all the $\varphi(m)$'s are multiple of p, the conclusion follows taking $t = p - 1$. □

Now let us apply the theory developed to some existence theorems:

THEOREM 4.3. Suppose that on $S = \{x \in E \mid \|x\| = 1\}$ a group G of finite order acts. Suppose that $f \in C^1(S, R)$ is a G-invariant function bounded from below which satisfy P.S. in $[m_0, m_\infty)$ where $m_0 = \min_S f$ and $m_\infty = \sup_S f$ (m_∞ is allowed to be $+\infty$).

Moreover, suppose that

(4.6) there exists $\gamma \geq 2$ such that every critical orbit has a cardinal multiple of γ.

Then f has infinitely many critical orbits.

Proof. Since S is contractible the $i(S) = 1$. We argue indirectly and suppose that f has only a finite number of critical orbits $O_1, \ldots, O_k$. Then by Proposition 4.1. and Corollary 2.5 we get

$$p \cdot \sum_{\ell=1}^{k} P_\ell(t) = 1 + (1+t)\, Q(t).$$

If you take $t = p - 1$, we get

$$p \cdot (\text{number}) = 1 + p\, Q(p-1)$$

and this is a contradiction. □

THEOREM 4.4. Suppose that $f \in C^1(E, R)$ satisfies (f_1) and (f_3) and it is invariant for the action of a finite group G. Moreover, suppose that

(4.7) $$\begin{cases} \text{(a) } 0 \text{ is the only fixed point of } G \\ \text{(b) } 0 \text{ is a critical point of } f \text{ and } i(0) = t^\alpha \text{ for some } \alpha \in N \\ \text{(c) (4.6) holds for any critical point different from } 0. \end{cases}$$

Then f has infinitely many critical orbits.

Proof. Take c small enough in order that, by lemma 2.7, we have

(4.8) $i(f_c) = 0.$

Now we argue indirectly and suppose that f has only a finite number of critical orbits $O_1, \ldots, O_k$ in f_c. Then by Corollary 2.5 we have

$$i(0) + \sum_{\ell=1}^{k} i(O_\ell) = i(f_c) + (1+t)\, Q(t).$$

By (4.7) (b), (4.8) and proposition 4.1 (with the assumption (4.6)), we get

$$t^a + \gamma \sum_{\ell=1}^{k} m_\ell \cdot p_\ell(t) = (1+t)\ \ Q(t).$$

Taking $t = \gamma - 1$ we get

$$(\gamma-1)^a = \gamma \cdot m \ \text{ where } \ m = Q(\gamma-1) - \sum_{\ell=1}^{k} m_\ell p_\ell(\gamma-1)$$

and this is a contradiction. □

REMARK 4.5. Theorem 4.4 is valid also without the assumpiton (4.7) (b). The proof is too involved and it will not be given here. However, it is not too hard to prove the following result.

THEOREM 4.6. Suppose that $f \in C^2(E)$ satisfies (f_1), (f_2) and it is invariant for the action of a finite group G. Moreover, suppose that (4.7) (a) holds and that every orbit different from 0 has a cardinality multiple of γ. Then f has infintely many critical orbits and the critical values are unbounded from above.

Proof. 0 is a critical value of f, since it is invariant for the flow by (4.7) (a). It could be degenerate but we can ignore this situation arguing as in theorem 4.19. Then the proof follows the same line of Theorem 4.4. In order to get the unboundedness of the critical values of f, argue as in Theorem 4.19. □

Example. Let $V(t, x) \in C^1(R \times R^n, R)$ be a T-periodic function, and consider the following system of ordinary differential equations

(4.9) $$\ddot{x} + V'(t,x) = 0 \ \text{ where } \ V'(t,x) = \left\{ \frac{\partial V}{\partial x_i}(t,x) \right\}.$$

We look for T-periodic solutions of (4.9). We make the following assumptions on V:

(4.10) $\exists R > 0$ and $p > 2$ such that $\langle V'(t,x), x\rangle \geq p\, V(x) > 0$ for all $x \in R^n$ with $|x| > R$ and all $t \in R$

(4.11) $V(t, x)$ is G-invariant, where $G \in O(n)$ is a finite group which satisfy the following:

(i) 0 is the only fix point of G

(ii) there exists $\gamma \geq 2$ such that every orbit O_x passing through $x \in R^n - \{0\}$ has a cardinality multiple of γ.

THEOREM 4.7. If (4.10) and (4.11) hold then (4.9) has infinitely many periodic solutions unbounded in L^∞.

Proof. We set

$$E = \{x \in H^1(0, T; R^n) \mid x(0) = x(T)\}$$

and

(4.12) $$f(x) = \int_0^T \left\{ \frac{1}{2} |\dot{x}|^2 - V(t,x) \right\} dt \qquad x \in E.$$

It is well known that the critical points of f correspond to T-periodic solutions of (4.9). We can apply theorem 4.6 to the functional (4.12). In fact, $f(x)$ has the form (f_1) with

$$\langle Lx, y\rangle_E = \int_0^T (\dot{x}\,\dot{y} + xy)\, dt \qquad x, y \in E$$

and

$$\psi(x) = \int_0^T (V(t,x) + x^2)\, dt.$$

Also f satisfies (f_3 by virtue of (4.10)). (a) and (b) of theorem 4.6 follow from (4.11). Then by Theorem 4.6 it follows that the critical values of the functional (4.12) are unbounded. Standard estimates (see e.g. [R2]) show that the corresponding critical points are unbounded in L^∞. □

Next we want to consider an example where the group G is continuous. We consider the group $S^1 = \{z \in c \mid |z| = 1\}$ with the multiplicative structure.

PROPOSITION 4.8. Let $\{\eta, \Gamma, S^1\}$ be an equivariant system relative to f. Take $A \in \Gamma$ such that

(i) A is S^1-invariant

(ii) $K(A) \cap \mathrm{Fix}(S^1) = \phi$

(iii) f is of class C^2 in a neighborhood of $K(A)$.

Then there exist a polynomium $P(t)$ with coefficients in Z such that

$$i(t, A) = (1+t)P(t).$$

Proof. We claim that $\exists\, p$ such that $Z_p \subset S^1$ acts freely on $K(A)$ for every p ($Z_p = \{e^{2\pi i/k} \mid i = 0, \dots, k-1\}$). To prove this we argue indirectly and suppose not. Then there exists a sequence $p(k) \to +\infty$ and points $x_k \in K(A)$ such that

(4.13) $$(g_{p(k)})^{\ell} \cdot x_k = x_k \qquad \ell = 0, \dots, p(k)-1$$

where

$$g_{p(k)} = e^{2\pi i/p(k)} \in Z_{p(k)}.$$

Since $K(A)$ is compact, we can suppose that x_k converges to some $\bar{x} \in K(A)$. Moreover, for every $g \in S^1$, there exists a sequence of $\ell(k)$'s such that

$$(g_{p(k)})^{\ell(k)} \to g \text{ for } k \to +\infty.$$

Then taking the limit in (4.13) with $\ell = \ell(k)$ we get

(4.14) $$g\bar{x} = \bar{x}.$$

Since g has been chosen arbitrarily, (4.14) implies that $\bar{x} \in \mathrm{Fix}(S^1) \cap K(A)$ against our assumptions. So the claim is proved.

Then, by proposition 4.2, for every p sufficiently large there exists $m(p) \in N$ such that

(4.15) $$i(p-1, A) = p \cdot m(p).$$

By Corollary 2.11, $i(t, A)$ is a polynomial. Then there exists a polynomial p and an integer number a_0 such that

$$i(t, A) = (1+t)P(t) + a_0.$$

Then, by (4.14) and the above formula, we get

$$\begin{aligned} a_0 &= i(t, A) - (1+t)P(t) && [\text{for every } t \in R] \\ &= i(p-1, A) - pP(p-1) && [\text{for every } p \geq \bar{p}] \\ &= p[m(p) - P(p-1)] && [\text{by (4.15)}]. \end{aligned}$$

Since $m(p) - P(p-1)$ is an integer number, the above formula implies that a_0 must be 0 (otherwise $|a_0| = +\infty$). □

Example. Let $V \in C^2(R^n, R)$ and consider the following equation

(4.16) $$\ddot{x} + V'(x) = 0 \qquad x \in R^n.$$

We have the following theorem.

THEOREM 4.19. If (4.10) holds, then the equation (4.16) has infinitely many T-periodic solutions for every $T > 0$, and the set of solutions is unbounded from above in the L^∞-norm.

Proof. First we make an extra assumption which we shall remove later, and we suppose that

(4.17) all the critical points of V, $x_1, \dots, x_2$, are nondegenerate in $\mathbb{R}^n$.

We want to show that (4.17) implies that V has an odd number of critical points. In fact, by Corollary 2.10 we get

$$\sum_{n=0}^{N} a(m)t^m = i(R^n) + (1+t)\,Q(t)$$

which for $t = 1$ gives

$$(\text{number of critical points of } V) = \sum_{n=1}^{\infty} a(m) = 1 + 2 \cdot Q(1) = (\text{odd number}).$$

Now, consider the functional

(4.18) $$f_T(x) = \int_0^T \{\dot{x}^2 - V(x)\, dt\}$$

defined on the space

$$E_T = \{x \in H^1(0, T; R^n) \mid x(0) = x(T)\}.$$

By (4.10), and well known results f_T satisfies (P.S.) on E. Moreover, f_T is invariant for an S^1-action, i.e. the action

$$(gx)(t) = x(\overline{t+s}) \text{ where } g = e^{2\pi i s/T}$$

and $t + s = t + s$ mod T. The fix points of S^1 are the constants. Therefore, $K(f_T) \cap \text{Fix } S^1 \cong$ {critical point of V in R^n} and this is true for every $T > 0$. Now take $T \in R$ such that

(4.19) $x \in K(f_T) \cap \text{Fix } S^1$ is not degenerate (i.e. $f''_T(x)$ invertible).

By (4.17), (4.19) is true for every $T > 0$ except than a discrete set.

Now choose a $T > 0$ such that (4.19) holds, and take two regular values of f, a and b such that

$$a < -\int_0^T V(x)\, dt < b \quad \text{for every } x \in R^n \text{ s.t. } V'(x) = 0.$$

Now take ϵ small enough such that $N_{2\epsilon}(x_k) \cap K(f_T) = \{x_k\}$ for every $x_k \in \text{Fix } S^1 \cap K(f_T)$. (This is possible by (4.19)). Now by theorem (2.4) we get

(4.20) $$\sum_{j=1}^{r} i(N_\epsilon(x_j)) + \sum_{j=1}^{s} i(U_j) = i(f_a^b) + (1+t)\, Q(t)$$

where $\bigcup_{j=1}^{s} U_j$ is a suitable neighborhood of $K(f_a^b) - \text{Fix}(S^1)$. Also we have

$$\begin{cases} \text{(a)}\ i_1(N_\epsilon(x_j)) = 1 \text{ by (4.19)} \\ \text{(b)}\ i_1(U_j) = \text{(even number) by Proposition (4.8).} \end{cases}$$

Then by the above formulas and (4.20) we get

(4.21) $i_1(f_a^b) =$ (odd number).

We claim that (4.21) holds even without assuming (4.17) and (4.19). In fact if f_T does not satisfy (4.17) and (4.19), there is always an arbitrarily close function which does. So (4.21) follows by Theorem 1.5.

Now we claim that the critical values of f are unbounded from above. We argue indirectly and suppose that they are bounded by a constant c_∞. Take c such that the conclusion of lemma 3.7 is satisfied. Then by theorem 1.4, we have:

(4.22) $$i(f_c^a) + i(f_a^b) + i(f_b^{c_\infty}) = i(f_c) + (1+t)\, Q(t).$$

But

$i_1(f_c^a) =$ even number	by Proposition 4.8
$i_1(f_a^b) =$ odd number	by (4.21)
$i_1(f_b^{c_\infty}) =$ even number	by Proposition 4.8
$i_1(f_c) = 0$	by the choice of c (cf. Lemma 3.7)

Then the inequality (4.22) gives a contradiction. Therefore there is a sequence c_n of critical values of f_T. Standard argument show that the corresponding critical points are unbounded in L^∞. □

REMARK 4.20. Other proofs of theorem 4.19 can be found in [B2] (with some extra assumptions) in [R2] and [BF2]. However, theorem 4.19 provides also estimates on the Morse index of the solutions.

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REFERENCES

[AZ] H. AMANN, E. ZEHNDER, Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 7 (1980), 539 - 603.

[AR] A. AMBROSETTI, P.H. RABINOWITZ, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* 14 (1973), 349 - 381.

[BBF] P. BARTOLO, V. BENCI, D. FORTUNATO, Abstract critical point theorems and applications to some problems with strong resonance at infinity, *Nonlinear Analysis T.M.A.*, 7 (1983), 981 - - 1012.

[B1] V. BENCI, A new approach to the Morse-Conley theory, to appear in *Recent Advances in Hamiltonian Systems*. G. Dell'Antonio Ed.

[B2] V. BENCI, A geometrical index for the group S^1 and some applications to the research of periodic solutions of O.D.E.'s, *Comm. Pure Appl. Math.*, 34 (1981), 393 - 432.

[B3] V. BENCI, On critical point theory for indefinite functionals in the presence of symmetries, *Trans. Amer. Math. Soc.*, 274 (1982), 533 - 572.

[BF1] V. BENCI, D. FORTUNATO, Subharmonic solutions of prescribed minimal period for nonautonomous differential equations, to appear in *Recent Advances in Hamiltonian Systems*. G. Dell'Antonio Ed.

[BF2] V. BENCI, D. FORTUNATO, Un teorema di molteplicitá per un'equazione ellittica non lineare su varietà simmetriche, in *Metodi Asintotici e Topologici in problemi differenziali non lineari.* L. Boccardo, A.M. Micheletti Ed.

[C] C.C. CONLEY, Isolated invariant sets and the Morse index, *CBMS Regional Conf. Ser. in Math.*, no. 38, Amer. Math. Soc., Providence, RI, 1978.

[CZ] C.C. CONLEY, E. ZEHNDER, Morse type index theory for flows and periodic solutions for Hamiltonian equations, *Comm. Pure Appl. Math.*, 37 (1984), 207 - 253.

[H] H. HOFER, A geometric description of the neighborhood of a critical point given by the Mountain Pass Theorem, *J. London Math. Soc.* (2) 31 (1985), 566 - 570.

[LS] A. LAZER, S. SOLIMINI, Nontrivial solutions of operator equations and Morese indices of critical points of minimax type, preprint.

[MP] A. MARINO, G. PRODI, Metodi perturbativi nella teoria di Morse, *Boll. U.M.I.*, 11, Suppl. Fasc. 3 (1975), 115 - 132.

[MeP] F. MERCURI, G. PALMIERI, Morse theory with low differentiability, preprint.

[R1] P.H. RABINOWITZ, Minimax methods in critical point theory with applications to differential equations, *C.B.M.S. Regional Conf. Ser. in Math.*, no. 65, Amer. Math. Soc., Providence, RI, 1986.

[R2] P.H. RABINOWITZ, On large norm periodic solutions of some differential equations, *Ergodic Theory and Dynamical Systems*, 2 Ed. E. Katok, Birkhauser (1982), 193 - 210.

[RY] K.P. RYBAKOWSKI, On the homotopy index for infinite dimensional semiflows, *Trans. Am. Math. Soc.*, 295 (1982), 351 - 381.

[SM] S. SMALE, An infinite dimensional version of Sard's theorem, *Amer. J. Math.*, 87 (1965), 861 - - 866.

[SP] E.H. SPANIER, *Algebraic Topology*, McGraw Hill (1966).

[SO] S. SOLIMINI, Existence of a third solution for a class of B.V.P. with jumping nonlinearities, *Nonlinear Analysis T.M.A.*, 7 (1984), 917 - 927.