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SOME APPLICATION OF THE MORSE-CONLEY THEORY TO THE STUDY OF PERIODIC SOLUTIONS OF SECOND ORDER CONSERVATIVE SYSTEMS

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1- The generalised Morse-Conley index for variational systems.

Let M be a Hilbert manifold and let $f \in C^2(M)$. We denote by $\eta(t,x)$ the flow relative to the differential equation

$$\frac{dx}{dt} = - \frac{f'(x)}{1 + \|f'(x)\|} .$$

When no ambiguity is possible, we shall write $x \cdot t$ instead of $\eta(t,x)$. If U is an open set in M we set

$$G^T(U) = \{x \in \overline{U} \mid x \cdot [-T,T] \subset \overline{U}\}$$

$$\Gamma^T(U) = \{x \in G^T(U) \mid x \cdot [0,T] \cap \partial U \neq \emptyset\}$$

where ∂U denotes the boundary of U.
Also we set

$$\Sigma = \{U \subset M \mid U \text{ is open and } \exists\, T > 0 \text{ such that } G^T(U) \subset U\}$$

S will denote the set of formal power series (in t) with nonnegative coefficients (or to be more precise with coefficients whith are cardinal numbers). The generalised Morse-Conley Index (GIM) is a map

$$i : \Sigma \longrightarrow S$$

defined as follows

$$(1\text{-}1) \qquad i_t(U) = \lim_{T \to \infty} \sum_{k=0}^{\infty} \dim \left[\overline{H}^k(G^T(U), \Gamma^T(U))\right] t^k$$

where $\overline{H}^*(\cdot,\cdot)$ denotes the Alexander-Spanier [Sp] cohomology with coefficients in some field, which in this paper will be $\mathbb{Q}$. The limit in (1-1) exists in a trivial sense; in fact in [B1], it is proved that, for T large

enough, $\overline{H}^*(G^T(U), \Gamma^T(U))$ does not depend on T.
When no ambiguity is possible we shall write $i(U)$ instead of $i_t(U)$.
Now we shall list some of the properties of the GIM which have been proved in [B1].

<u>Theorem 1.1</u> The GIM satisfies the following properties

(i) if $U \in \Sigma$ then $G^T(U) \in \Sigma$ and $i(G^T(U)) = i(U) \quad \forall \; T > 0$

(ii) if $U \in \Sigma$ then $\eta(T,U) \in \Sigma$ and $i(\eta(T,U)) = i(U) \; \forall \; T > 0$

(iii) if $U,V \in \Sigma$ and $T > 0$ such that $G^T(U) \subset V$ and $G^T(V) \subset U$, then $i(U) = i(V)$

(iv) if $x \in U$ and for every $x \in U$, $\exists t > 0$ such that $x \cdot t \notin U$, then $i(U) = 0$

(v) if $U \in \Sigma$ is contractible and positively invariant, then $i(U) = 1$

(vi) if $U,V \in \Sigma$ and $\overline{U} \cap \overline{V} = \Phi$, then $i(U \cup V) = i(U) + i(V)$

(vii) if η_i is a semiflow on M_i $(i=1,2)$, then a semiflow $\eta_1 \times \eta_2$ is defined on $M_1 \times M_2$; in this case if $U_i \in \Sigma(\eta_i)$ $(i=1,2)$, then $U_1 \times U_2 \in \Sigma(\eta_1 \times \eta_2)$ and

$$i(U_1 \times U_2, \eta_1 \times \eta_2) = i(U_1, \eta_1)\, i(U_2, \eta_2)$$

<u>Def. 1.2</u> Let $U_1, U_2 \in \Sigma$ with $U_1 \cap U_2 = \Phi$. We say that U_2 is <u>over</u> $\overline{U}_1$ if there exists $T > 0$ such that $U_1 \cap G^T(U_1 \cup U_2)$ is positively invariant with respect to $G^T(U_1 \cup U_2)$.

<u>Def.1.3</u> Let $U \in \Sigma$. A family of sets $\{U_k\}_{k \leq N}$ is called a Morse decomposition of U if

(i) $\overline{U} = \bigcup_{k=1}^{N} \overline{U}_k$

(ii) $U_k \in \Sigma$ for $k = 1,\dots,N$

(iii) $U_k \cap U_h = \Phi$ for $k \neq h$

(iv) U_{h+1} is over $\bigcup_{j=1}^{h} U_j$ for $h = 1,\dots,N-1$

<u>Example</u> Let f be a Liapunov function for (M,η) (i.e.function strictly

decreasing on non-stationary trajectories), and let $c_1 < c_2 < \dots < c_{N-1}$ be a sequence of regular values for f(i.e. $f(x) = c_i \Longrightarrow f'(x) \neq 0$ $i=1,\dots,N-1$). Now set $c_o = -\infty$ and $c_N = +\infty$, and

$$U_k = \{ x \in U \mid c_{k-1} < f(x) < c_k \} \quad k = 0,\dots,N \quad U \in \Sigma$$

It is easy the check that $\{U_k\}$ is a Morse decomposition of U.

Theorem 1.4. If $\{U_k\}_{k \leq N}$ is a Morse decomposition of U, then there exists $Q \in S$ such that

$$\sum_{k=1}^{N} i(U_k) + (1+t)Q(t) .$$

Now let $\Gamma \subset \Sigma$ be a family of sets which satisfies the following properties

(1-2) if $U \in \Gamma$ then any sequence $\{x_n\} \subset U$ such that $f'(x_n) \longrightarrow 0$ has a converging subsequence.

The property (1-2) is related to the well known condition of Palais-Smale.

Def.1.5 We say that $f \in C^1(M)$ satisfies P.S. if any bounded sequence $\{x_n\} \subset M$ such that $f(x_n)$ is bounded and $f'(x_n) \longrightarrow 0$ has a converging subsequence.

Then if f satisfies P.S. it follows that

(1-3) $\Gamma = \{U \in \Sigma \mid f|_U \text{ is bounded }\}$.

The couple $\{\eta,\Gamma\}$ is called variational system. In [B1] and [B2] there is a detailed study of variational systems.

As we will see the sets $U \in \Gamma$ are sets for which the main properties of the Morse-Conley theory which are true in finite dimension are preserved.

Before recalling these properties some notation is necessary.

$$f_a^b = \{ x \in M \mid a < f(x) < b\}$$

$$f_a = f_a^{+\infty} \quad ; \quad f^b = f_{-\infty}^b$$

$$K(U) = \{x \in U \ : \ f'(x) = 0 \} \qquad U \subset M$$

$$\mathcal{K} = \{K \subset M \mid K = K(U) \text{ for some } U \in \Sigma, \text{ and } K \text{ is connected}\}$$

For $x \in M$, $f''(x)$ can be regarded as a bounded selfadjoint operator on the tangent space of M at x. We assume that the nonpositive part of the spectrum of $f''(x)$ consists of isolated eigenvalues of finite multeplicity. Then, for $x \in K(M)$, we set

m(x) = dimension of the space spanned by the eigenvectors of f corresponding to negative eigenvalues

(1-4 $\quad n(x) = \dim [\ker f(x)]$

$$m^*(x) = m(x) + n(x)$$

m(x) is called the Morse index of x and n(x) the nullity of x. If $n(x) = 0$ then x is called "non-degenerate".

For $K \subset K(M)$ we set

$$m(K) = \min_{x \in K} m(x)$$

$$m^*(K) = \max_{x \in K} m^*(x)$$

Proposition 1.6.

(i) if f satisfies P.S. and $f|_U$ is bounded below $(U \in \Sigma)$, then $i(U) = 0$ implies $K(U) = \phi$

(ii) if f satisfies P.S. and $a, b \in \mathbb{R}$ are regular values of f, then $f_a^b \in \Sigma$ and $i(f_a^b) = \sum_{q=0}^{\infty} \dim [H_q(f^b, f^a)] t^q$

where H_* denotes the singular homology with coefficients in Q.

(iii) if $U \in \Gamma$ then $i(U)$ is finite (i.e. $i_t(U)$ is a polynomial in t with nonnegative coefficients)

(iv) let $K \in \mathcal{K}$ and let $U,V \in \Gamma$ with $K(U) = K(V) = K$. Then $i(U) = i(V)$.

Proposition 1.6. (iv) suggests the following definition

Def.1.7. If $K \in \mathcal{K}$, then we set

$$i(K) = i(U)$$

where U is a sufficiently small neighborood of K such that $K(U) = K$.
In particular the index of an isolated critical point x_o is defined (identifying x_o with $\{x_o\}$). Moreover we have

Prop.1.8. If x_o is a nondegenerate critical point then we have

$$i(x_o) = t^{m(x_o)}$$

If x_o is degenerate, we get some information from the following proposition

Prop.1.9. If $U \in \Gamma$ and $K = K(U)$, we have

$$i(U) = \sum_{q=m(K)}^{m^*(K)} a_q t^q$$

where the a_q's are nonnegative numbers.
In particular if $K \in \mathcal{K}$ we have

$$i(K) = \sum_{q=m(k)}^{m^*(k)} a_q t^q$$

Def.1.10. If $K \in \mathcal{K}$ we define the multeplicity of K the integer number $i_1(K)$.

If $i_1(K) = 1$ we say that K is topologically non-degenerate.
If a point x_o is non-degenerate, then $\{x_o\} \in \mathcal{K}$ and by Prop.1.8. x_o is topologically non-degenerate.
The definition 1.10 is justified by the following proposition:

Prop.1.11. Let $K \in \mathcal{K}$ with $i(K) = \sum_{q=m(K)}^{m^*(K)} a_q t^q$, and let U be a sufficiently small neighborood of K.
Then every sufficiently C^2 small perturbation g which satisfies P.S.and whose critical points in U are non-degenerate, has al least

$$\sum_{q=m(x)}^{m^*(x)} a_q$$

critical points. Moreover, at least a_q of them have Morse index q (for $q = m(x), m(x)+1, \dots, m^*(x)$).
Notice that a generic perturbation of f has all non-degenerate critical points. Therefore the conclusion of Prop.1.10 holds for a generic perturbation of f which satisfies P.S.
Now we can state the "Morse relations" for variational systems as defined above.

Def.1.12 Let $X \in \Gamma$ and let $K = K(X)$.
A family of sets $\{U_j\}_{j\in I}$ is called ε-Morse covering of K if

(i) $\overline{U}_j$ is connected for $j \in I$

(ii) $K \subset \bigcup_{j=1}^{N} U_j \subset N_\varepsilon(K)$

(iii) $U_j \in \Gamma$ and $\sum_{j\in I} i(U_j) = i(X) + (1+t)Q(t) \quad Q \in S$

The above definition is justified by the following theorem

Theorem 1.13 If $X \in \Gamma$, then for every $\varepsilon > 0$, there exists a finite ε-Morse covering of $K(X)$.

From Def.1.7. and the above theorem we get the following Corollary

Corollary 1.14. If $U \in \Gamma$ and $K(U)$ consists of a finite number of connected components $K_1, \dots, K_N$, then

$$\sum_{j=1}^{N} i(K_j) = i(U) + (1+t)Q(t)$$

From Corollary 1.14., the classic Morse relations follow

Corollary 1.15. Let $U \in \Gamma$ and suppose that $K(U)$ contains only nondegenerate critical points. Then they are a finite number.

Moreover if $a(q)$ denotes the number of critical points having Morse index q, we have

$$\sum_{q=m(K(U))}^{m^*(K(U))} a(q)\, t^q = i(U) + (1+t)Q \qquad Q \in S$$

The next theorem generalises the Morse relations to a set where f is not bounded above.

Theorem 1.16. Let f be a function which satisfies P.S. and let $K = K(f_c)$. Then, for every $\varepsilon > 0$ there exists an ε-Morse covering of K.

Notice that, in theorem 1.16., the series $\sum_{j\in I} i(U_j)$ and $Q(t)$ (which appears in (iii) of def.1.2) may have some coefficients equal to $+\infty$.

Proof. Let $c_n > c$ be increasing sequence of regular values of f diverging to $+\infty$. By theorem 1.4 we have, for every $n \in \mathbb{N}$,

$$(1\text{-}4) \qquad i(f_c^{c_n}) + i(f_{c_n}^{\infty}) = i(f_c) + (1+t)Q_n^1(t) \qquad Q_n^1 \in S$$

By theorem 1.13 (with $X = f_c^{c_n}$) we have

$$\sum_{j=1}^{k_n} i(U_j) = i(f_c^{c_n}) + (1+t)Q_n^2(t) \qquad Q_n^2 \in S$$

Comparing the above formula with (1-4) we get

$$(1\text{-}5) \qquad \sum_{j=1}^{k_n} i(U_j) + i(f_{c_n}) = i(f_c) + (1+t)Q_n \qquad Q_n = Q_n^1 + Q_n^2$$

Now if $p = \sum_{n=0}^{\infty} a_\ell t^\ell \in S$, we set $\{p\}_\ell = a_\ell$

Then (1-5) reads

$$(1\text{-}6) \qquad \{\sum_{j=1}^{k_n} i(U_j)\}_\ell + \{i(f_{c_n})\}_\ell = \{i(f_c)\}_\ell + \{(1+t)Q_n^{(t)}\}_\ell$$

The theorem is proved if we can take the limit in (1-6) for every $\ell \in \mathbb{N}$.

We consider two cases

(a) $\quad \{i(f_{c_n})\}_\ell = 0$ for n large enough

(b) $\quad \{i(f_{c_n})\}_\ell \neq 0$ for a subsequence $c'_n \to +\infty$.

If (a) holds we have done, since we can take the limit in (1-6) (notice that the sequence $\sum_{j=1}^{k_n} i(U_j)$ is monotonically increasing as $n \to +\infty$).

If (b) holds, then, by proposition 1.6 (iv), we have that

$$H_\ell(M, f^{c'_n}) \neq 0 \qquad \text{for the subsequence } c'_n.$$

Let $[\alpha]$ denote the support of a nontrivial homology class $\alpha \in H_\ell(M, f^{c'})$ and let $c'_m > \max_{x\in[\alpha]} f(x)$.

Consider the exact homology sequence:

$$\dots \to H_\ell(f^{c'_m}, f^{c'_n}) \xrightarrow{i_\ell} H_\ell(M, f^{c'_n}) \xrightarrow{j_\ell} H_\ell(M, f^{c'_m}) \xrightarrow{\partial_\ell} \dots$$

By our choice of c'_m, $j_\ell(\alpha) = 0$; then by the exactness of the sequence $\exists\, \beta \in H_\ell(f^{c'_m}, f^{c'_n})$ s.t. $j_\ell(\beta) = \alpha$.

This fact shows that

$$\{ i(f^{c'_m}_{c'_n}) \}_\ell \neq 0$$

and by theorem (1.3) there exists $U_m \subset f^{c'_m}_{c'_n}$ such that $U \in \Gamma$ and

$$\{ i(U_m) \}_\ell \neq 0.$$

Since this is true for all the terms of the subsequence c'_n defined by (b), it follows that taking the limit in (1-6)

$$\{ \sum_{j=1}^{k_n} i(U_j) \}_\ell$$

diverges to $+\infty$.

Thus the equality (iii) of def.1.12 is satisfied also in this case. //

2 - The Maslov index and the rotation number.

For $\sigma \in S^1 = \{ z \in \mathbb{C} \mid |z| = 1\}$ we set

$$L^2_{\sigma,T} = \{x \in L^2_{loc}(\mathbb{R}, \mathbb{C}^N) \mid x(t+T) = \sigma\cdot x(t)\}$$

where $L^2_{loc}(\mathbb{R}, \mathbb{C}^N)$ in the set of function $x : \mathbb{R} \longrightarrow \mathbb{C}^N$ which are measurable and whose square is locally integrable.

$L^2_{\sigma,t}$ is a Hilbert space if it is equipped with the following scalar product

(2-1) $$(x,y)_{L^2_{\sigma,T}} = \frac{1}{T}\int_0^T (x(t),\, y(t))_{\mathbb{C}^N}\, dt$$

Now let A(t) be a family of real simmetric N x N matrices depending continuously on t and periodic of period τ and consider the following ordinary differential equation

(2-2) $\ddot{y} + A(t)\, y = -\lambda y \qquad y \in \mathbb{C}^N,\ \lambda \in \mathbb{R}$

with the condition

(2-3) $y(t+T) = \sigma\cdot y(t) \qquad \sigma \in S^1,\ T = k\tau,\ k \in \mathbb{N}$

Now let $W^2_{loc}(\mathbb{R}, \mathbb{C}^N)$ denote the space of functions having two square locally integrable derivative.

If $\mathcal{L}_{\sigma,T}$ is the extension to $W^2_{loc}(\mathbb{R}, \mathbb{C}^N) \cap L^2_{\sigma,T}$ of the operator

(2-3') $$-\ddot{y} - A(t)\, y$$

then it is well known that $\mathcal{L}_{\sigma,T}$ is a selfadjoint unbounded operator on $L^2_{\sigma,T}$. Then the eigenvalue problem (2-2), (2-3) becames

(2-4) $\mathcal{L}_{\sigma,T}\, y = \lambda y \qquad y \in D(L_{\sigma,T}) = L^2_{\sigma,T} \cap W^2_{loc}(\mathbb{R}, \mathbb{C}^N)$

It is easy to check from elemetary facts of spectral theory that $\mathcal{L}_{\sigma,T}$ has discrete spectrum with only a finite number of negative eigenvalues.

This fact allows us to define a function

$$j(T,\cdot)\ S^1 \longrightarrow \mathbb{N}$$

as follows.

$j(T,\sigma)$ = {number of negative eigenvalues of $\mathcal{L}_{\sigma,T}$ counted with their multeplicity}.

We shall call the function $j(T,1)$ the Maslov index relative to the equation $\ddot{y} + A(t)y = 0$ in the interval $[0,T]$.

Now let W(t) be the Wronskian matrix relative to equation

$$\ddot{y} + A(t)\, y = 0$$

i.e. the matrix which sends the initial data $\begin{bmatrix} x_o \\ v_o \end{bmatrix}$ to $\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$.

The map $W(T) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ (where T is the period of A(t)) is called the Poincaré map or the monodromy map.

The eigenvalues of W(T) are usually called Floquet multipliers(relative to the interval (O,T)).

Proposition 2.1. The function $\dot{J}(T,\sigma)$ satisfies the following properties

(i) $\dot{J}(T,\bar{\sigma}) = \dot{J}(T,\sigma)$ where $\bar{\sigma}$ is the complex coniugate of σ

(ii) if $\dot{J}(1,\sigma)$ is discontinuous at the point σ^* then σ^* is a Floquet multiplier

(iii) $|\dot{J}(T,\sigma_2) - \dot{J}(T,\sigma_1)| \leq \ell \;\forall\; \sigma_1, \sigma_2 \;\; S^1 - \{+1,-1\}$ where 2ℓ is the number of non-real Floquet multiplies on S^1 counted with their multeplicity. Thus, in particular

$$|\dot{J}(T,\sigma_2) - \dot{J}(T,\sigma_1)| \leq N \qquad \text{for every } \sigma_1, \sigma_2 \in S^1 - \{+1,-1\}$$

(iv) $\dot{J}(kT,\theta) = \sum_{\sigma^k=\theta} \dot{J}(T,\sigma)$.

Proof. (i) if y(t) is an eigenfunction of $\mathcal{L}_{\sigma,T}$, the comlex coniugate function $\overline{y(t)}$ is an eigenfunction of $\mathcal{L}_{\bar{\sigma},T}$ corresponding to the some eigenvalue. Therefore $\mathcal{L}_{\sigma,T}$ and $\mathcal{L}_{\bar{\sigma},T}$ have the some number of negative eigenvalues.

(ii) The eigenvalues of $\mathcal{L}_{\sigma,T}$ depend continuously on σ. Since $\mathcal{L}_{\sigma,T}$ is selfadjoint, they are all real. Therefore the number of the nonpositive eigenvalues $\dot{J}(T,\sigma)$ can change only for those σ^* such that O is an eigenvalue of $\mathcal{L}_{\sigma^*,T}$. This mens that if σ^* is a discontinuity of $\dot{J}(\sigma,T)$ i.e.the following problem

(2-5) $\ddot{y} + A(t)\, y = 0$

(2-6) $y(t+T) = \sigma^* y(t)$

has a nontrivial solution y(t). If W(t) is the Wronskian matrix of

then $\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = W(t) \begin{bmatrix} x \\ v \end{bmatrix}$ for some $x, v \in \mathbb{C}^N$.

Then the condition (2-6) for t = O reads

$$W(T) \begin{bmatrix} x \\ v \end{bmatrix} = \sigma^* \begin{bmatrix} x \\ v \end{bmatrix}$$

Therefore σ^* is an eigenvalue of W(T).

(iii) W(T) is a sympletic matrix; then if λ is an eigenvalue of W(T), also $\bar{\lambda}$ λ^{-1}, $\bar{\lambda}^{-1}$ are eigenvalues of W(T). Therefore

(a) the number of eigenvalues of W(T) different from ± 1 is even.

(b) the sum of the multiplicity of the eigenvalues +1 and -1 is even (if ± 1 are not eigenvalues then their multiplicity has to be assumed O in order to make sense of the above statement).

In particular the eigenvalues of W(T) on $S^1 - \{+1,-1\}$ is an even number 2ℓ. We can assume that all the eigenvalues are simple (otherwhise use a perturbation argument).

Therefore $\dot{J}(T,\sigma)$ has at most 2ℓ points of discontinuity and at each of them the jump of $\dot{J}(T,\sigma)$ is ± 1 since we have assumed that all the eigenvalues are simple.

Now

$$|\dot{J}(T,\sigma_2) - J(T,\sigma_2)| = |\dot{J}(T,e^{i\omega_1}) - \dot{J}(T,e^{i\omega_2})| \quad \text{with } \omega \in (-\pi,\pi) - \{0\} .$$

By (i) we have that $\dot{J}(T,e^{i\omega_1}) = \dot{J}(T,e^{-i\omega_2})$. Then we can assume that $\omega \in (0,\pi)$.

But the function $\dot{J}(T,e^{i\omega})$ has at most ℓ jumps and this proves the statement.

(iv) if $v \in L^2_{\sigma,T}$ then v has the following series expansion :

$$v(t) = e^{i\omega t} \sum_{n=-\infty}^{+\infty} c_n\, e^{2\pi i n/T} \quad \text{with } \omega \in [0,2\pi] \text{ with } e^{i\omega t} = \sigma.$$

Using the above formula we have that $v \in \bigoplus_{\sigma^k=\theta} L^2_{T,\sigma}$ has the following expansion :

$$v(t) = \sum_{\ell=0}^{k-1} e^{i\omega_\ell t} \sum_{n=-\infty}^{+\infty} c_{\ell,n} e^{2\pi i n/T} \qquad \text{with } e^{i\omega_\ell T} = \sigma_\ell,\ \sigma_\ell = \sqrt[k]{\theta} \quad \ell = 0,\dots,k-1$$

But we have $\omega_\ell = \omega_o + 2\pi\ell/kT$ $\quad \ell = 1,\dots,k-1$, with $e^{i\omega_o t} = \theta$. Then

$$v(t) = e^{i\omega_o T} \sum_{k=0}^{k-1} \sum_{n=-\infty}^{+\infty} c_{n,\ell}\, e^{2\pi i(kn+\ell)/kT}$$

and rearranging the terms we have

$$v(t) = e^{i\omega_o T} \sum_{m=-\infty}^{+\infty} a_m\, e^{2\pi i m/kT}$$

The above formula shows that

$$L^2_{kT,\theta} = \bigoplus_{\sigma^k=\theta} L^2_{T,\sigma}$$

Now the operator $\mathcal{L}_{\theta,kT}$ leaves invariant the spaces $L^2_{T,\sigma}$ (if $\sigma^k = \theta$).
Now if $\sigma^-(L)$ denotes the negative spectrum of L we have:

$$\sigma^-(\mathcal{L}_{\theta,kT}) = \bigcup_{\sigma^k=\theta} \sigma^-(\mathcal{L}_{\theta,kT}\big|_{L^2_{\sigma,T}}) = \bigcup_{\sigma^k=\theta} \sigma^-(\mathcal{L}_{\sigma,T})$$

From the above formula the conclusion follows. //

Now we can define the rotation number as follows:

$$\rho = \frac{1}{2\pi\tau} \int_{S^1} j(\tau,\sigma)\, d\sigma = \frac{1}{2\pi\tau} \int_0^{2\pi} j(\tau,e^{i\omega})\, d\omega$$

Proposition 2.2. The rotation number satisfies the following properties:

(i) $\rho = \lim_{\substack{T\to+\infty \\ T=k\tau}} \frac{1}{T}\, j(T,1)$

(ii) $\rho = \frac{\pi}{2T} \int_{S^1} j(T,\sigma)\, d\sigma \qquad T = k\tau$

(iii) $|T\rho - j(T,\sigma)| \le \ell$ for every $\sigma \in S^1-\{+1,-1\}$, $T = k\tau$

2ℓ is the number of Floquet multiplier on $S^1-\{+1,-1\}$

(iv) for every $\sigma \in S^1$ we have $\lim_{\substack{T\to+\infty \\ T=k\tau}} \frac{1}{T}\, j(T,\sigma) = \rho$

Proof. (i) By Proposition (2.1) (iv) we have

$$(2\text{-}7) \qquad \lim_{k\to+\infty} \frac{1}{k\tau}\, j(kT,1) = \lim_{k\to+\infty} \frac{1}{k\tau} \sum_{\ell=0}^{k-1} j(T,e^{2\pi i\ell/k})$$

By the definition of the Cauchy integral we have

$$(2\text{-}8) \qquad \lim_{k\to+\infty} \frac{2\pi}{k} \sum_{\ell=0}^{k-1} j(\tau,e^{2\pi i\ell/k}) = \int_0^{2\pi} j(\tau,e^{i\omega})\, d\omega = 2\pi\tau\rho$$

Then by (2.7) and (2.8) we have

$$\lim_{\substack{T\to+\infty \\ T=k\tau}} \frac{1}{T}\, j(k\tau,1) = \frac{1}{2\pi\tau} \lim_{k\to+\infty} \frac{2\pi}{k} \sum_{\ell=0}^{k-1} j(\tau,e^{2\pi\ell/k}) = \rho$$

(ii) from (i) it follows that ρ is indipendent on $T = k\tau$

(iii) $|T\rho - j(T,\sigma)| = \frac{1}{2\pi} \left| \int_{S^1} j(T,\theta)\, d\theta - \int_{S^1} j(T,\sigma)\, d\theta \right| \le$

$\le \frac{1}{2\pi} \int_{S^1-\{+1,-1\}} |j(T,\theta) - j(T,\sigma)|\, d\theta \le \ell$ by Proposition 2.1(iv)

(iv) it follows from (i) and (iv). //

Example. Consider the equation

$$\ddot{y} + Ay = 0$$
$$y(0) = y(T)$$

where A is a time indipendent real simmetric matrix with ℓ positive eigenvalues $\omega_1^2, \dots, \omega_\ell^2$ and $N-\ell$ negative aigenvalues.

Then the negative eigenvalues of $-\ddot{y} - Ay$ on $L^2_{1,T}$ are

$$\lambda_{n,j} = \left(\frac{2\pi}{T}\right)^2 n^2 - \omega_j^2 \quad \text{with } n \in N,\ \ell = 0,\dots,k-1 \text{ and } n < \frac{\omega_j T}{2\pi}$$

Notice that for $n \geq 1$ they have double multiplicity. Therefore

$$j(T,1) = \ell + 2 \cdot \# \{(n,j) \mid n \leq \frac{\omega_j T}{2\pi}\} = \ell + 2 \sum_{j=1}^{\ell} \left[\frac{\omega_j T}{2\pi} \right]$$

Then by Proposition 2.2 (i) we have

$$\rho = \lim_{T\to+\infty} \frac{1}{T} j(T,1) = \lim_{T\to+\infty} \left\{ \frac{\ell}{T} + \frac{2}{T} \sum_{j=1}^{\ell} \left[\frac{\omega_j T}{2\pi} \right] \right\} = \frac{1}{\pi} \sum_{j=1}^{\ell} \omega_j$$

3 - The generalized Morse-Conley index for periodic solutions of second order conservative systems.

In this section we consider the following system of ordinary differential equations

(3-1) $\ddot{x} + V'(t,x) = 0 \qquad x \in \mathbb{R}^N$

With $V \in C^2(\mathbb{R} \times \mathbb{R}^N)$. We suppose that $V(t,\cdot)$ is T-periodic.

We set

$$W^T = \{ x \in W^2_{loc}(\mathbb{R}, \mathbb{R}^N) \mid x \text{ is T-periodic} \}$$

W^T is an Hilbert space if it is equipped with the following scalar product :

$$(x,y)_{W^T} = \frac{1}{T} \int_0^T (\dot{x} \cdot \dot{y} + x \cdot y)\, dt$$

Where " $\cdot$ " denotes the scalar product in $\mathbb{R}^N$.
The equation (3.1) are the Euler-Lagrange equations corresponding to the functional

(3-2) $f(x) = \frac{1}{T} \int_0^T \{ \frac{1}{2} |\dot{x}|^2 + V(t,x) \}\, dt \qquad x \in W^T$

It is well known that $f(x)$ is a functional of class C^2 on W^T.
Therefore, any T-periodic solution of (3.1) can be interpreted as a critical

point of the functional (3-2).

If we apply the theory of section 1, we can define a Morse index for every T-periodic solution $\bar{x}$ of (2-9) (cf.Def.(1.4)) which we shall denote by $m(\bar{x},T)$ to enphatise the fact that the Morse index is computed in the space W^T.
Of course we can also define the nullity $n(\bar{x},T)$ and the number $m^*(\bar{x},T) = m(\bar{x},T) + n(\bar{x},T)$ as in Def.(1.4) . Now let us consider the linearization of the equation (3-1) at $\bar{x}$:

(3-3) $\ddot{y} + V''(t,\bar{x}(t))y = 0$

It is easy to check that $m(\bar{x},T)$ is the number of negative eigenvalues of the selfadjoint operator

(3-3') $y \longrightarrow -\ddot{y} - V''(t,\bar{x}(t))y \quad \text{in} \quad L^2((0,T),\mathbb{R}^n)$

$n(\bar{x},T)$ is the multiplicity of the eigenvalue 0 of (3-3') and hence it is the number of indipendent solutions of equation (3-3).
A T-periodic solution $\bar{x}$ of (3-1) is called nondegenerate if it is nondegenerate as critical point of the functional (3-2) i.e. if $n(x,T) = 0$.
Clearly $\bar{x}$ is nondegenerate if and only if the linear system (3-3) does not have any nontrivial T-periodic solution, or, if you like, if 1 is not a Floquet multiplier of the equation (3-3) relative to the interval (0,T).
We recall that a number $\alpha \in \mathbb{C}$ is called a Floquet exponent if e^{α} is a Floquet multiplier.

<u>Definition 3 1</u>.Let $\bar{x}$ be a T-periodic solution of the equation (3-1) and let $2\pi i\omega_j$ ($j=1,\ldots,\ell \leq N$) be the purely immaginary Floquet exponent of the linearised equation (3-3). Then if $\omega_j \notin \mathbb{Q}$ for $j=1,\ldots,\ell$ we say that $\bar{x}$ is nonresonant.

It is easy to check that if $\bar{x}$ is a nonresonant τ-periodic solution, then $\bar{x}$ is T-nondegenerate for every $T = k\tau$ $k \in \mathbb{N}$.

If $\bar{x}$ is a T-degenerate solution of (3-1) then the definition 1.10 can be applied to define the multeplicity of $\bar{x}$.

We can associate to the equation (3-3) a Maslov index $\dot{J}(T,\sigma)$ as in section 2 where $A(t) = V''(t,\bar{x}(t))$ and consequently a rotation number $\rho(\bar{x})$.

Proposition 3.2. If $\bar{x}$ is a T-periodic solution of (3-1) ($T = k\tau$, $k \in \mathbb{N}$) then

(i) $m = (\bar{x},T) = \dot{J}_{\bar{x}}(T,1)$

Moreover if $\bar{x}$ is not degenerate

(ii) $T\cdot\rho(\bar{x}) - N \leq m(\bar{x},T) \leq T\cdot\rho(\bar{x}) + N$.

Proof. (i) is a trivial consequence of the definitions.

(ii) Since 1 is not a Floquet multiplier, then for σ_1 very close to 1 ($\sigma \in S^1$) σ_1 is not a Floquet multiplier and

$$m(T,\bar{x}) = \dot{J}_{\bar{x}}(T,\sigma) \qquad \text{by Prop.2.1(ii).}$$

Then the conclusion follows from proposition 2.2(iii). //

Now let Γ^T be the family of subsets of W^T defined in (1.2).

Now we want to examine the relationship between the index of a set U ($U \in \Gamma^T$) and the rotation number of the solution of (3-1) contained in U.

Proposition 3.3. Let $U \in \Gamma^T$ and let $i(U) = \sum_{\ell=m_1}^{m_2} a_\ell t^2$ with $a_m \neq 0$ ($m_1 \leq m \leq m_2$). Then $\exists\, x \in U$ such that

$$\frac{m-N}{T} \leq \rho(x) \leq \frac{m+N}{T}$$

Proof. Since $U \in \Gamma^T$ we can apply proposition 1.11. Then for every $\varepsilon > 0$ there exists $g_\varepsilon > 0$ such that i(U) relative to g_ε is the some than the index relative to f and all the critical points of g_ε in U are nondegenerate.

Then, since $a_m \neq 0$, there exists x_ε, critical point of g_ε, such that

$$\frac{1}{T}(m-N) \leq \rho(x_\varepsilon) \leq \frac{1}{T}(m+N)$$

(we have used Prop.3.2 (ii)).

Now, letting $\varepsilon \to 0$, $x_\varepsilon \to x$ and $\rho(x_\varepsilon) \to \rho(x)$ and this proves our assumption. //

Corollary 3.4. Let x_o be a degenerate critical point whose index is

$$i(x_o) = \sum_{\ell=m_1}^{m_2} a_\ell t^\ell \qquad (a_{m_1} \neq 0, \quad a_{m_2} \neq 0, \quad m_1 \leq m_2)$$

Then

$$\frac{1}{T}(m_2-N) \leq \rho(x) \leq \frac{1}{T}(m_1+N) .$$

Proof. Apply Prop.3.4. //

Next we shall examine some facts which occur in the autonomous case i.e. we consider the equation

$$\text{(3-4)} \qquad \ddot{x} + V'(x) = 0 \qquad x(t) \in \mathbb{R}^N$$

In this situation every critical point $x \in \mathbb{R}^N$ of V is a constant solution of (3-4).

Proposition 3.5. Let $U \in W^T$ ($U \subset \Gamma$) be a set which does not contain constant solutions. Then there exists a polynomial P(t) with integer (but not necessarily positive) coefficients such that

$$i(U) = (1+t)\, P(t)$$

Proof. Sec. [B2] Prop.4.8. //

4 - Some applications in the nonautonomous case.

In this section we try to get some information on the structure of the periodic solutions of the equation (3-1).
We suppose that $V(t,x)$ satisfies the following asymptotic conditions.

(4-1) there exists $R > 0$ and $p > 2$ such that

$$0 < V(t,x) \le \frac{1}{p} V_x(t,x)\cdot x \quad \forall\, t \in \mathbb{R} \quad \forall\, x \ \text{ with } |x| > R.$$

Condition (4-1) implies that $V(t,x)$ grows more than $|x|^2$ as $|x| \longrightarrow +\infty$.
Moreover this condition implies the following facts :

Lemma 4.1. Suppose that V satisfies (4-1). Then the functional (3-2) satisfies P.S.
Proof. See e.g. [R]. //

Lemma 4.2. Let

$$f_c = \{x \in W^T \mid f(x) > c\}$$

Then there exists $c_o \in \mathbb{R}$ such that

$$f_c \in \Sigma \quad \text{and} \quad i(f_c) = 0 \quad \text{for every } c \le c_o.$$

Proof. See [B2] lemma 3.7. //

Theorem 4.3. Suppose that V satisfies (4-1) and let x_o be a nonresonant τ-periodic solution of 3-1.
Then, for every $\varepsilon > 0$ there exists a T-periodic solution $x \ne x_o$ (with $T = k\tau$, $T < \tau + \frac{2N+1}{\varepsilon}$) such that

$$|\rho(x) - \rho(x_o)| \le \varepsilon$$

Proof. Take $T = k\tau$ with $\frac{2N+1}{\varepsilon}$ $T <$ $+ \frac{2N+1}{\varepsilon}$. Since x_o is nonresonant, there is a neighborhood $N_\delta(x_o)$ in W^T which does not contain periodic solutions of (4-1). Now take a δ-Morse covering $\{U_\ell\}$ of f_c (where f_c is as in lemma 4.2, $c \le c_o$). Then, by Th. 1.16

$$i(x_o) + \sum_{\ell \in I} i(U_\ell) = (1+t)Q(t).$$

By the above formula there exists $\ell \in I$ such that either

(4-2) $$i(U_\ell) = t^{m+1}$$

or

$$i(U_\ell) = t^{m-1}$$

where m is the Morse index of x_o (i.e. $i(x_o) = t^m$).
We consider the first possibility (if the second one holds we argue in the same way).
By Prop. 3.2 (ii) we have

(4-3) $$i(x_o) = t^m \quad \text{with} \quad \rho(x_o)T - N \le m \le \rho(x_o)\cdot T + N .$$

By Prop. 3.3 and (4-2), there exists $x \in U_\ell$ such that

(4-4) $$\frac{1}{T}(m+1-N) \le \rho(x) \le \frac{1}{T}(m+1+N).$$

Comparing (4-3) and (4-4) we get

$$|\rho(x) - \rho(x_o)| \le \frac{1}{T}(2N+1) \le \varepsilon . \qquad //$$

The next theorem we are going to prove has stronger assumptions and gives a better information about the T-periodic solution of equation (3-1).

<u>Theorem 4.4</u>. Suppose that V satisfie (4.1). Let $T = p\tau$ with p prime number, and suppose that all the T-periodic solution of (3-1) are isolated (as pointe in W^T). Let $x_1, x_2, \dots, x_n, \dots$ be the periodic solutions of equation (3-1). We suppose that they are T-nondegenerate and ordered by increasing rotation number.

$$\rho(x_1) \leq \rho(x_2) \dots \leq \rho(x_n) \leq \dots$$

Then for every number $\rho \in [\rho(x_{2n-1}), \rho(x_{2n})]$ $(2n < p)$ there is a T-periodic solution $\bar{x}$ such that

$$|\rho(\bar{x}) - \rho| \leq \frac{N+1}{T}$$

<u>Proof</u>. By the theorem 1.16 relative to the space W^T we have

$$(4\text{-}5) \qquad \sum_{j \in J} i(x_j) + \sum_{j \in I} i(U_j) = (1+t)Q(t) \qquad \text{with } Q(t) = \sum_\ell q_\ell t^\ell$$

where $\{U_j\}_{j \in I}$ is an ε-Morse covering of the T-periodic solutions of (3-1) which are not τ-periodic and $\{x_j\}_{j \in J}$ is the set of τ-periodic solutions.

Now fix $\rho \in [\rho(x_{2n+1}) + T\cdot(N+1), \ \rho(x_{2n}) - T\cdot(N+1)]$ and take

$$m = \{\text{integer part of } \rho\cdot T\} .$$

Consider only the terms of (4-5) of order less or equal to m :

$$(4\text{-}6) \qquad \sum_{\ell=1}^{m} a_\ell t^\ell + \sum_{\ell=0}^{m} b_\ell t^\ell = (1+t) \sum_{\ell=0}^{m-1} q_\ell t^\ell + q_m t^m$$

where

$$(4\text{-}7) \qquad \sum_{\ell=1}^{m} a_\ell t^\ell = \sum_{j=1}^{2n-1} i(x_j)$$

and the therm $\sum_{\ell=0}^{m} b_\ell t^\ell$ comes from the ε-Morse covering relative to the solutions which are not τ-periodic.

Since we have supposed that these solutions are isolated, by Proposition 4.1 of [B2] we have that

$$b_\ell = p\beta_\ell \qquad \text{for some } \beta_\ell \in \mathbb{N}$$

Then rewiting (4-6) for t = -1, we get

$$(4\text{-}8) \qquad \sum_{\ell=1}^{m} (-1)^\ell a_\ell + p \sum_{\ell=0}^{m} (-1)^\ell \beta_\ell = (-1)^m q_m .$$

By (4-7), the first term of (4-8) is an odd number less or equal to 2n-1, and by our assumption less than p.

Thus the sum of the two terms of the left hand side of (4-8) is different from . Thus $q_m \neq 0$. Then, by (4-5), there exists U_j such that

$$i(U_j) = t^m + \text{possible other terms.}$$

Proposition 3.4 implies that there exists $\bar{x} \in U_j$ such that

$$\frac{1}{T}(m - N) \leq \rho(\bar{x}) \leq \frac{1}{T}(m + N)$$

and by the definition of m we have that

$$\rho - \frac{N+1}{T} \leq \rho(\bar{x}) \leq \rho + \frac{N+1}{T} .$$

Thus the theorem is proved for $\rho \in [\rho(x_{2n-1}) + T(N+1), \ \rho(x_{2n}) - T(N+1)]$.

Considering also the solutions x_{2n-1} and x_{2n} the theorem is proved for every $\rho \in [\rho(x_{2n-1}), \rho(x_{2n})]$. //

We conclude this section with a theorem which is the analogous of Th.4.3

in the asymptotically quadratic case.

We say that $V(t,x)$ is asymptotically quadratic if there exists a matrix $A_\infty(t)$ such that

(4-9) $$V_x(t,x) = A_\infty(t)x + O(|x|) \qquad \text{as} \quad |x| \longrightarrow +\infty .$$

If V is asymptotically quadratic we can consider the linearised system at ∞

(4-10) $$\ddot{y} + A_\infty(t)y = 0$$

and associate to (4-10) a rotation number ρ_∞.

Then we have the following result :

<u>Theorem 4.5</u>. Suppose that V satisfies (4-9) and suppose that (4-10) has no T-periodic solution different from 0.

Let x_o be a nondegenerate T-periodic solution of (3-1) with rotation number $\rho(x_o)$ such that

(4-11) $$|\rho(x_o)-\rho_\infty| > \frac{2N}{T} .$$

Then the system (3-1) has a T-periodic solution $\bar{x}$ such that

$$|\rho(\bar{x}) - \rho(x_o)| < \frac{2N+1}{T}$$

<u>Sketch of the proof</u>. If we take a ball in W^T of sufficiently large radious R, arguing as in [B2] , we have that

$$B_R \in \Gamma^T \qquad \text{and} \qquad i(B_R) = t^{m(\infty)} .$$

It is easy to check that

(4-12) $$T\cdot\rho_\infty - N \leq m(\infty) \leq T\cdot\rho_\infty + N .$$

Then the Morse relation take the form

$$i(x_o) + \sum_{\ell\, I} i(U_\ell) = t^{m(\infty)} + (1+t)Q(t)$$

Let $i(x_o) = t^m$.

Then, by (4-11) and (4-12)

$$|m - m(\infty)| \neq 0.$$

Therefore we have that $Q(t) \neq 0$.

From now on we can argue as at the end of the theorem 4.3. //

5 - One application to the autonomous case.

Now we consider the autonomous equation 3-4. We restrict ourselves to the superlinear case i.e. we still assume that V satisfies (4-1).

In this case the theorems 4.3 and 4.4 do not apply since every solution of equation (3-4) is degenerate.

In fact if x is a T-periodic solution of (3-4), $y = \dot{x}$ is a T-periodic solution of the linearised equation

$$\ddot{y} + V''(x(t))y = 0 .$$

Let $\rho_o = \max \{\rho(x) \mid x \text{ is a constant solution of } (3\text{-}4)\}$.

<u>Theorem 5.1</u>. For every $\rho \geq \rho_o$ there is a T-periodic solution $\bar{x}$ such that

$$|\rho - \rho(\bar{x})| \leq \frac{N+1}{T} .$$

<u>Proof</u>. For sake of simplicity we will suppose that the constant solutions $x_1, \dots, x_n$ of (3-4), i.e. the critical points of $V(x)$ are T-nondegenerate

solutions.

The general case can be treated using a perturbation argument of the type used in [B2] Th. 4.19.

By theorem 1.16 relative to the space W^T we have

$$(5\text{-}1) \qquad \sum_{j=1}^{n} i(x) + \sum_{j\in I} i(U_j) = (1+t)Q(t) \quad \text{with } Q(t) = \Sigma_\ell q_\ell t^\ell \quad q_\ell \geq 0$$

where $\{U_j\} \cup \{\sum_{j=1}^{n} B_\varepsilon(x_j)\}$ is an ε-Morse covering.

Now we claim that n is a add number.

In fact the critical points of V(which we are supposed non degenerate) satisfy the following Morse relation

$$\Sigma_\ell a_\ell t^\ell = i(\mathbb{R}^n) + (1+t)Q(t) \ .$$

Since $i(\mathbb{R}^n) = 1$ by our assumption on the potential V, taking the above relation with $t = 1$ we get

$$\text{number of critical points of } V = 1 + 2\cdot Q(1)$$

which proves our claim.

By Proposition 3.5

$$i(U_j) = (1+t)\ P_j(t) \ .$$

Then equation (5-1) can be written as follows

$$(5\text{-}2) \qquad \sum_{j=1}^{n} i(x_j) + (1+t)\ \Sigma_\ell b_\ell t^\ell = (1+t)\ \Sigma q_\ell t^\ell \quad \text{where } \Sigma_\ell b_\ell t^\ell = \Sigma_j\ P_j(t).$$

Now take $\rho > \bar{\rho} + \frac{N+1}{T}$ and let

$$m = \text{integer part of } \rho T$$

The equation (5-2) up to the order m reads

$$(5\text{-}3) \qquad \sum_{j=1}^{n} i(x_j) + (1+t) \sum_{\ell=1}^{m-1} b_\ell t^\ell + b_m t^m = \sum_{\ell=1}^{m-1} q_\ell t^\ell + q_m t^m \ .$$

Now taking $t = -1$, from the above equation we get

$$\sum_{j=1}^{n} i_1(x) + (-1)^m\ b_m = (-1)^m q_m \ .$$

Since $\sum_{j=1}^{n} i_1(x)$ is an add number, it follows that b_m (or q_m) is different from zero.

In either case, from equation (5-1) it follows that there exists U_j such that

$$i(U_j) = t^m + \text{other possible terms.}$$

Then by Prop.3.4, there exists $\bar{x}$ such that

$$\frac{m-N}{T} \leq \rho(\bar{x}) \leq \frac{m+N}{T}$$

The conclusion follows from the definition of m.

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