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BIFURCATION FROM THE ESSENTIAL SPECTRUM FOR SOME NON-COMPACT NON-LINEARITIES

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1. Introduction

We consider the bifurcation of solutions for equations having the form,

$$Su - F(u) = \lambda u \tag{1.1}$$

where S is a self-adjoint operator and F is the gradient of a real-valued function defined on a dense subset of a real Hilbert space H. Supposing that $F(0) = 0$ and that F is of order higher than linear near $u = 0$, there is a line of trivial solutions $\{(\lambda, 0) \in R \times H : \lambda \in R\}$. In this setting the main result gives conditions on F implying that 0 is a bifurcation point for (1.1) and that the bifurcation is towards negative values of λ. These conditions require S to be positive with $0 = \inf \sigma(S)$ where $\sigma(S)$ is the spectrum of S. However, 0 is not necessarily an eigenvalue of S and in fact the primary object is to deal with situations where S may have no eigenvalues, $S - \lambda I$ being injective but not surjective for all $\lambda \in R$. This kind of problem arises in the L^2 - theory of semi-linear elliptic equations on unbounded domains.

Earlier results in this direction (1-4) deal with cases where F satisfies a compactness condition as a mapping from the graph space of $S^{1/2}$ into its dual. More recently this compactness has been relaxed to the requirement that the natural energy associated with (1.1) on spheres in H can be increased by making a compact perturbation of F.

The basic definitions and hypotheses are set out in section 2, followed by the main result and an indication of its proof in section 3. The application of this result to semi-linear elliptic equations on R^N is described in section 4.

2. Definitions and hypotheses

Let H denote a real Hilbert space with scalar product $(\ ,\ )$ and norm $|\ |$.

(H1) $S:D(S)\subset H \longrightarrow H$ is a positive self-adjoint operator with $0 = \inf\sigma(S)$.

In this case S has a unique positive self-adjoint square-root $S^{\frac{1}{2}}$ whose domain $D(S^{\frac{1}{2}})$ equipped with its graph norm:

$$|u|_1 = (|u|^2 + |S^{\frac{1}{2}}u|^2)^{\frac{1}{2}}$$

is a Hilbert space denoted by H_1.

(H2) $f\in C^2(H_1,R)$ with $f(0) = 0$ and $f'(0) = 0$.

Identifying H with its dual and using $(\ ,\ )$ to denote both the duality between H and its dual as well as that between H_1 and its dual, we have that $H_1\subset H\subset H_1{}^*$. Furthermore S has a unique extension to a bounded linear operator $L : H_1 \longrightarrow H_1{}^*$ such that $(Lu,v) = (S^{\frac{1}{2}}u,S^{\frac{1}{2}}v)$ for all $u,v\in H_1$. In this setting the gradient of f is denoted by $F : H_1 \longrightarrow H_1{}^*$ where $(F(u),v) = f'(u)v$ for $u,v \in H_1$ and $F(0) = 0$.

Under these hypotheses a solution $(\lambda,u)\in R\times H_1$ of

$$Lu - F(u) = \lambda u \quad \text{or, equivalently}$$

$$(S^{\frac{1}{2}}u,S^{\frac{1}{2}}v) - (F(u),v) = \lambda(u,v) \quad \text{for all } v\in H_1 \quad (2.1)$$

constitutes a generalised solution of (1.1).

Let $E = \{(\lambda,u)\in R\times H_1 : u \neq 0 \text{ and } Lu - F(u) = \lambda u\}$. There is bifurcation to the left for (2.1) at $\lambda\in R$ if there is a sequence $\{(\lambda_n,u_n)\}\subset E$ such that

$$\lambda_n < 0\ ,\quad \lambda_n \longrightarrow 0 \quad \text{and} \quad |u_n|_1 \longrightarrow 0 \qquad (2.2).$$

In this definition of bifurcation we have used the norm of H_1. A different notion of bifurcation would be obtained by replacing $|u_n|_1$ by $|u_n|$ in (2.2). It is one consequence of the following hypotheses that, for $\lambda\leqslant 0$, these two notions of bifurcation point are equivalent.

(H3) There exist constants $m\in N$, $K>0$, $a_i\in[0,2)$ and $b_i > 2-a_i$ such that for all $u\in H_1$

$$0 \leqslant 2f(u) \leqslant (F(u),u) \leqslant K\sum_{i=1}^{m} |S^{\frac{1}{2}}u|^{a_i}\,|u|^{b_i} .$$

(H4) $F:H_1 \longrightarrow H_1{}^*$ is demi-continuous in the sense that $F(u_n) \longrightarrow F(u)$ weakly in $H_1{}^*$ whenever $u_n \longrightarrow u$ weakly in H_1.

It follows from (H1) to (H4) that $F'(0) = 0$ and that F takes bounded sets in H_1 to bounded sets in $H_1{}^*$.

3. Main result

Under the hypotheses (H1) and (H2), we introduce the following notation:

$S(r) = \{u \in H_1 : |u| = r\}$ where $r > 0$,

$J_f(u) = \frac{1}{2}|S^{\frac{1}{2}}u|^2 - f(u)$ where $u \in H_1$,

$M_f(r) = \inf\{J_f(u) : u \in S(r)\}$ where $r > 0$.

Lemma 3.1 Let S and f satisfy the conditions (H1) to (H4). Suppose that there exists $u \in S(r)$ with $J_f(u) < 0$. Then there exist sequences $\{u_n\} \subset S(r)$ and $\{\lambda_n\} \subset R$ such that

$J_f(u_n) \leqslant J_f(u)$ for all $n \in N$,

$\lambda_n \leqslant 2J_f(u_n)/r^2$ for all $n \in N$,

$Lu_n - F(u_n) - \lambda_n u_n \longrightarrow 0$ strongly in H_1^*.

This result is proved by using u as an initial condition for the differential equation associated with the projection of the gradient of J_f onto the tangent space of $S(r)$.

Theorem 3.2 Let S and f satisfy the conditions (H1) to (H4) and suppose that there exists a function g satisfying (H2) such that:

(i) $g-f : H_1 \longrightarrow R$ is weakly sequentially lower semi-continuous;

(ii) There exists $A>0$ such that, for all $r \in (0,A)$, $M_f(r) < M_g(r) \leqslant 0$.

Then there is bifurcation to the left for (2.1) at $\lambda = 0$.

Fixing $r \in (0,A)$, there is a sequence $\{v_n\} \subset S(r)$ such that $J_f(v_n) \longrightarrow M_f(r) < 0$. Using Lemma 3.1 we can suppose that there is a sequence $\{\lambda_n\} \subset R$ such that $\lambda_n \leqslant 2J_f(v_n)/r^2$ and $Lv_n - F(v_n) - \lambda_n v_n \longrightarrow 0$ strongly in H_1^*.

Passing to a subsequence we can assume that $v_n \longrightarrow u_r$ weakly in H_1 and that $\lambda_n \longrightarrow \lambda_r$ where $\lambda_r \leqslant 2M_f(r)/r^2$. Thus $|u_r| \leqslant r$ and since $J_g(v_n) = J_f(v_n) - (g-f)(v_n)$, it follows from (i) that $\limsup J_g(v_n) \leqslant M_f(r) - (g-f)(u_r)$. From (ii) we see that $(f-g)(u_r) > 0$ and consequently $u_r \neq 0$ because $f(0) = g(0) = 0$. Using (H4) we then find that $(\lambda_r, u_r) \in E$ with $\lambda_r < 0$. Finally we deduce from (H3) that $\lambda_r \longrightarrow 0$ and $|u_r|_1 \longrightarrow 0$ as $r \longrightarrow 0$.

Remarks 1. If f is weakly sequentially continuous we can set $g = 0$ and the conditions (i) and (ii) amount to the requirement that $M_f(r) < 0$ for $0<r<A$.

2. Using G to denote the gradient of g, the condition (i) is implied by the compactness of $G-F : H_1 \longrightarrow H_1^*$.

3. The assumption that $0 \in \sigma(S)$ in (H1) is not used explicitly. However if S is positive definite and f satisfies (H2) and (H3), then $M_f(r) \geqslant 0$ for r near 0 and (ii) cannot hold.

4. For $(\lambda,u) \in E$, it follows from (H3) that

$$|S^{\frac{1}{2}}u|^2 \leqslant (F(u),u) \leqslant K\sum_{i=1}^{m} |S^{\frac{1}{2}}u|^{a_i}|u|^{b_i} \quad \text{when } \lambda \leqslant 0$$

and hence if the constants a_i were $\geqslant 2$ there would be no bifurcation to the left at $\lambda = 0$.

4. A semi-linear elliptic equation

In this section we consider the equation,

$$\Delta u(x) + \lambda u(x) + q(x)h(|u(x)|)u(x) = 0 \text{ for } x \in R \quad (4.1)$$

where the functions q and h satisfy the following conditions:

(A1) $q \in L^{\infty}(R^N)$, $q \geq 0$ a.e. on R^N and $q(x) \longrightarrow Q$ as $|x| \longrightarrow \infty$. If $Q > 0$ we require that $\int_{|x|=c} q(x)-Q\,dx \geq$ but $\not\equiv 0$ for $c > 0$. Also,

either (i) $N \geq 1$ and there exist $A>0$ and $t \in [0,2)$ such that $q(x) \geq A(1+|x|)^{-t}$ a.e. on R^N;

or (ii) $N=1$ and $\int_R q(x)\,dx > 0$, possibly infinite.

(A2) $h \in C^1((0,\infty))$ with $h' \geq 0$ on $(0,\infty)$ and $h(s) \longrightarrow 0$ as $s \longrightarrow 0+$. Furthermore, there exist $\alpha > 0$ and $\beta \in (0,4/N)$ such that $\lim_{s \to 0+} h'(s)/s^{\alpha-1} = B > 0$ and $\limsup_{s \to \infty} h'(s)/s^{\beta-1} < \infty$ where $0 < \alpha < 2(2-t)/N$ if (A1)(i) holds and $0 < \alpha < 2$ if (A1)(ii) holds.

To apply the previous results to (4.1), we take H to be $L^2(R^N)$ with its usual norm. Then $-\Delta$ with domain the Sobolev space $H^2(R^N)$, defines a positive self-adjoint operator S in H and $\sigma(S) = [0,\infty)$. In this case, H_1 is the Sobolev space $H^1(R^N)$ with its usual norm.

Assuming that (A1) and (A2) are satisfied, we denote by P the primitive of $h(|s|)s$ such that $P(0) = 0$ and set $f(u) = \int_{R^N} q(x)P(|u(x)|)\,dx$ for $u \in H_1$.

It follows that (H1) and (H2) are satisfied and that solutions of (2.1) are weak solutions of (4.1) in the usual sense.

Theorem 4.1 Let the functions q and h satisfy the conditions (A1) and (A2). Then there is bifurcation to the left for (4.1) at $\lambda = 0$ in $H^1(R^N)$

We have already noted that (H1) and (H2) are satisfied. For (H3) we observe that $h' \geq 0$ implies that $0 \leq 2f(u) \leq (F(u),u)$ for $u \in H_1$. Furthermore, $h = h_1 + h_2$ where $0 \leq h_1(s) \leq C|s|^{\alpha}$ and $0 \leq h_2(s) \leq C|s|^{\beta}$ for all $s>0$. From this and the multiplicative Sobolev inequalities we can show that (H3) is satisfied with $m=2$ and $a_1 = \alpha N/2$, $b_1 = 2 + \alpha - a_1$, $a_2 = \beta N/2$ and $b_2 = 2 + \beta - a_2$. Thus (H3) is satisfied and using standard facts about Nemytskii operators we find that (H4) also holds. Using test functions of the form $\exp(-k|x|)$, normalised so as to lie on $S(r)$ and with k small and positive, we find that $M_f(r) < 0$ for small positive r. If $Q = 0$, we set $g = 0$ and observe that conditions (i) and (ii) of Theorem 3.2 are fulfilled. For $Q > 0$, we set $g(u) = Q\int_{R^N} P(|u(x)|)\,dx$. By Schwarz symmetrisation it follows that there exists $v \in S(r)$ such that $J_g(v) = M_g(r) < 0$ and furthermore, $J_f(v) < J_g(v)$. Since $g-f : H_1 \longrightarrow R$ is weakly sequentially continuous, we again find that conditions (i) and (ii) are satisfied. The result now follows from Theorem 3.2.

Remarks 1. Details of the proofs of the results in sections 3 and 4 are given in (5). Related work on the existence of solutions of elliptic equations in situations where compactness is lacking is contained in (6 - 7).

2. The "a priori" bounds in (8) show that in general the restrictions on the exponents cannot be relaxed.

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