INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SECOND SCHOOL ON ADVANCED TECHNIQUES
IN COMPUTATIONAL PHYSICS
(18 January - 12 February 1988)

SMR.282/31

ACCELERATOR PHYSICS

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COMPUTING IN ACCELERATOR PHYSICS

A. Wrulich
SINCROTRONE TRIESTE

COMPUTING → Accelerator Design
Accelerator Operation

1. Lesson:

Accelerators, an introduction
Basic equations for transverse motion
Computation of linear optics

2. Lesson:

Basic equations for longitudinal motion
Nonlinear transverse motion
Resonance effects in circular accelerators
Simulation of particle motion, tracking

3. Lesson:

Chaotic trajectories of transverse motion
Collective effects
Beam-beam effect
Beam-environment interaction
Numerical field calculations

ACCELERATORS (introduction)

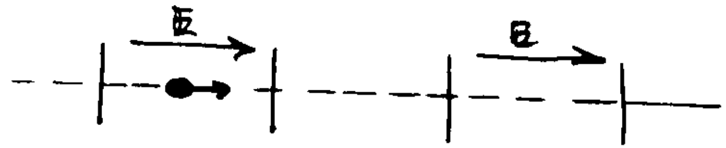
used for →

- Elementary particle physics,
  - colliders
- Light sources
- Health service
- Industry (compact sources)

types of accelerators →

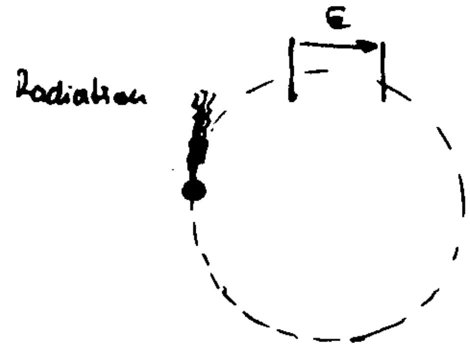
- Linear
- Circular (synchrotron, storage ring, cyclotron, microtron, ...)

Linear Accelerator:



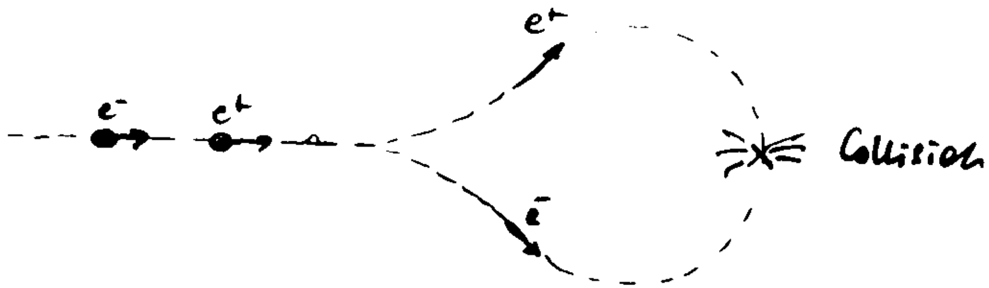
Circular Accelerator:

Storage rings
Synchrotrons



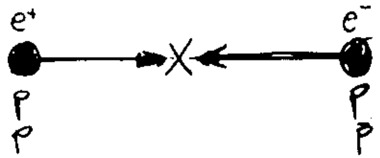
Problems: (e^+e^-)
Synchrotron Radiation
Beam beam effect
(p) large bending fields

SLC (Stanford):

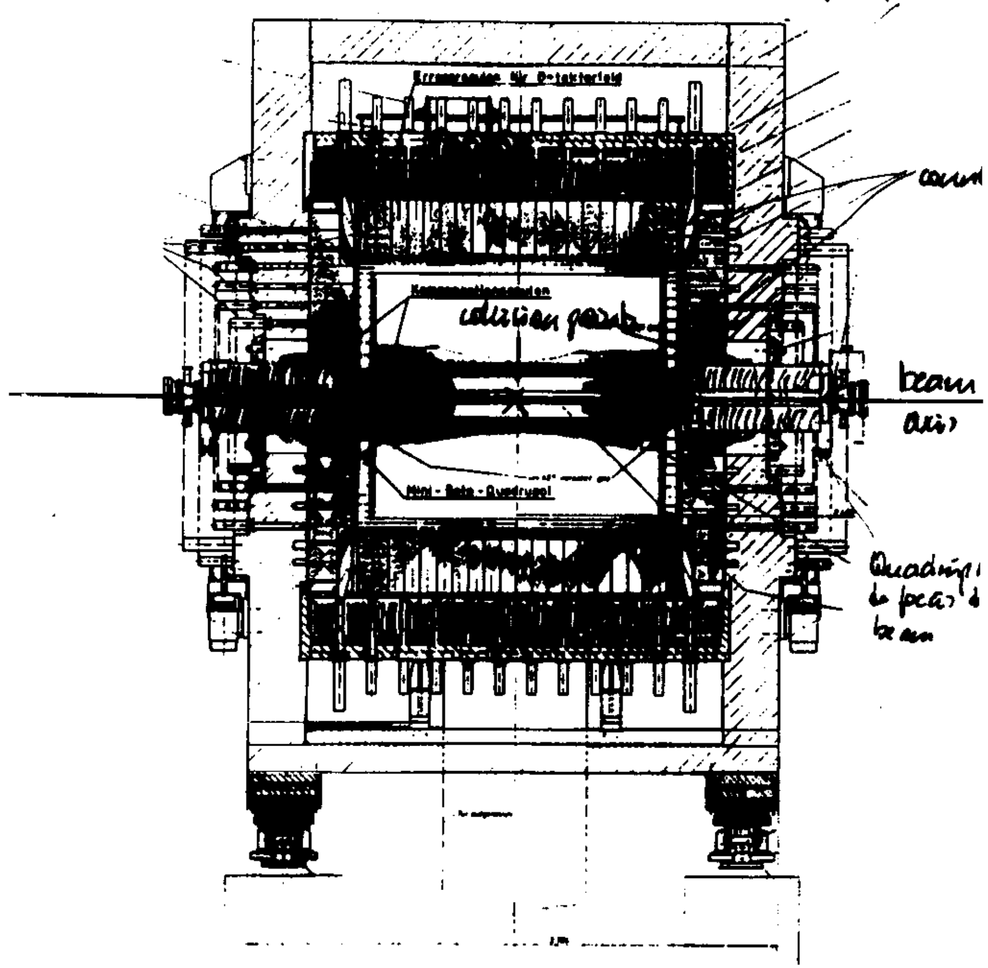


Accelerators for High Energy Physics

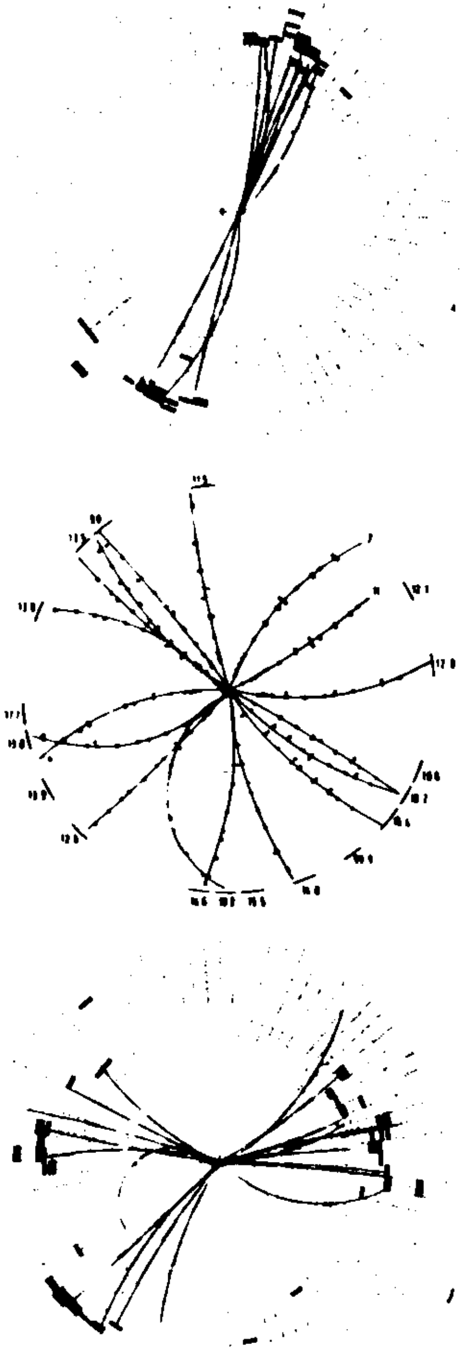
Energy of colliding particles



is used to create showers of new particles. Their traces are detected by large apparatuses (DETECTORS) and evaluated by computers.

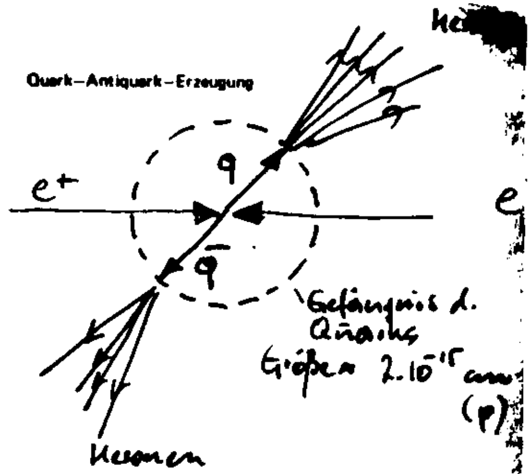


ARGUS at DORIS II (DESY)



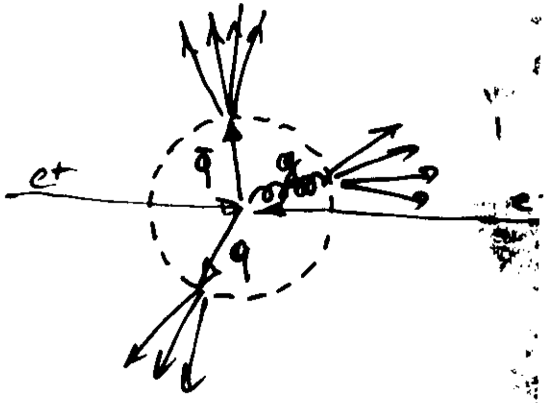
2-jetevent

Quark–Antiquark–Erzeugung



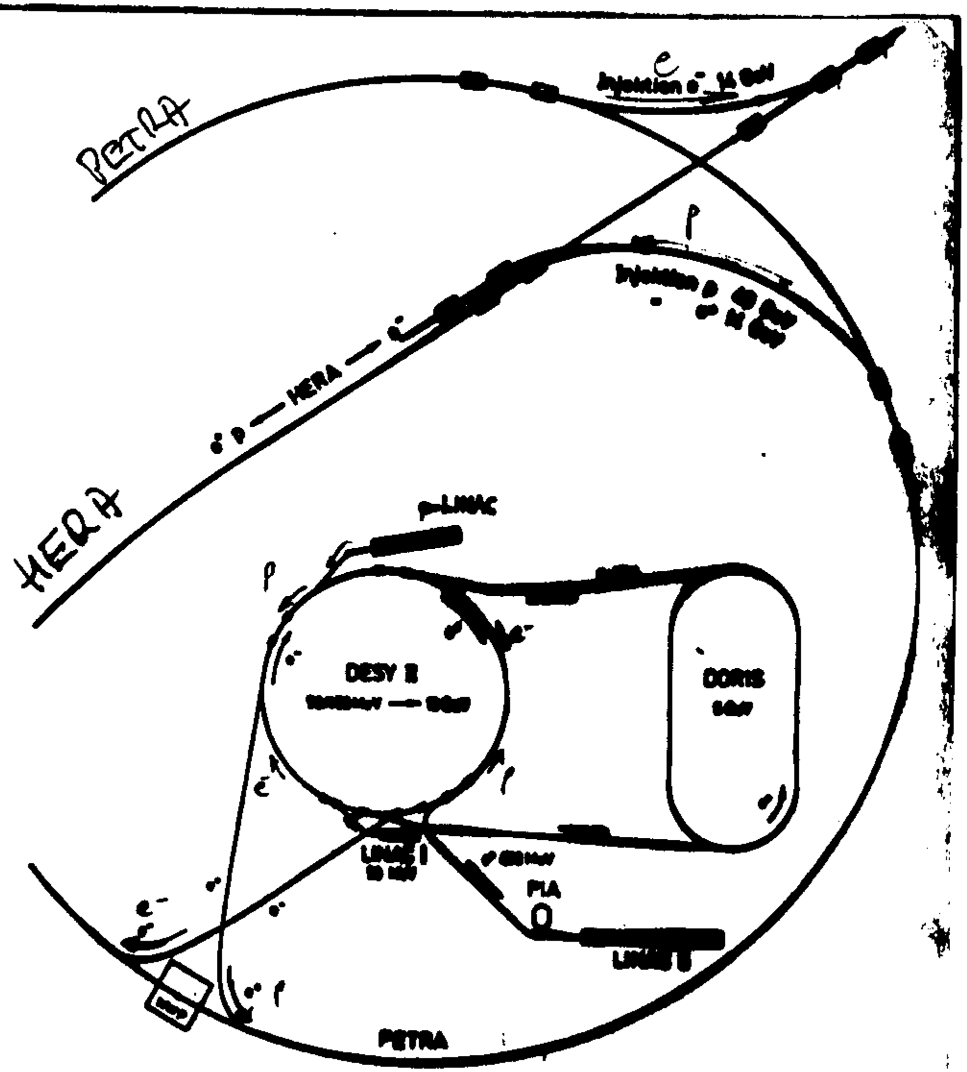
"Multi-Hadron"–Erzeugung

3-jetevent

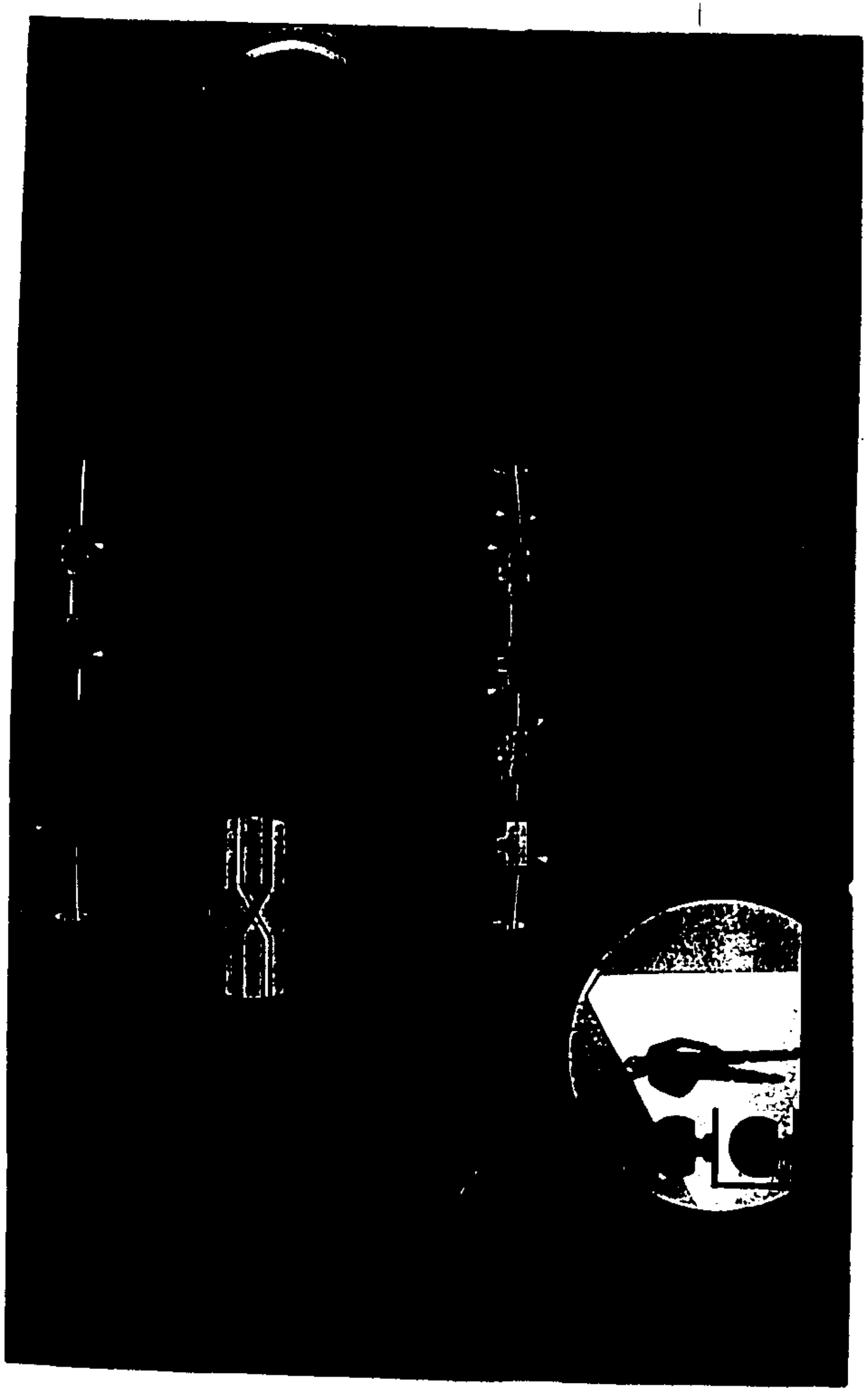


Quark–Antiquark–Erzeugung mit abgestrahltem Gluon

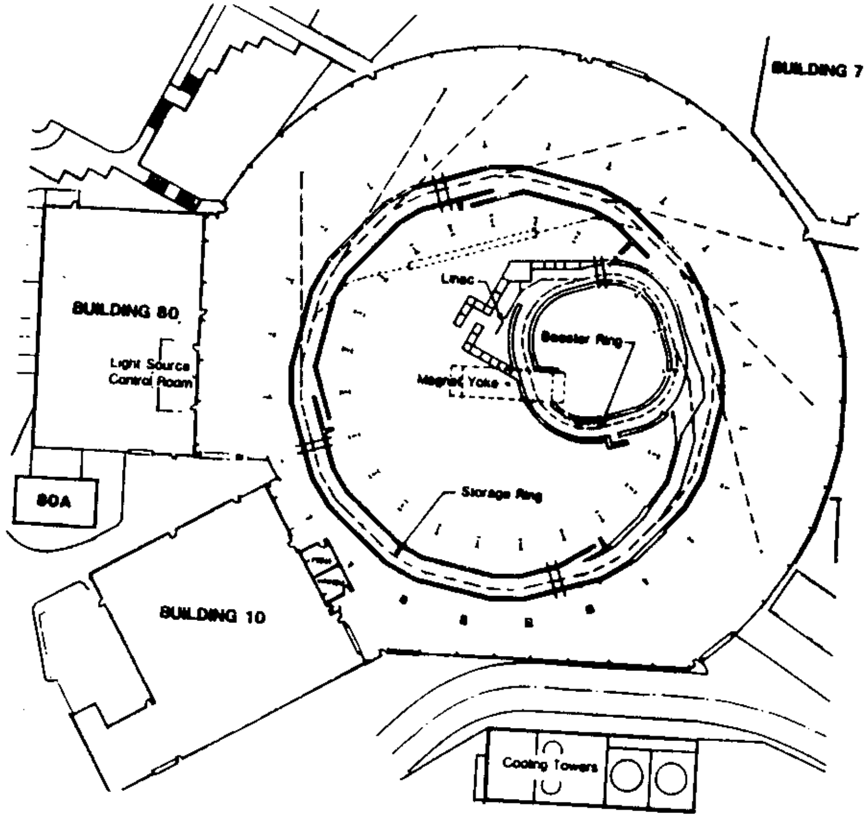
Am Speicherring PETRA registrierte Computer-Rekonstruktionen von "Ereignissen" bei der Kollision von Elektronen mit Positronen.



DESY – accelerators



Radiation Sources:

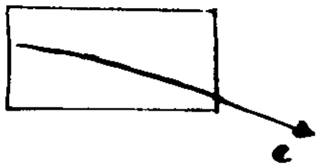


Berkeley 1.5 GeV Light Source

RADIATION

Basic principle: Radiation is emitted by relativistic charged particles circulating in a magnetic field.

I. BENDING MAGNETS:



$\Psi_x \approx 30\text{-}50$ mrad

SR is emitted with →

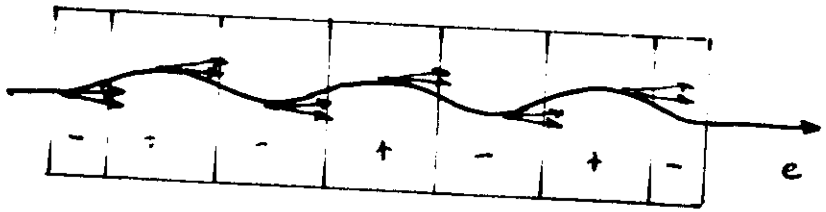
Horizontal divergence: $\Psi_x \approx 30 - 50$ mrad

Vertical divergence: $\Psi_z = \pm \frac{1330}{\gamma} \left(\frac{\lambda}{\lambda_c}\right)^{0.43} \ldots$

$\approx \pm \frac{1}{\gamma}$

$E = 2\,GeV$ ($\gamma \approx 4000$) $\Rightarrow \Psi_z = \pm\, 0.33$ mrad

II. WIGGLER



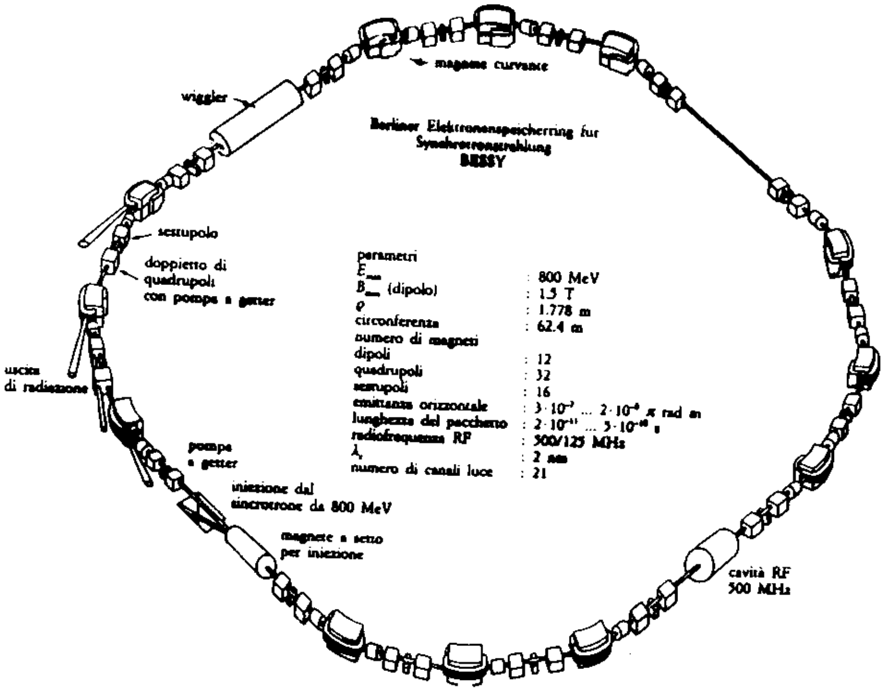
$\varepsilon_c\,[keV] = 0.665 \cdot E^2\,[GeV] \cdot B\,[T]$

↑ can be chosen independent of storage ring energy

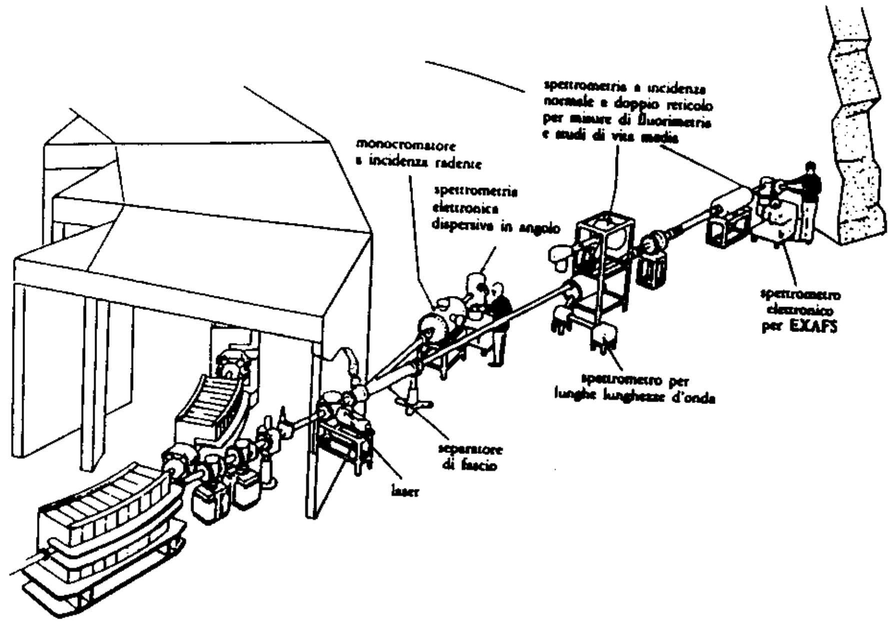
FLUX is increased due to superposition

inductance Trieste:

$\left.\begin{array}{l} E = 2\,GeV \\ B = 1.2\,T \end{array}\right\} \Rightarrow \underline{\underline{\varepsilon_c = 3.2\ keV}}$



BESSY - Radiation Source



Beam line of radiation source

EQUATIONS OF MOTION

Principle: Charged particles moving in a (constant) magnetic field are bend on a circular orbit

LORENTZ FORCE → $\boxed{\vec{K} = e[\vec{v} \times \vec{B}]}$

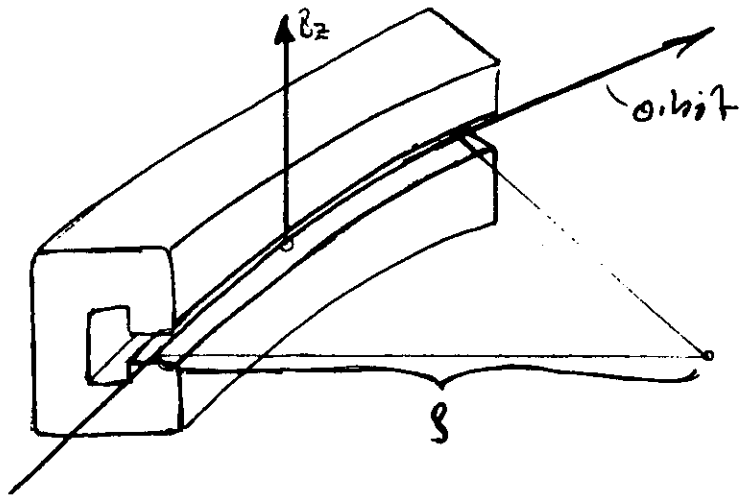
Basic magnetic elements of an accelerator →

BENDING MAGNETS

Quadrupoles

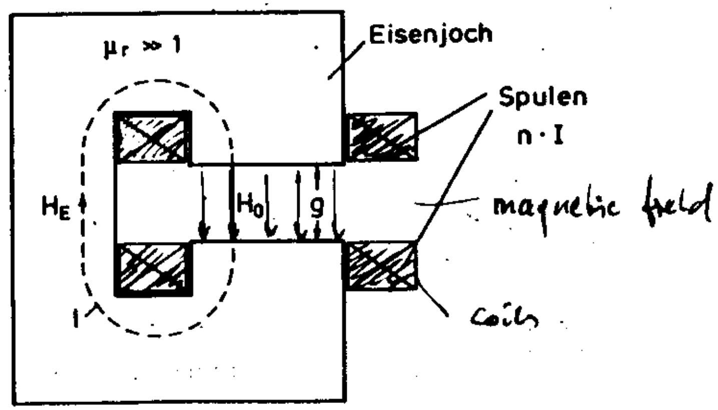
Sextupoles

BENDING MAGNET → Guiding Field



$\vec{v} \perp \vec{B}$: $\quad K = e[\vec{v} \times \vec{B}] = \frac{mv^2}{\rho} \Rightarrow$

$$\boxed{\frac{e}{p_0} B_z = \frac{1}{\rho}}$$

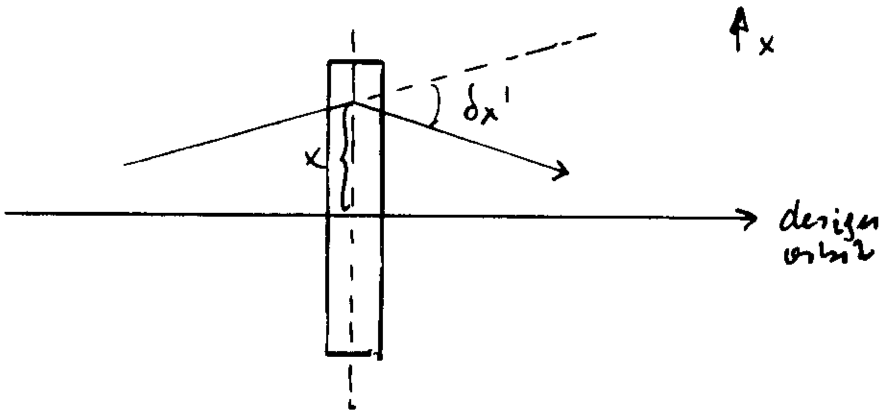


Guiding field only is not sufficient!

→ Original deviations of the particle from the design orbit would lead to a loss after several turns.

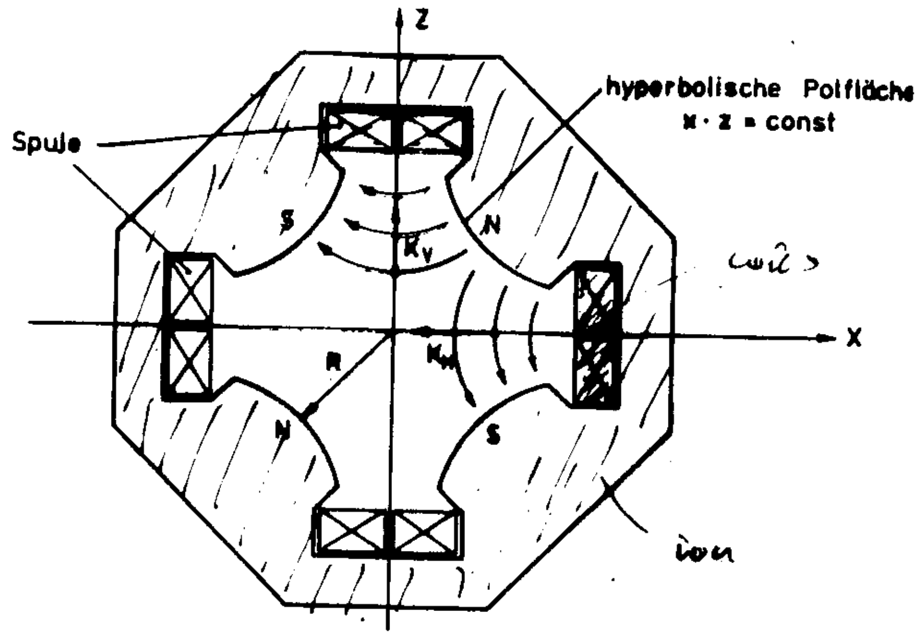
THEREFORE →

Force is needed to bend the particle back to the design orbit, with strength proportional to the deviation from the design orbit:



FOCUSING: $\delta_x' = -g \cdot x$

QUADRUPOLES → Focusing



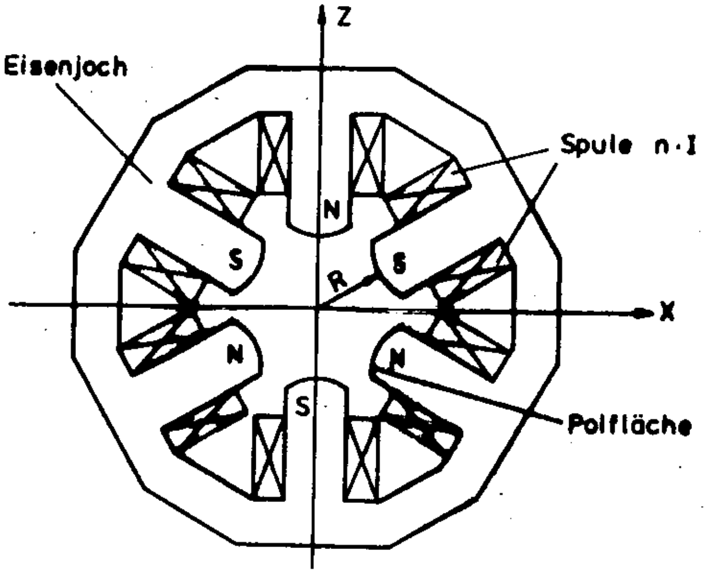
$$\boxed{\text{Potential: } \quad V = -g \cdot x \cdot z}$$

$$\frac{dV}{dz} = -g x = -\frac{\partial B_z}{\partial x} \cdot x$$

DEFLECTION: $\boxed{\delta_x' = -\frac{e}{p_0} \frac{\partial B_z}{\partial x} L_Q \cdot x}$

SEXTUPOLES → for chromatic corrections

TO AVOID TUNE SPREAD
HEAD TAIL INST.



$$\delta x'' = m \cdot x^2 = \left(\frac{1}{2}\frac{e}{p_0}\frac{\partial^2 B_z}{\partial x^2} l_s\right) \cdot x^2$$

$\frac{\Delta p}{p}$ → orbit deviation $x_0 = \frac{\Delta p}{p} D$ D... dispersion

$g \to g/(1+\frac{\Delta p}{p})$

$$\delta x' = m(x - x_0)^2 = m x^2 - (2 m x_0) x + m x_0^2$$

↓

quadrupole

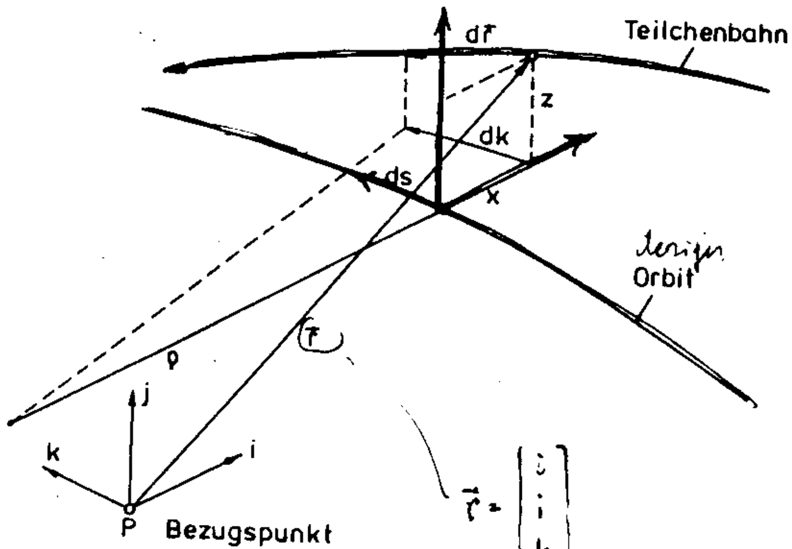
If in addition field errors are taken into account, the general expression for the field can be expanded in a Taylor series (for small deviations from the design orbit) →

ONE PLANE:

$$\frac{1}{\rho} = \frac{e}{p} B_z(x) = \underbrace{\frac{e}{p} B_{z0} + \left(\frac{e}{p}\frac{\partial B_z}{\partial x}\right) x + \left(\frac{1}{2}\frac{e}{p}\frac{\partial^2 B_z}{\partial x^2}\right) x^2}_{\text{Regular elements of circular accelerators}} + \dots$$

Equation of Motion:

$$\boxed{\dot{\vec p} = e\,[\dot{\vec r} \times \vec B]}$$



$$\vec r = \begin{bmatrix} j \\ i \\ k \end{bmatrix}$$

→ transformation to reference system moving with the particle

⇒

After changing the independent variable from

$$t \rightarrow s$$

we find after a straightforward evaluation of the equations of motion in the new reference frame →

$$\boxed{\vec r\,'' = \frac{e}{p}\left(1 + \frac{x}{\rho}\right) \begin{bmatrix} -B_z \\ B_x \\ x' B_z - z' B_x \end{bmatrix}}$$

– for transverse magnetic fields only!

For a linear machine, i.e.

$$\begin{aligned} \frac{e}{p} B_z &= \frac{1}{\rho} + g\,x \\ \frac{e}{p} B_x &= -g\,x \end{aligned}$$

focusing

bending magnet deflection

→ Equations of motion:

$$\boxed{\begin{aligned} x'' + \left(k + \frac{1}{\rho^2}\right)x &= 0 \\ z'' - k \cdot z &= 0 \end{aligned}}$$

General:

$$\boxed{\frac{d^2x}{ds^2} + K(s) \cdot x = 0}$$

<u>HILL's differential equation</u>

Solution of any second order differential equation →

$$x(s) = C(s,s_0)\,x(s_0) + S(s,s_0)\,x'(s_0)$$

In matrix form, including the derivation →

$$\begin{bmatrix} x \\ x' \end{bmatrix}_s = \begin{bmatrix} C & S \\ C' & S' \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{s_0}$$

To solve the equation, we follow the historical way:

Substitute in the differential equation:

$$u = \frac{x}{\sqrt{\beta}} \quad , \quad \phi' = \frac{1}{\beta}$$ Floquet transformation

$$\frac{d^2x}{ds^2} + K(s)\,x = 0 \quad \rightarrow \quad \boxed{\frac{d^2u}{d\phi^2} + u = 0}$$

↓

Harmonic oscillator with angular frequency 1

'β' - betafunction is defined by

$$\frac{d^2}{ds^2}(\beta^{1/2}) + K(s)(\beta^{1/2}) - \frac{1}{\beta^{3/2}} = 0$$

Solution of the Harmonic oscillator:

$$u(\phi) = A\cos\phi + B\sin\phi$$

Develope the constants by defining the initial conditions → $\phi = 0 \rightarrow u_0, u_0'$

$$u = \frac{x}{\sqrt{\beta}} \qquad \rightarrow \quad \underline{\underline{A = u_0}}$$

$$\left(\frac{x}{\sqrt{\beta}}\right)' = u' = \frac{1}{\sqrt{\beta}}\left(x' + \alpha\frac{x}{\beta}\right) \quad \rightarrow \quad \underline{\underline{B = u_0'\beta}} \qquad \alpha = -\tfrac{1}{2}\beta'$$

$$\begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} \cos\phi & \beta\sin\phi \\ -\frac{1}{\beta}\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}_0$$

starting vector

→ backtransformation $(u, u') \rightarrow (x, x')$

$$\boxed{\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\phi + \alpha_0\sin\phi) & \sqrt{\beta\beta_0}\sin\phi \\ -\frac{1}{\sqrt{\beta\beta_0}}[(\alpha-\alpha_0)\cos\phi + (1+\alpha\alpha_0)\sin\phi] & \sqrt{\frac{\beta_0}{\beta}}(\cos\phi - \alpha\sin\phi) \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}}$$

α, β .. Courant Snyder parameters

By using the trigonometric identity

$$\cos^2\phi + \sin^2\phi = 1$$

we can derive the

Courant Snyder invariant

$$\boxed{I = \frac{x^2 + (x\alpha + x'\beta)}{\beta} = \frac{x_0^2 + (x_0\alpha_0 + x_0'\beta_0)^2}{\beta_0}}$$

- Equation of ellipse with area

$$\varepsilon = \pi \cdot I$$

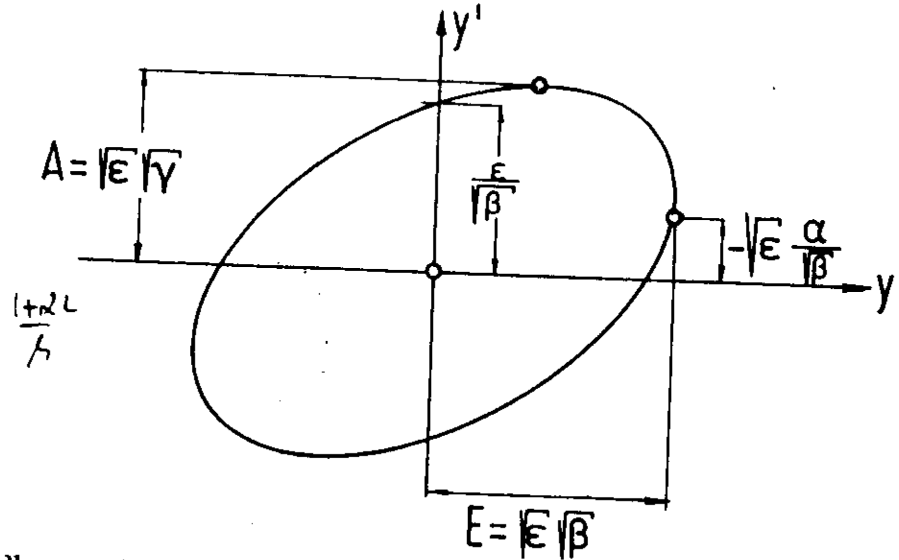
$\uparrow$

Emittance

If we substitute $x = A\cos(\phi + \delta)$ we find

$$A \equiv \sqrt{\varepsilon\beta}$$

amplitude of betatron oscillations



hase plane ellipse

nce the area of the phase-plane ellipse
given the maximum amplitude of
a betatron oscillation E becomes smaller
arger) if the beta function becomes
maller (larger).

β = measure of beam cross section

Periodic structures.

$$\left.\begin{aligned} \beta &= \beta_0 \\ \alpha &= \alpha_0 \\ \phi &= 2\pi Q = \mu \end{aligned}\right\}$$

$\beta_0, \alpha_0, \phi_0$ at s_0; β, α, μ at $s_0 + L$

$\phi - \phi_0 = 2\pi Q = \mu$

Q .. tune

$$\begin{bmatrix} x \\ x' \end{bmatrix}_1 = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\frac{1+\alpha^2}{\beta}\sin\mu & \cos\mu - \alpha\sin\mu \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_0$$

$$\vec{X}_1 = \left[\mathbb{I}\sin\mu + \mathbb{J}\cos\mu\right]\vec{X}_0$$

$$\mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbb{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}$$

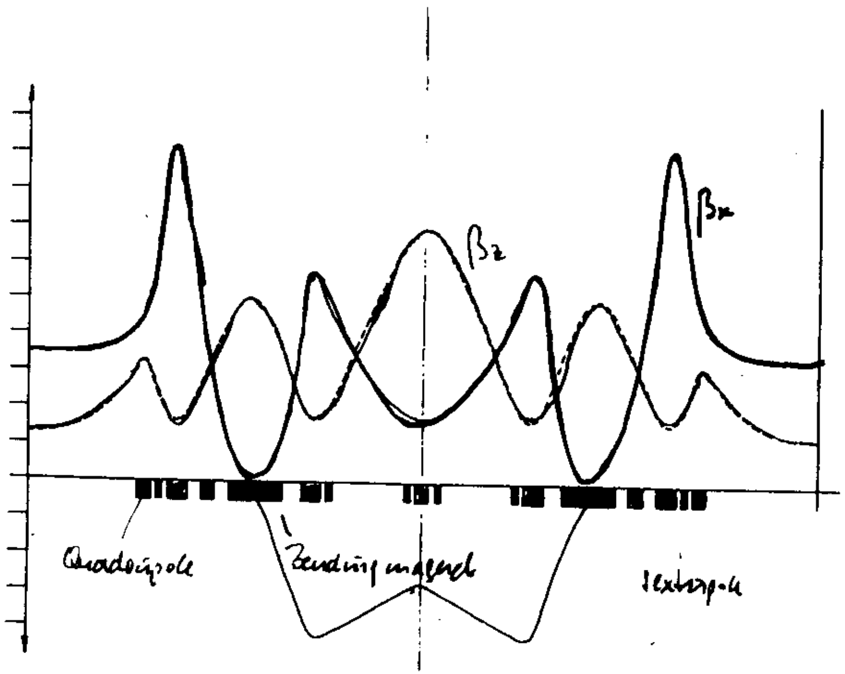
$$\gamma = \frac{1+\alpha^2}{\beta}$$

$$\boxed{\begin{aligned} \vec{X}_n &= \mathbb{M}^n \vec{X}_0 \\ &= \left[\mathbb{I}\sin n\mu + \mathbb{J}\cos n\mu\right]\vec{X}_0 \end{aligned}}$$

... initial vector $\vec{X}_0$ transformed over
n turns in a circular accelerator!

SINCROTRONE TRIESTE

– lattice functions



The Courant-Snyder parameter are the characteristica of a magnet lattice!

How to calculate the structure functions β and α?

Phase-plane ellipse in parameter form:

$s = s_0$: $\vec{X}(s) = \sqrt{\varepsilon}\,\left[\,\vec{X}_1(s_0)\cos\delta + \vec{X}_2(s_0)\sin\delta\,\right]$

$$0 < \delta \leq 2\pi$$

Ellipse defined by linear independent vectors

$$\vec{X}_1 = \begin{bmatrix} x_1 \\ x_1' \end{bmatrix} \quad \text{and} \quad \vec{X}_2 = \begin{bmatrix} x_2 \\ x_2' \end{bmatrix}$$

To get the phase plane area ε, they must be normalized to give

$$\tilde{\vec{X}}_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{X}_1 = 1$$

i.e. Wronski-determinante $= 1$

...e particle is moving through
... magnet structure, the form
... ellipse is changed

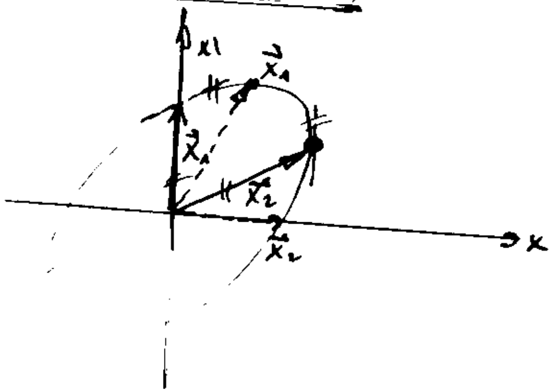
$$) = \sqrt{\varepsilon}\left[\vec{X}_1(s)\cos\phi + \vec{X}_2(s)\sin\phi\right]$$

$$\boxed{\vec{X}_i(s) = M(s|s_0)\,\vec{X}_i(s_0)}$$

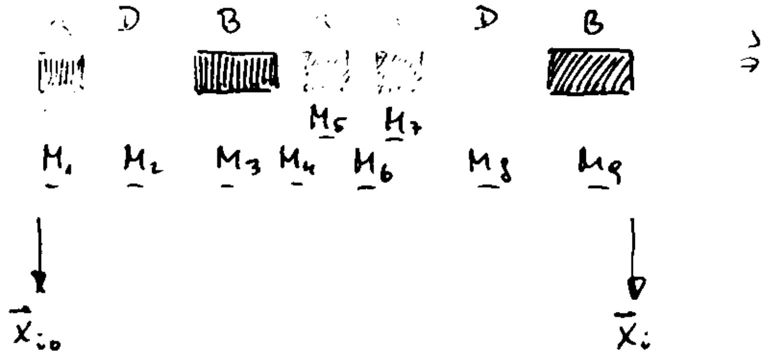
evaluated numerically
by transforming element
by element

$x_1^2 + x_2^2$

$-(x_1x_1' + x_2x_2')$



To calculate the [illegible] functions,
the conjugated vectors are transformed
through the structure →



$$\vec{X}_i = M_9 M_8 M_7 M_6 M_5 M_4 M_3 M_2 M_1 \vec{X}_{i0}$$

To get the transformation matrices the
Differential equation is solved piecewise

$$x'' + K(s)\,x = 0$$

$K(s) = k$ ... Quadrupoles

$\frac{1}{\rho^2}$ ... Bending magnets

0 ... Drift spaces

Transformation matrix $\underline{M}$ →

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$ DRIFT

$$\begin{bmatrix} \cos\frac{s}{\rho} & \rho\sin\frac{s}{\rho} \\ -\frac{1}{\rho}\sin\frac{s}{\rho} & \cos\frac{s}{\rho} \end{bmatrix}$$ BENDING MAGNET

$$\begin{bmatrix} \cos s\sqrt{|k|} & \frac{1}{\sqrt{|k|}}\sin s\sqrt{|k|} \\ -\sqrt{|k|}\sin s\sqrt{|k|} & \cos s\sqrt{|k|} \end{bmatrix}$$ QUADRUPOLE (FOC)

How to get the initial values of the conjugated vectors ?

Linear machines → $\vec{x}_1, \vec{x}_2$ is given by the initial beam conditio

Circular machine → Phase plane ellipse must transform into itse after 1 turn

Rewrite parameter form of phase plane ellipse:

$$\vec{x}(s_0) = \sqrt{\varepsilon}\,\{\vec{x}_1(s_0)\cos\delta + \vec{x}_2(s_0)\sin\delta\}$$

$$= \frac{\sqrt{\varepsilon}}{2}\,\{[x_1(s_0) - i\vec{x}_2(s_0)]e^{i\delta} + [x_1(s_0) + i\vec{x}_2(s_0)]e^{-i\delta}$$

We have shown before that the transformation of a vector over one closed turn changes only the phase, i.e. →

*) $e^{i\delta} = \cos\delta + i\sin\delta$

$$\underbrace{\underline{M}(s_0+L, s_0)\left[\vec{x}_1(s_0) - i\vec{x}_2(s_0)\right] = e^{-i2\pi Q}\left[x_1(s_0) - i x_2(s_0)\right]}$$

Eigenwert-equation of revolution matrix $\underline{M}$

$$\boxed{\underline{M}\,\vec{y} = \lambda\,\vec{y}}$$

The motion is stable, if $|\lambda| = \left|e^{-i2\pi Q}\right| = 1$,

i.e. Q = real !

ALTERNATIVE METHOD:

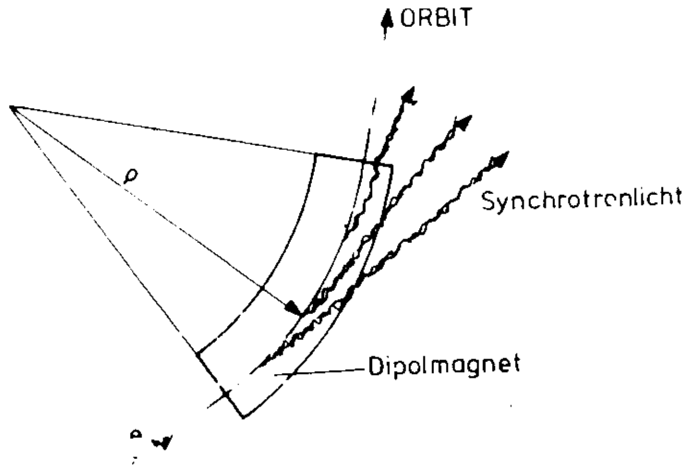
To find the initial conjugate vectors, solve the Eigenwert-equation of revolution matrix.

Transform the vectors through the structure to calculate the structure function (β)!

→ defines required magnet structure!

LONGITUDINAL MOTION:

Emission of Synchrotron Light →



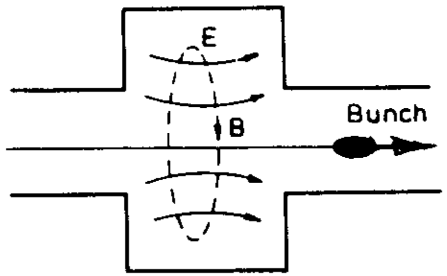
Radiated power:

$$\boxed{P_\gamma = \frac{2e^2c^3 r_e}{3\,(m_0c^2)^3} \cdot E^2 B^2}$$

With

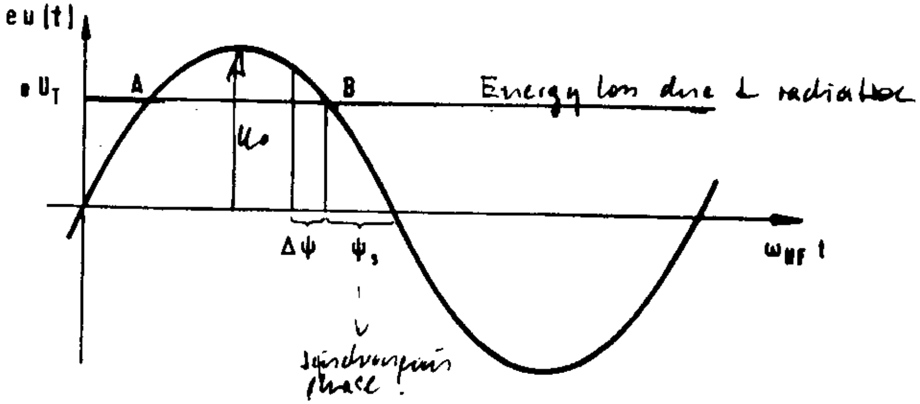
$$\frac{1}{\rho} = \frac{ecB}{E} \Rightarrow \boxed{P_\gamma = 88.5\,\frac{E^4\,[\mathrm{GeV}]}{\rho\,[\mathrm{m}]}}$$

Energy lost by synchrotron radiation must be compensated in 'accelerating sections' (cavities)



→ cavity with oscillating field
(350 MHz, 500 MHz, 1-3 GHz)

Accelerating voltage in the cavities →



eU_T ... necessary to compensate the radiation losses

Energy gain by accelerating voltage:

$$E = e U_0 \sin(\psi_s + \Delta\psi) \qquad \Delta\psi \text{ .. phase deviation}$$

$$E_0 = e U_0 \sin \psi_s \qquad \text{exact phase!}$$

$$\frac{E - E_0}{T_0} \approx \frac{d\Delta E}{dt} = e U_0 f_0 \left[\sin(\psi_s + \Delta\psi) - \sin\psi_s\right]$$

Period of synchrotron oscillation ≫ revolution period!

Phase deviation for particles with wrong revolution time, i.e.

$$T = T_0 + \Delta T$$

$\Rightarrow$

$$\Delta\varphi = -\omega_{RF} \cdot \Delta T$$

$$\frac{d\Delta\varphi}{dt} = -\omega_{RF} \frac{\Delta T}{T} = -\omega_{RF} \frac{\Delta L}{L}$$

$$\downarrow$$

$$\frac{\Delta L}{L} = \alpha_c \frac{\Delta E}{E}$$

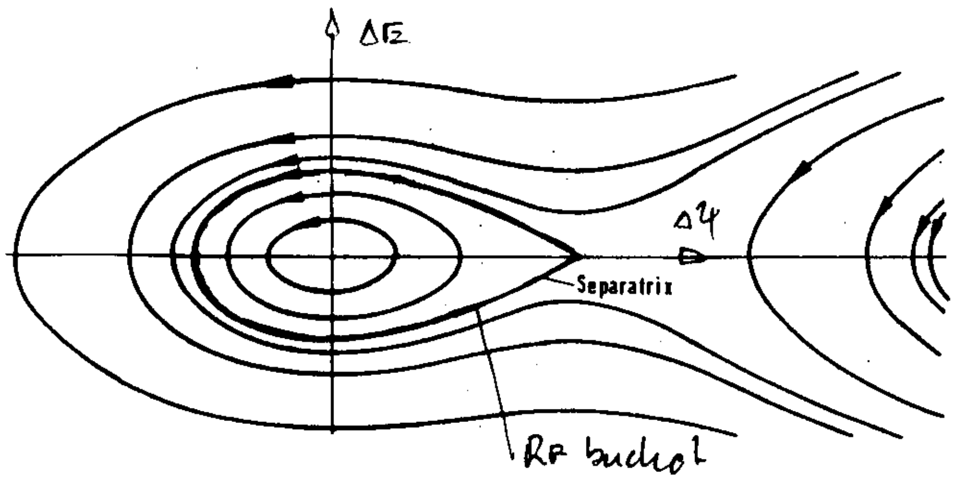
α_c .. momentum compaction

$$\boxed{\frac{d\Delta\varphi}{dt} = -\omega_{RF}\, \alpha_c \frac{\Delta E}{E}}$$

ΔE substituted from previous equation gives $\rightarrow$

$$\boxed{\frac{d^2\Delta\varphi}{dt^2} + \omega_0 \omega_{RF} \frac{\alpha_c e U_0}{2\pi E_0}\left[\sin(\varphi_s + \Delta\varphi) - \sin\varphi_s\right] = 0}$$

Phase-plane of longitudinal motion $\rightarrow$



For small amplitudes $\Delta\varphi$ $\rightarrow$

Harmonic oscillation

$$\sin(\varphi_s + \Delta\varphi) - \sin\varphi_s \approx (\cos\varphi_s)\,\Delta\varphi$$

$$\boxed{\frac{d^2\Delta\varphi}{dt^2} + \Omega^2 \Delta\varphi = 0}$$

$$\Omega^2 = \omega_0 \omega_{RF} \frac{\alpha_c e U_0 \cos\varphi_s}{2\pi E_0}$$

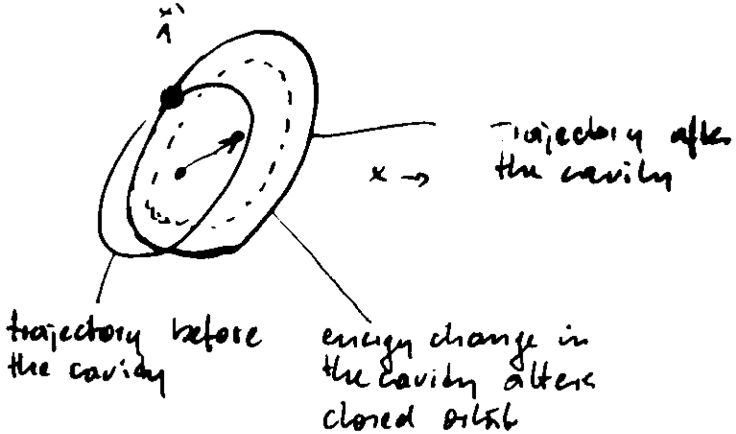
Coupling between transverse (betatron) motion and longitudinal (synchrotron) motion:

Sources: uncompensated chromaticity
fields in accelerating structures
beam beam interaction (crossing angle)
dispersion in the cavity
⋮

EXAMPLE: Dispersion in the cavity
i.e. orbit displacement in the cavity varies with the momentum of the particle.

$X = $ [illegible]

Transverse phase space:



⇒ OSCILLATION AMPLITUDE IS INCREASED!

Must be numerically evaluated:

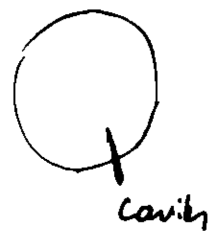
CAVITY-transformation:

$$E_{n+1} = E_n + eU\,[\sin(\psi_s + \psi_n) - \sin\psi_s]$$

$$\psi_{n+1} = \psi_n$$

$$x_{n+1} = x_n - D\,\frac{E_{n+1} - E_n}{E_0}$$

$$x'_{n+1} = x'_n - D' \quad \text{—}$$



MAGNET STRUCTURE-transformation:

$$E_{n+2} = E_{n+1}$$

$$\psi_{n+2} = \psi_{n+1} - 2\pi q\left[\alpha_c\,\frac{E_{n+1} - E_0}{E_0} + \underbrace{f(x_{n+1}, x'_{n+1}, D)}_{=0 \text{ for } D=0}\right]$$

$$x_{n+2} = x_{n+1}\cos\mu + x'_{n+1}\,\beta\sin\mu$$

$$x'_{n+2} = -\frac{x_{n+1}}{\beta}\sin\mu + x'_{n+1}\cos\mu$$

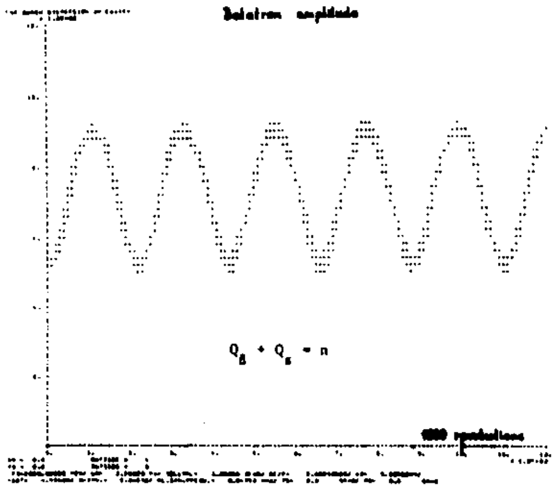


Fig 1

$$A_\beta^2 + q A_{syn}^2 = const$$

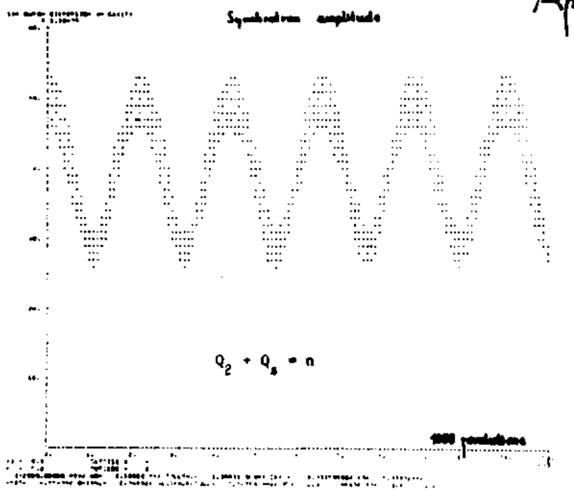


Fig 2

Exchange of energy only!

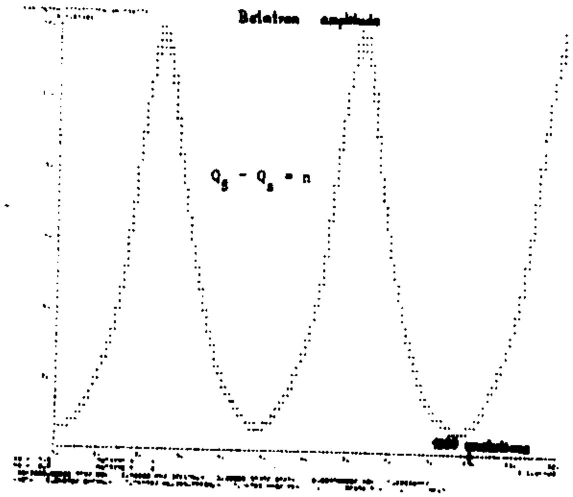


Fig 3

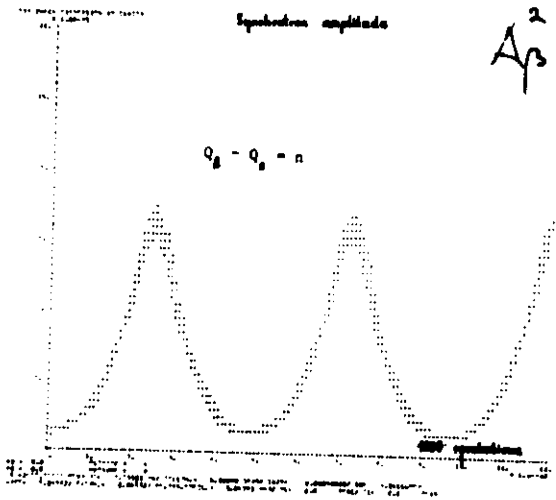


Fig 4

$$A_\beta^2 - q A_s^2 = const$$

Both amplitudes increase!

$Q_\beta < n$

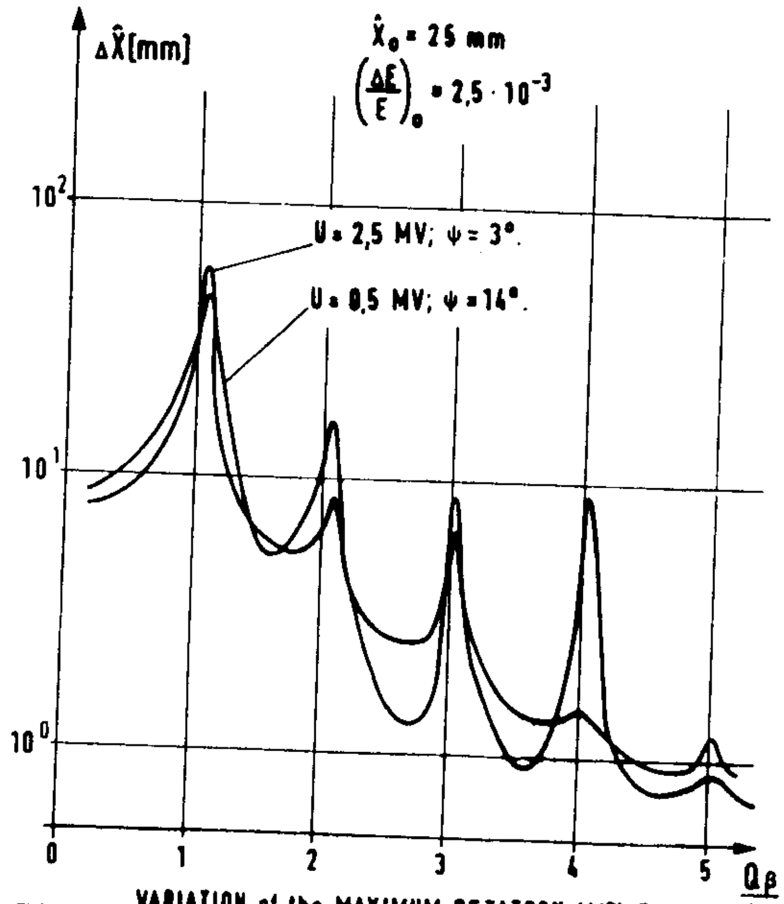


Fig 6: VARIATION of the MAXIMUM BETATRON AMPLITUDE with BETATRON FREQUENCY.

NONLINEAR MOTION (transverse)

→ due to sextupoles
or field errors

Equations of motion:

$$\boxed{\frac{d^2x}{ds^2} + K(s)\,x = \frac{e}{p_0} \sum_{n=0} b_n(s)\, x^n}$$

$$b_n = \frac{1}{n!} \frac{\partial^n B}{\partial x^n}$$

b_0 .. dipole
b_1 .. quadrupole
b_2 .. sextupole
⋮

In superconducting magnets the still relevant maximum orders are 24-poles (usually!)

Transformation to normalized coordinates (as before in the linear case) $\rightarrow$ Floquet transform.

$$u = \frac{x}{\sqrt{\beta}} \quad , \quad \phi = \int \frac{ds}{\beta}$$

$\Rightarrow$

$$\boxed{\frac{d^2u}{d\phi^2} + u = \frac{e}{p} \sum_n b_n(\phi) \beta^{\frac{n+3}{2}}(\phi)\, u^n}$$

Periodic function $\rightarrow$

$$\frac{e}{p} b_n(\phi) \beta^{\frac{n+3}{2}}(\phi) = \sum_k f_{nk} \cos 2\pi k \cdot \frac{\phi}{2\pi Q}$$

$\mu = 2\pi Q$

represented by Fourier series with period $\underline{2\pi Q}$

(Assumption here: even function with zero mean)

Fourier coefficient:

$$f_{nk} = \frac{1}{\pi Q} \int_0^{2\pi Q} \left[\frac{e}{p} b_n(\phi) \beta^{\frac{n+3}{2}}(\phi) \right] \cos \frac{k}{Q}\phi \, d\phi$$

For small distortions we can substitute the homogeneous solution for u at the right hand side of the differential equation:

$$u = u_0 \cos\phi + \underbrace{u_0 \sin\phi}_{=0}$$

$$\frac{d^2u}{d\phi^2} + u = \sum_n u^n \sum_k f_{nk} \cos \frac{k}{Q}\phi$$

$\downarrow$

$$u^n = u_0^n (\cos\phi)^n = \frac{u_0^n}{2^{n-1}} \sum_{p=0}^{\frac{n-1}{2}} \binom{n}{p} \cos(n-2p)\phi$$

$$\boxed{\frac{d^2u}{d\phi^2} + u = \sum_n \frac{U_0^n}{2^{n-1}} \sum_{p=0}^{\frac{n-2}{2}} \binom{n}{p} \cos(n-2p)\phi \sum_k f_{nk} \cos\frac{k}{Q}\phi}$$

take one term only →

$$\frac{d^2u}{d\phi^2} + u = \frac{U_0^n}{2^{n-1}} f_{nk} \underbrace{\cos n\phi \cos\frac{k}{Q}\phi}_{\downarrow} \qquad (p=0)$$

$$\frac{1}{2}\left[\underline{\cos\left(n-\frac{k}{Q}\right)\phi} + \cos\left(n+\frac{k}{Q}\right)\phi\right]$$

only →

$$\boxed{\frac{d^2u}{d\phi^2} + u = A\cos\left(\frac{k}{Q}-n\right)\phi} \qquad A = \frac{u_0^n}{2^n} f_{nk}$$

Solution :

$$u = u_H + u_P \qquad u_H = u_0\cos\phi$$

$$u_P = \underbrace{\frac{A}{1-\left(\frac{k}{Q}-n\right)^2}}_{\text{Resonance term !}} \cos\left(\frac{k}{Q}-n\right) + \ldots$$

i.e. betatron amplitude increases, if

$$\left(\frac{k}{Q}-n\right) = 1 \qquad \text{or}$$

$$\boxed{(n+1)\,Q = k}$$

Resonance condition for 1-dimensional resonance

e.g. $n = 2$ → Sextipole → $\left(\frac{1}{2}\frac{\partial^2 B}{\partial x^2}\right)$

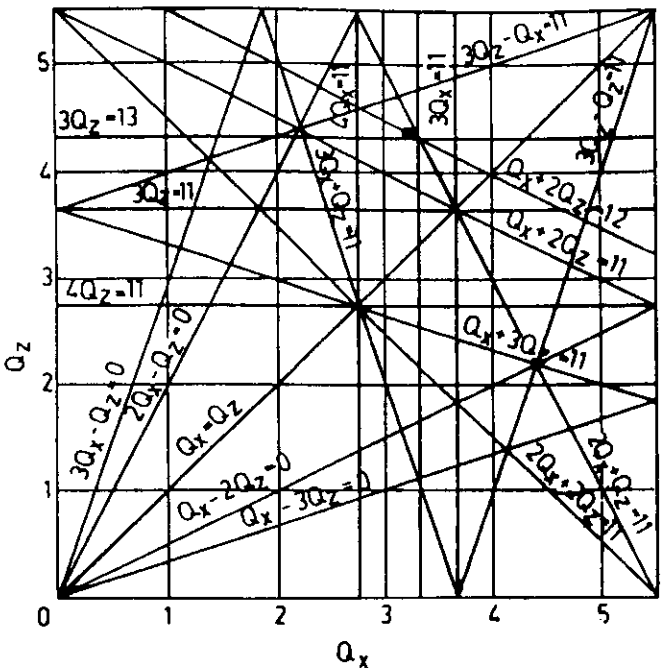
$\underline{3Q = k}$

In the general 2-dimensional case we have the resonances →

$$\left.\begin{array}{l} 3Q_x \\ 2Q_x \pm Q_z \\ Q_x \pm 2Q_z \\ 3Q_z \end{array}\right\} = \underline{\text{integer!}} \qquad \left| n_x Q_x + n_z Q_z = p \right|$$

Resonances in the presence of sextipole fields!

$N = |n_x| + |n_z|$ .. order of resonance



Resonance lines in 2-dimensional tune diagram
(up to 4th order)

$n_x Q_x + n_z Q_z = p$ $\qquad N = |n_x| + |n_z|$ .. order of resonance!

Maximum stable betatron amplitude can be calculated analytical for one isolated resonance.

For a realistic pattern of magnet field errors one has to simulate the motion with a computer →

TRACKING

Tracking codes:

PATRICIA
MAD
RACETRACK
DIAMAT
TRANSPORT
⋮

KICK CODES !

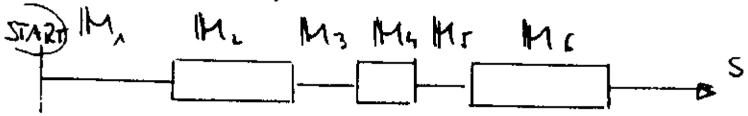
Principle structure of these codes:

INPUT:

Magnet structure
Magnet strengths
Field errors
Tracking parameters:
- # of turns
- # of particles
- initial amplitude

↓

Find Revolution Matrix:
(linear optics)



$$\boxed{M = \prod_{n=1}^{N} M_n}$$

M_n ... single element transformation (6×6)

↓

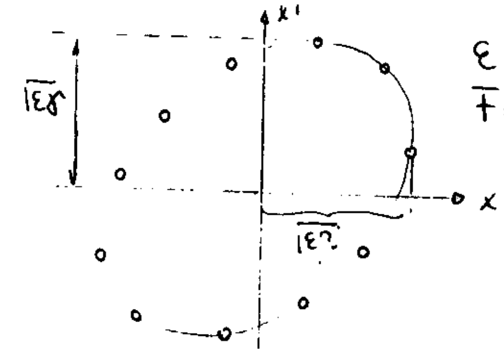
Calculate optical parameters at the starting point,
by solving the Eigenwert-equation of the revolution matrix

↓

Initial coordinates:

$\gamma = \frac{1+\alpha^2}{\beta}$

$\alpha = -\frac{1}{2}\frac{d\beta}{ds}$



$\varepsilon = \text{const}$
$F = \varepsilon\pi$

↓

Tracking of particle ensemble through the structure:

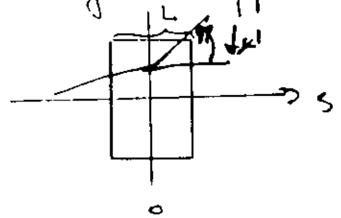
① Linear element: $\vec{x}_f = M\,\vec{x}_i$

② Nonlinear element:

$$\frac{d^2x}{ds^2} = \frac{e}{p_0} \sum_{n=0} b_n(s_i)\, x_i^n$$

thin magnet approximation

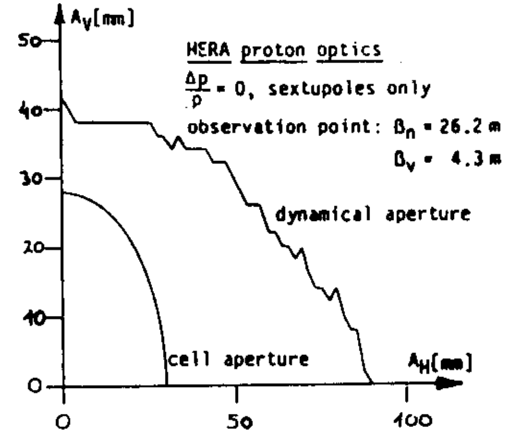
→



Kick:

$$\frac{dx}{ds} = \Delta x' = \sum_n (b_n L_n)\, x_i^n$$

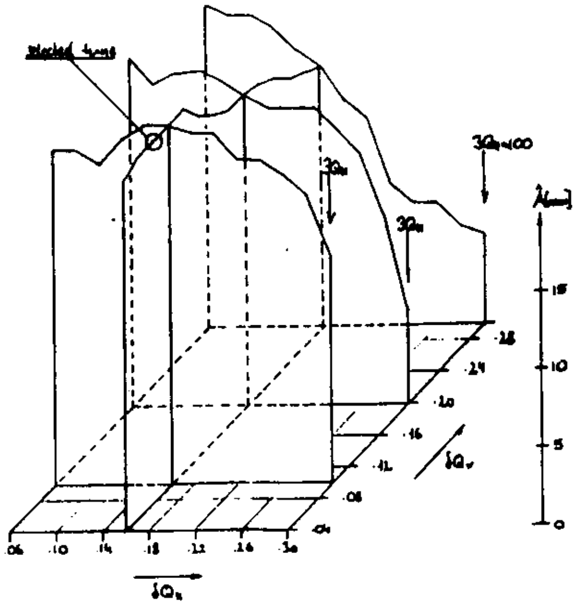
Find maximum stable betatron amplitude by varying the initial value

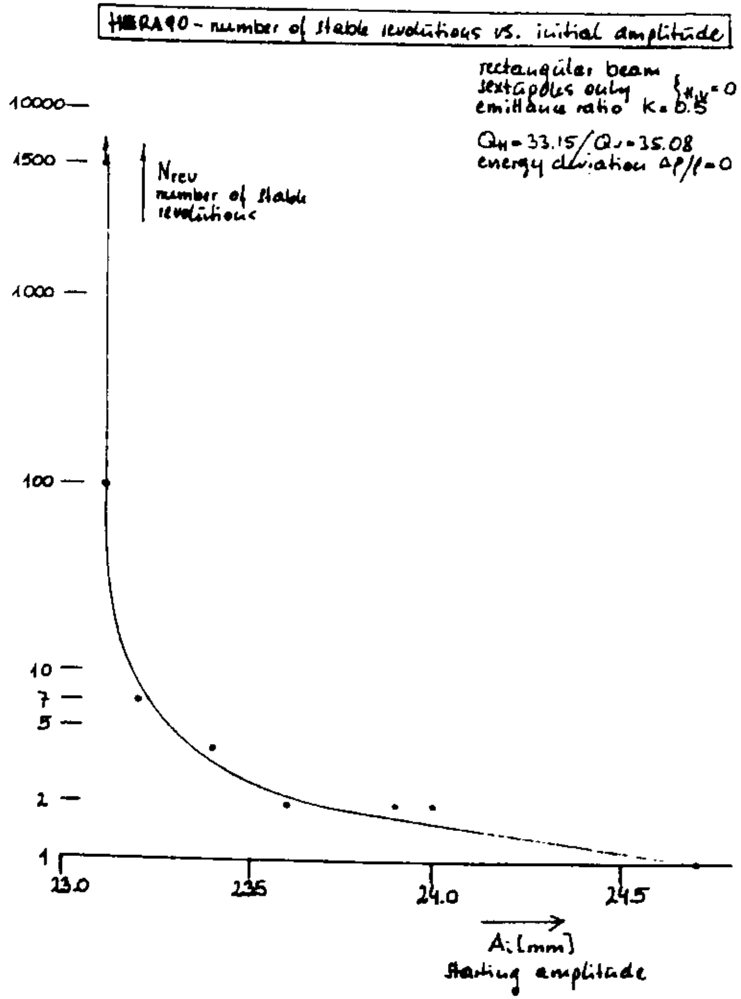


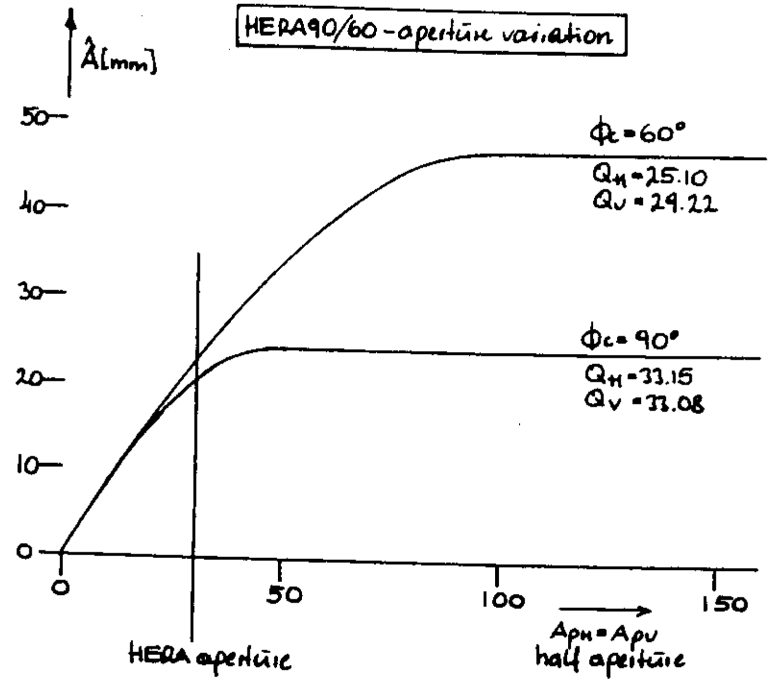
HERA90 - tune variation

Maximum stable initial amplitude vs. tune

rectangular beam
sextupoles only $\xi_{H,V} = 0$
emittance ratio $K = 0.5$
energy deviation $\Delta p/p = 0$
$Q_H = 33 + \delta Q_H$ / $Q_V = 35 + \delta Q_V$
$N_{iter} = 100$





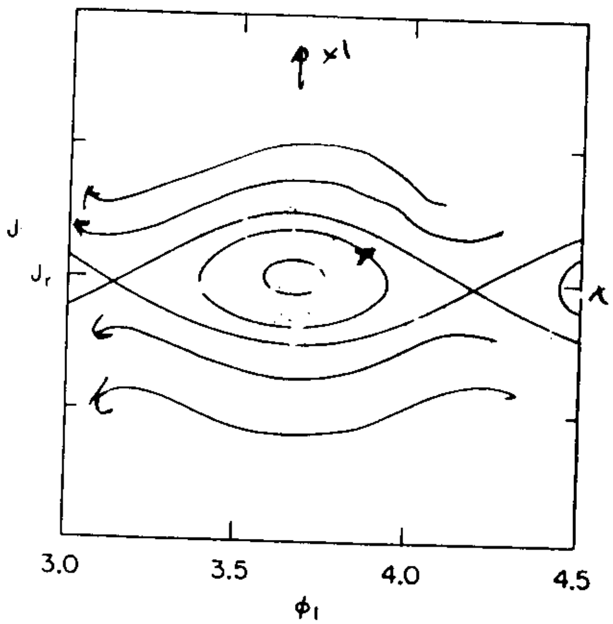


rectangular beam
sextupoles only
emittance ratio K=1.0
no energy deviation

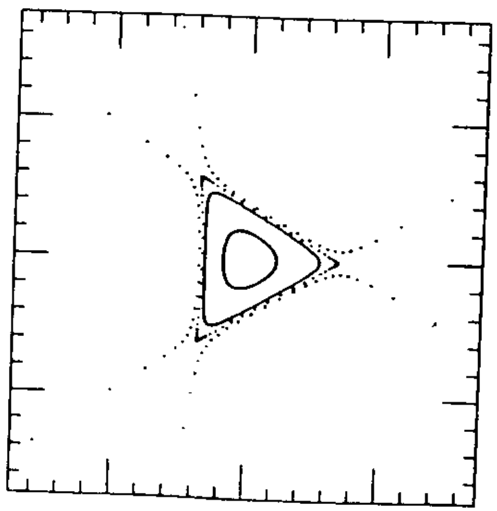
$N_{rev}=30$!

Phase space motion:

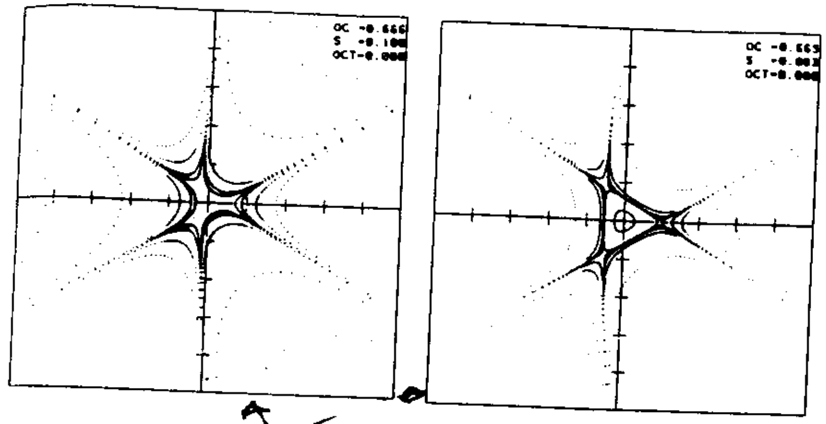
Pendulum



Motion near a third integer resonance in a circular accelerator:

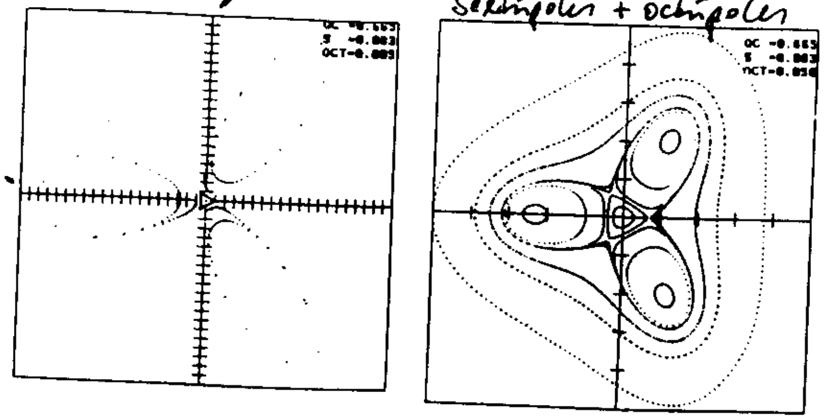


Unstable motion due to sextupole nonlinearities, stabilized by octupoles →



sextupole resonances for various tunes

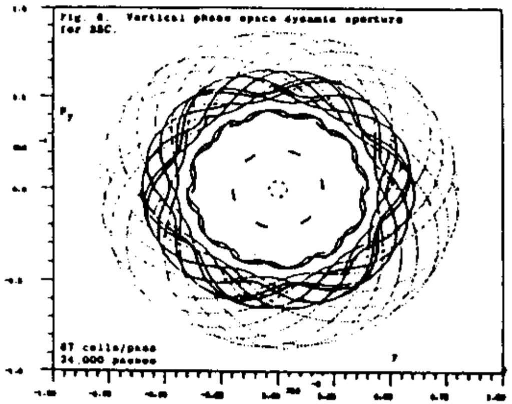
sextupoles + octupoles !!



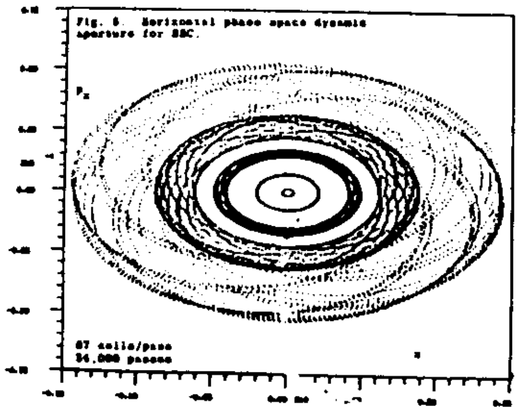
(Landau damping due to octupoles!)

Tune variation with amplitude → particle falls out of the resonance!

Prom: MARYLI

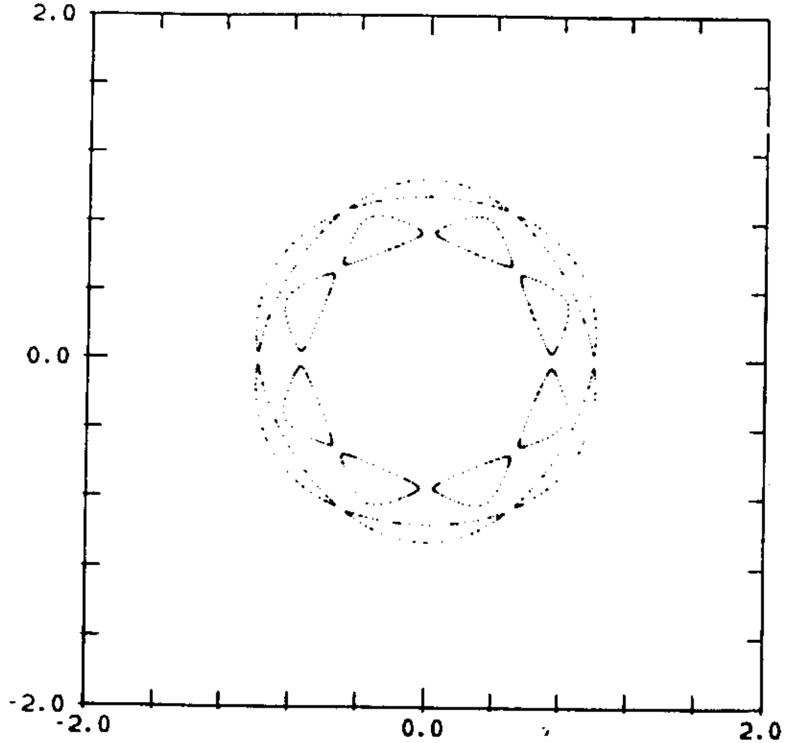


Vertical



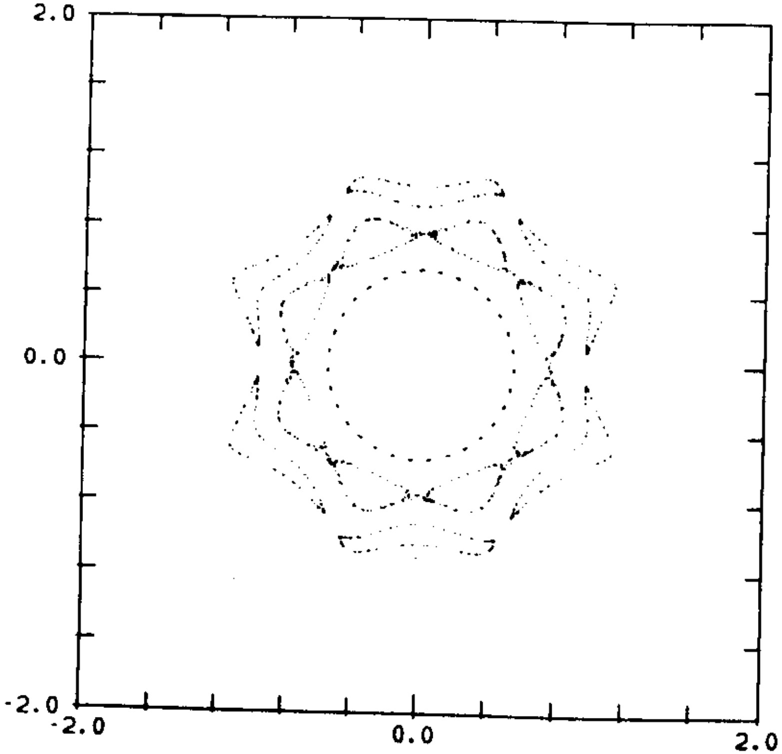
Horizontal

Isolated resonances plotted together →

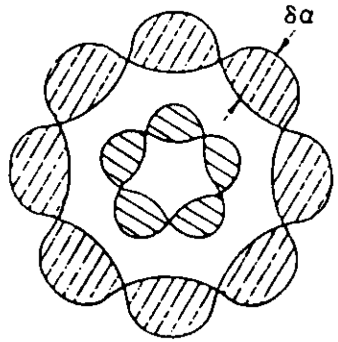


Situation becomes complicated if the particle motion is influenced by more than one resonance!

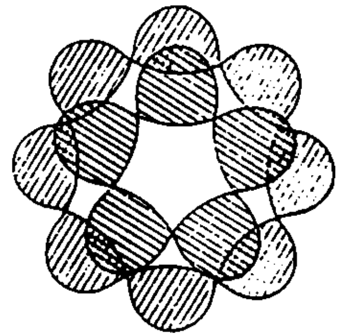
Actual phase space for these resonances →



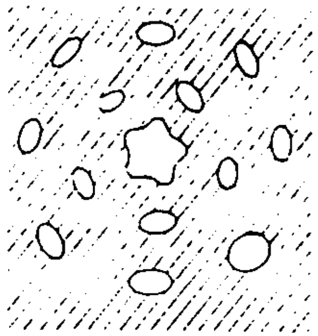
If the resonances start to overlap, the particle can easily become <u>chaotic</u> →



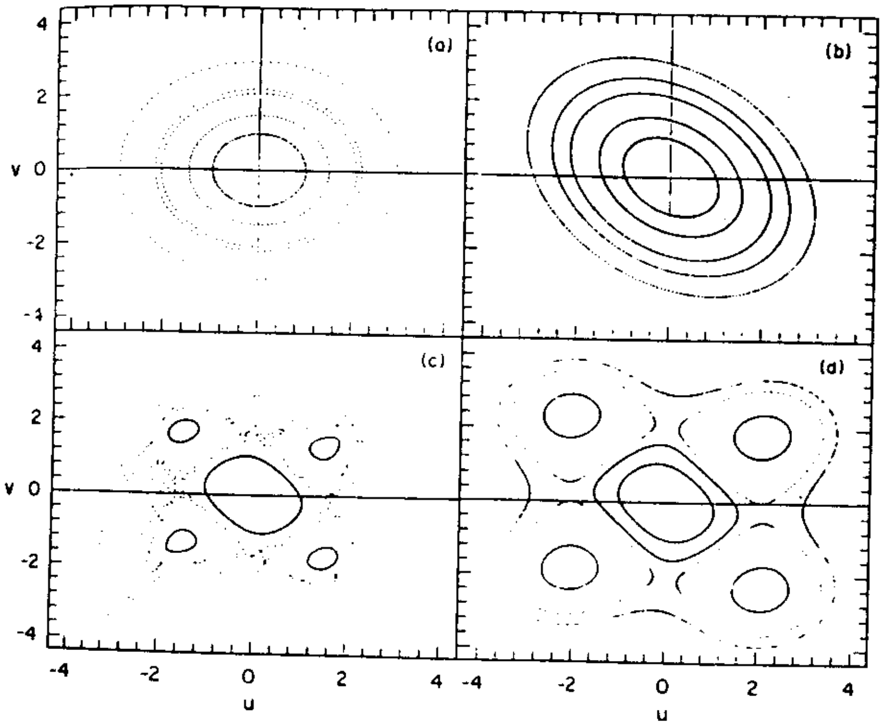
Single Resonances



Overlapping Resonances



Actual phase space

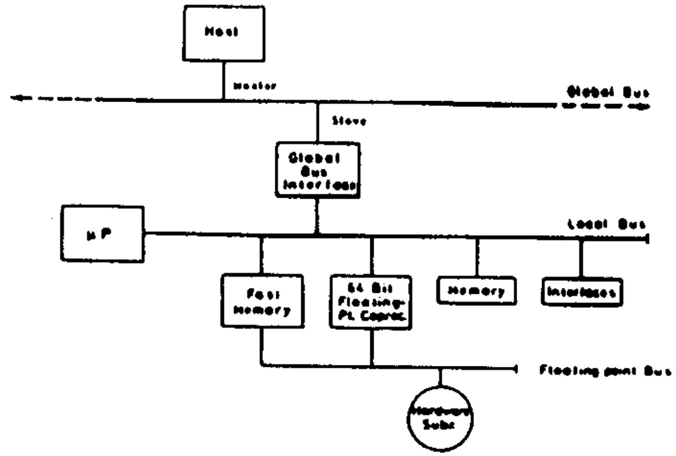


Regular motion → concentric curves or islands

Chaotic motion → 'randomly' distributed orbits

To perform long time tracking, a dedicated computer has been developed → parallel microprocessors (one for each particle

Conceptual design →



Host computer – programs compiled and linked then can directly run on the μP

μP – connected with global bus interface to host, local bus to other interface

μP: MC 68000
double precision floating point unit controller
memory with 128 kBytes

Long-term stability

Even though particles may be chaotic, they may nevertheless survive for the limited number of revolutions normally accessible

To distinguish regular from chaotic motion use concept of characteristic Lyapunov exponent →

$$\boxed{\chi = \lim_{\substack{d_0 \to 0 \\ N \to \infty}} \left(\ln \left| \frac{d_N}{d_0} \right| \right)}$$

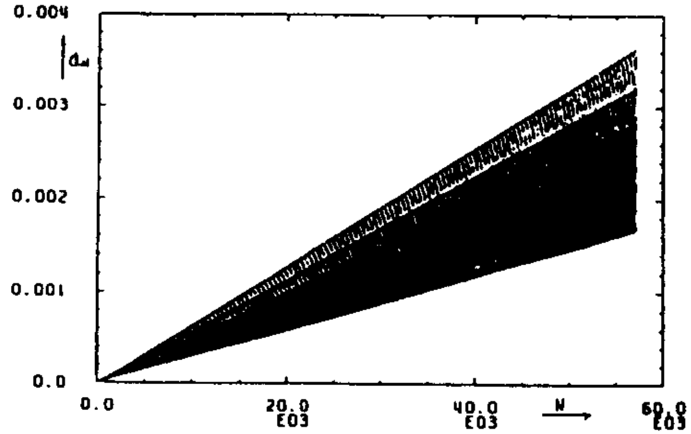
d.. separation in phase space
N.. revolution number

For chaotic motion the distance in phase space of two neighbouring trajectories increases exponentially with the Lyapunov exponent.

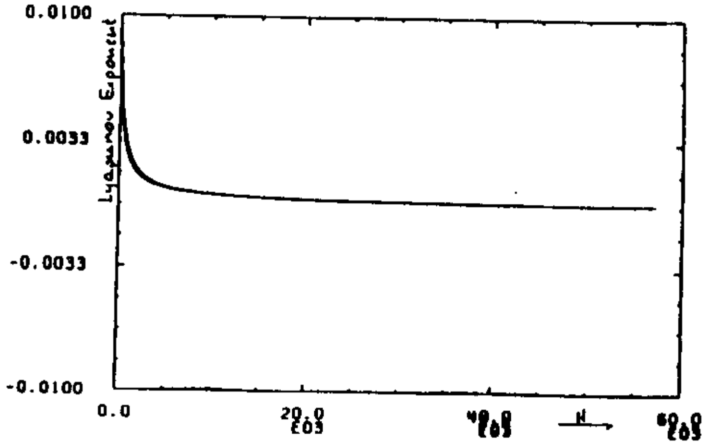
→ BREAK OF STRONG CAUSALITY !

Regular motion

→ distance in phase space increases linearly
→ Lyapunov exponent = 0

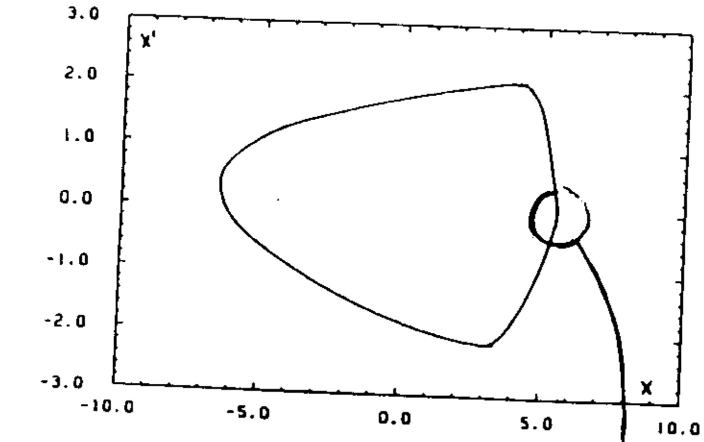


Trajectory separation for regular motion.

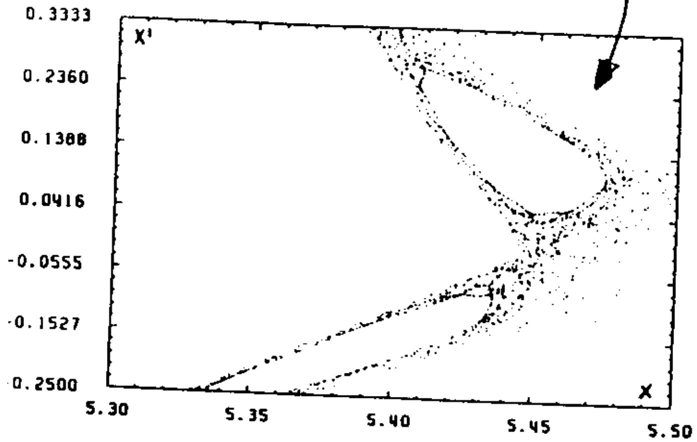


Lyapunov exponent for regular motion.

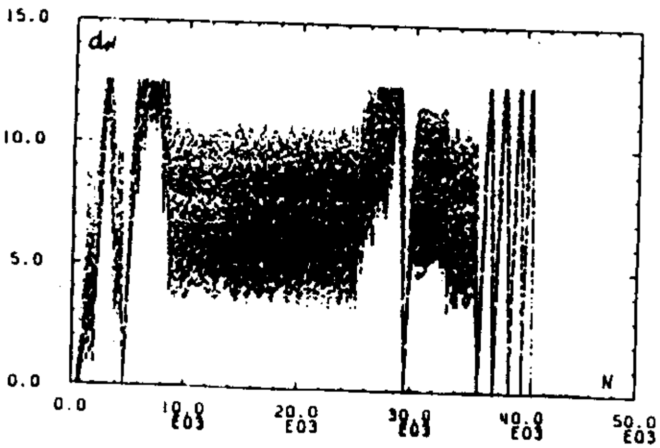
Regular motion



Trajectory close to the dynamic limit

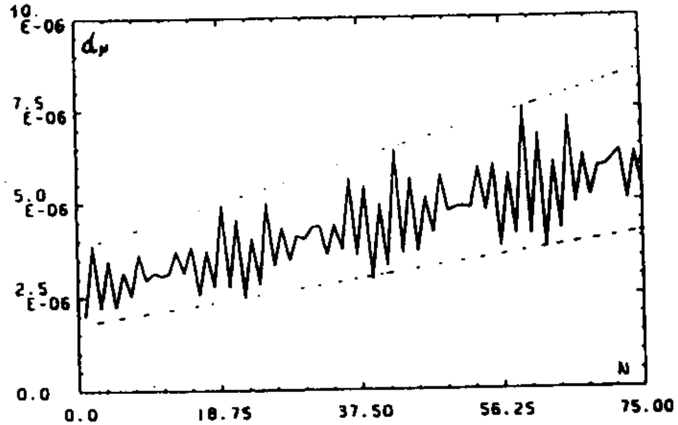


Chaos

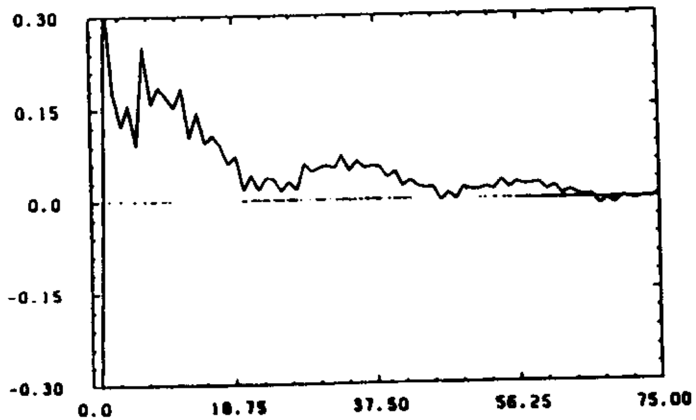


Distance in phase space!

~~Lyapunov-Exponent~~

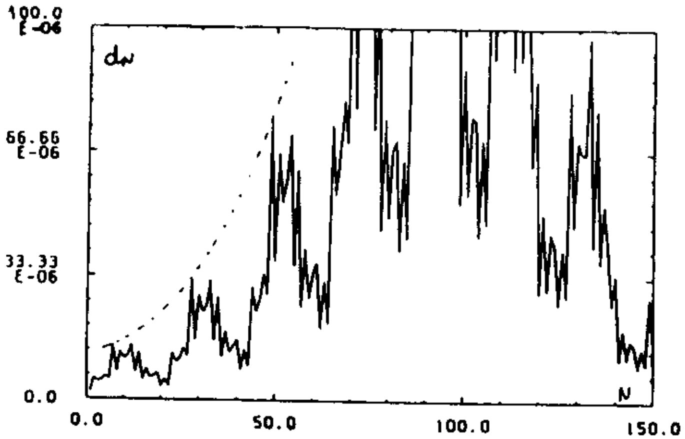


Distance of trajectories in phase space for regular motion

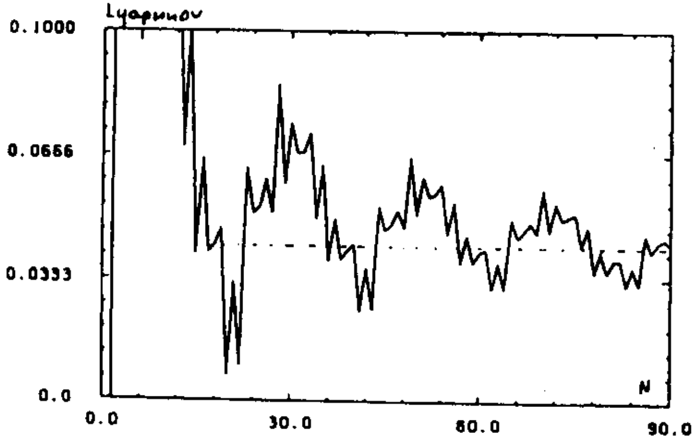


Lyapunov exponent for regular motion.

Chaotic motion



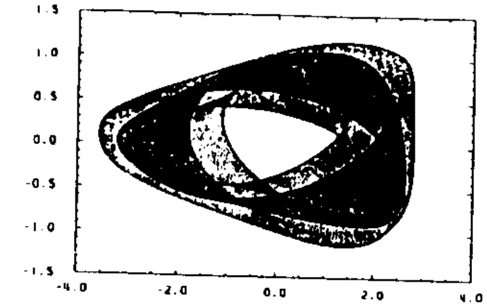
Distance of trajectories in phase for chaotic motion



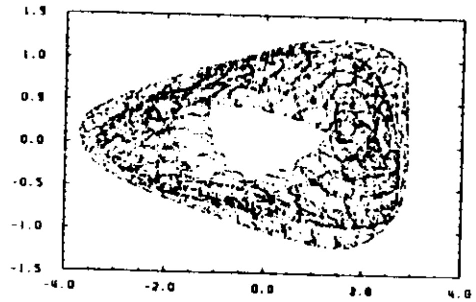
Lyapunov exponent to chaotic motion.

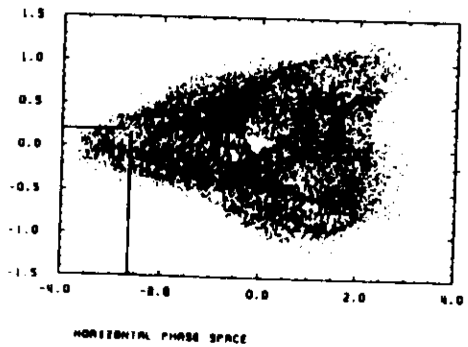
4-dimensional phase space

Projection onto the horizontal plane



increasing nonlinearities





CHAOS

For systems with $N > 2$, degrees of freedom the stochastic layers are connected together and form the ARNOLD WEB

For an initial condition within the WEB, the chaotic motion on the WEB can carry the particle to large amplitudes

$\Rightarrow$ ARNOLD DIFFUSION

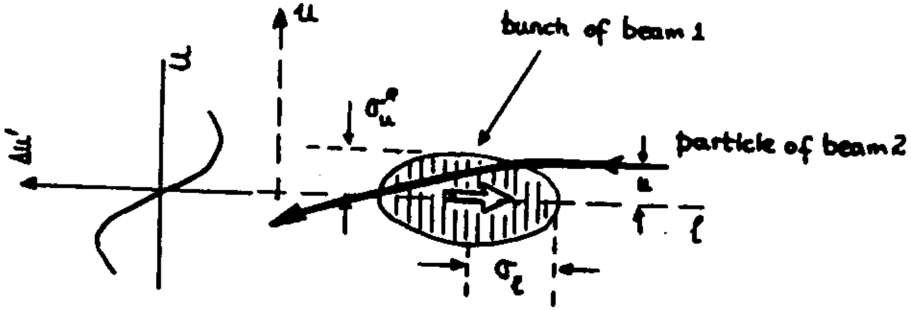
(Some) $\boxed{\text{Collective Effects}}$

So far $\rightarrow$ only single particle dynamics has been considered

BUT! Particle ensemble (bunch) as a whole acts via the environment (vacuum chamber) or directly on particles of the same bunch or particles in other bunches in the ring.

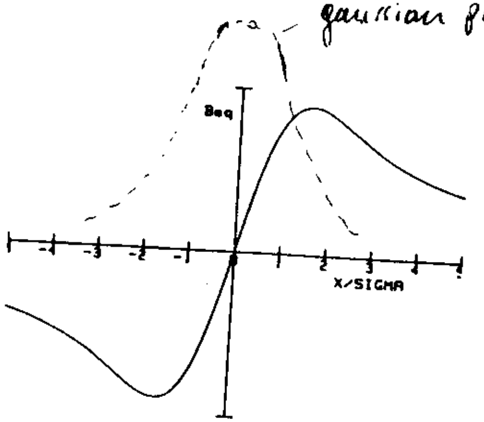
FORCE $\sim$ Intensity, i.e. number of particles in the bunch

① Beam-beam effect in colliders



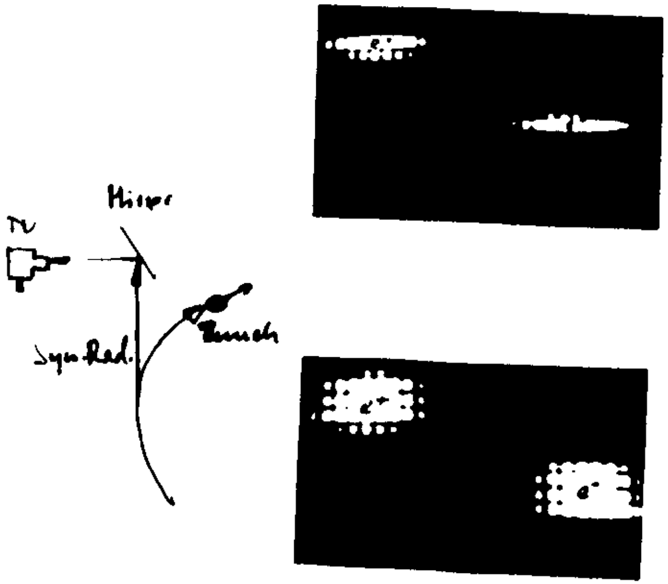
Particles experience a transverse deflecting force due to the charge of the colliding bunch →

gaussian particle distribution!



Beam-beam force

For high intensities (number of particles per bunch) the beams blow up transversally →



beams separated

colliding beams

i.e. the Luminosity is decreased!

Events/sec $\frac{dN}{dt} = L \cdot \sigma$

Luminosity cross section

Low beta optics for high luminosity

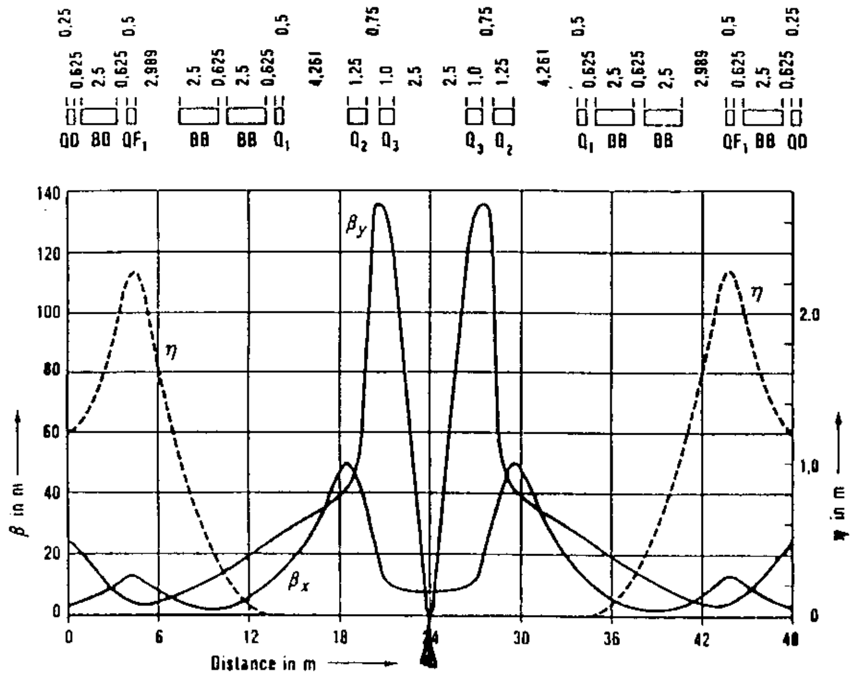


Fig. 3: Betatron functions β_x, β_y (left scale) and momentum function η

interaction point

Luminosity is inversely proportional to beam cross section →

$$\boxed{L = \underbrace{\frac{1}{4\pi e^2 f_0}}_{\text{const.}} \cdot \frac{I^+ I^-}{\sigma_x \sigma_z}}$$

needed:
High current
small beam sizes

Deflection kick due to the beam-beam force:

In e^--rings the unperturbed charge distribution is gaussian →

$$\rho(x,z,s) = \frac{N_B \cdot e}{2\pi^{3/2}\sigma_x\sigma_z\sigma_s} \exp\left\{-\frac{x^2}{2\sigma_x^2}-\frac{z^2}{2\sigma_z^2}-\frac{s^2}{2\sigma_s^2}\right\}$$

$\sigma_x, \sigma_z, \sigma_s$ -- rms beam dimensions
N_B -- particles per bunch

Electric potential:

$$V(x,z,s) = \frac{Ne}{4\pi^{\frac{3}{2}}\,\varepsilon_0} \int_0^\infty \frac{\exp\left\{-\frac{x^2}{(2a^2+q)} - \frac{z^2}{(2b^2+q)} - \frac{s^2}{(2c^2+q)}\right\} dq}{\left\{(2a^2+q)(2b^2+q)(2c^2+q)\right\}^{\frac{1}{2}}}$$

Integrated transverse deflection →

$$\Delta\left(\frac{dz}{ds}\right) = \frac{e}{\gamma m_0 c^2} \int_{-\infty}^{+\infty} E_z \, ds = -\frac{e}{\gamma m_0 c^2} \int_{-\infty}^{+\infty} \frac{\partial V}{\partial z} \, ds$$

→

$$\Delta z' = \frac{2Nr_e}{\gamma} \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\exp\left\{-\frac{x^2}{(2a^2+q)} - \frac{z^2}{(2b^2+q)} - \frac{s^2}{(2c^2+q)}\right\}}{(2a^2+q)^{\frac{1}{2}}\,(2b^2+q)^{\frac{3}{2}}\,(2c^2+q)^{\frac{1}{2}}} \, dq \, ds$$

⇓ (Eriksson, Bassetti)

$$\boxed{\Delta z' = \frac{Nr_e}{\gamma} \sqrt{\frac{2\pi}{a^2-b^2}}\, R\left[w\left(\frac{x+iz}{\sqrt{2(a^2-b^2)}}\right) - \exp\left\{\frac{-x^2}{2a^2} - \frac{z^2}{2b^2}\right\} \cdot w\left(\frac{x\frac{b}{a} + iz\frac{a}{b}}{\sqrt{2(a^2-b^2)}}\right)\right]}$$

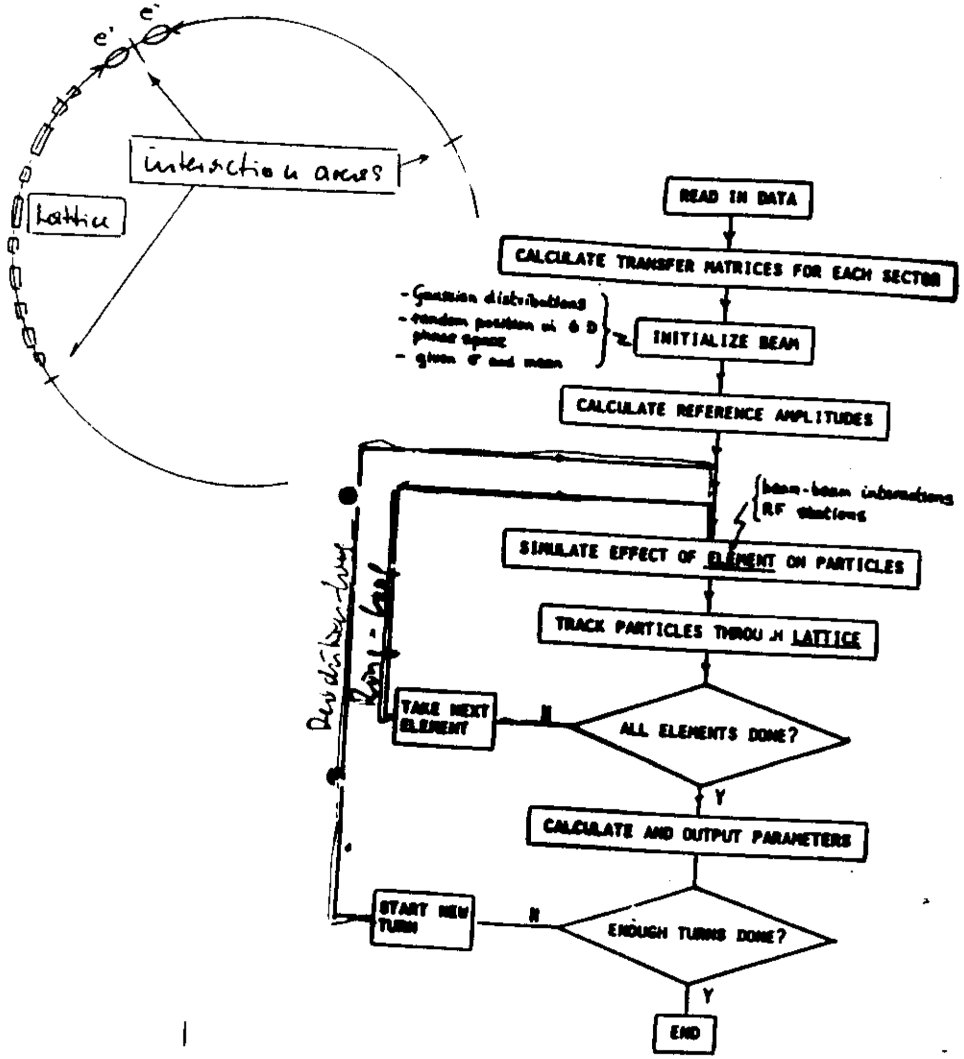
$a = \sigma_x$

$b = \sigma_z$

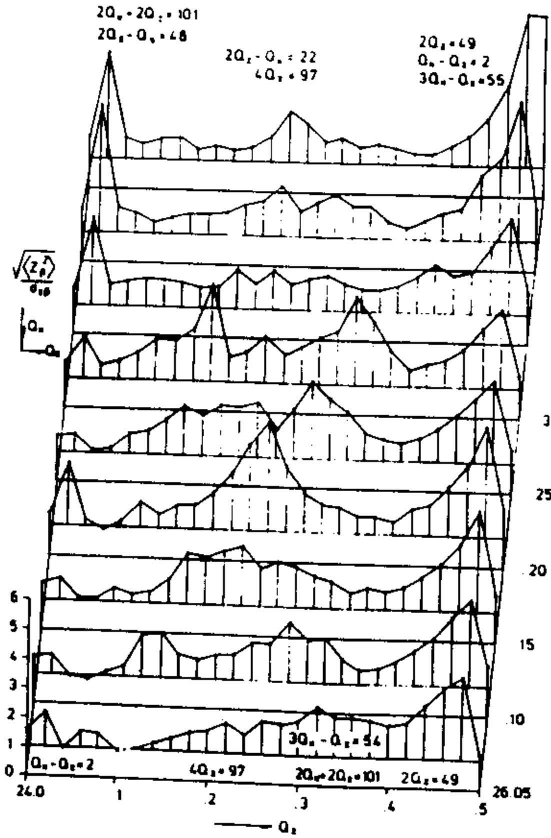
w .. complex error function !

Beam-beam effect must be simulated, usually several hundred of particles are tracked over several thousands of turns.

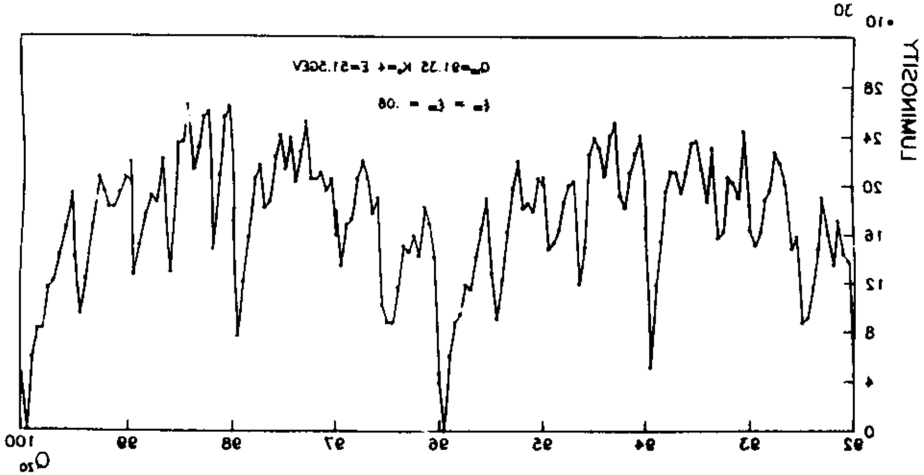
Principle of simulation:



Purpose of simulations →
prediction of suitable working parameter for minimum beam-beam blow up:

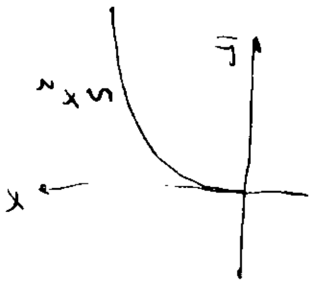


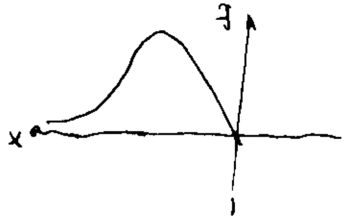
Luminosity vs. vertical drive



Beam-beam force: $\overline{F} \to 0$ for $x \to \infty$

Magnet nonlinearities: $\overline{F} \to \infty$ for $x \to \infty$





Beam-beam resonances:

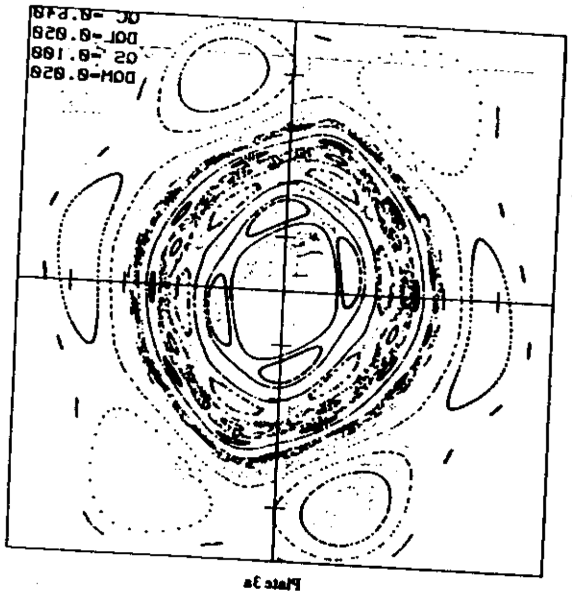


Plate 3a

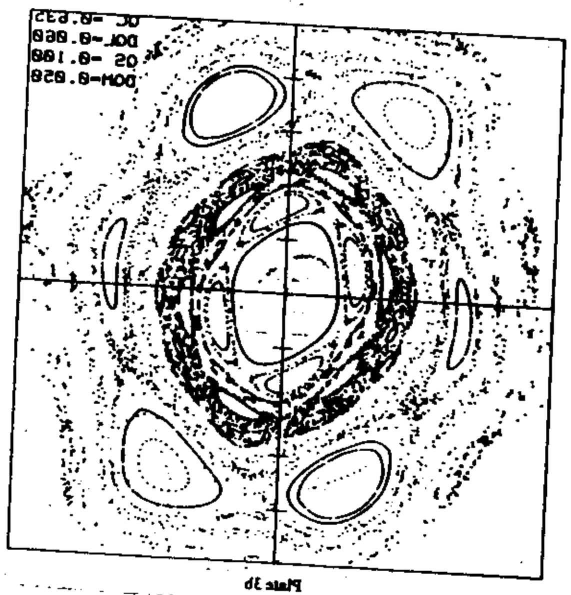
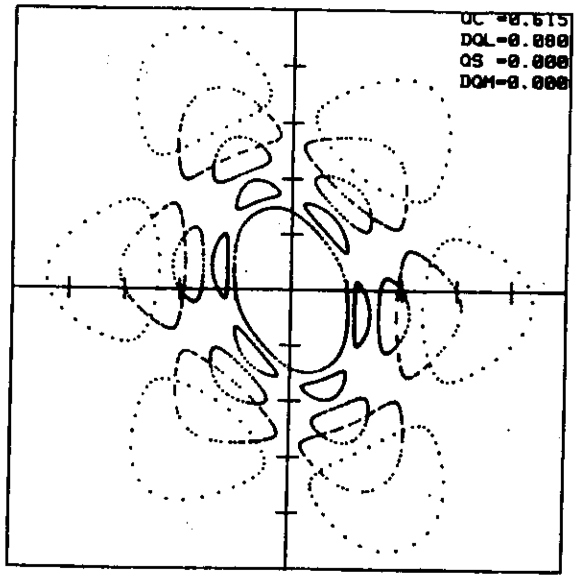


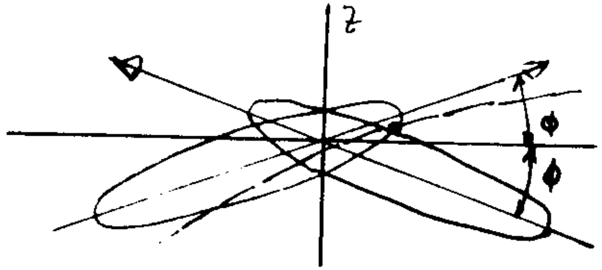
Plate 3b

…atic layers
…nnected via
Arnold web.

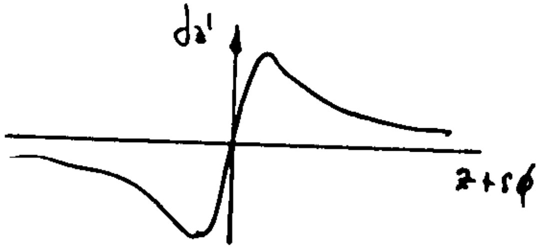
Phase space topology near a 6th order beam-beam resonance as the tune is changed:



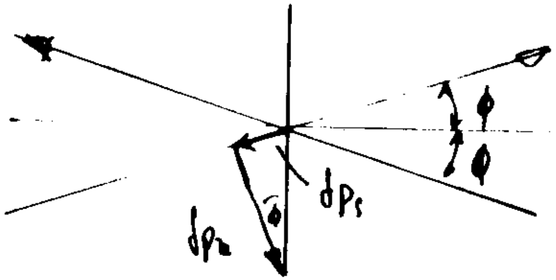
Beam-beam interaction at crossing angle



Deflection : $\boxed{dz' = f(z+s\phi)}$



Energy change:



$$dp_s = \phi\, dp_z$$

$$\boxed{\frac{dE}{E} \approx \frac{dp}{p} = \phi \frac{dp_z}{p_z} = \phi\, dz' = \phi\, f(z+s\phi)}$$

COMPUTER SIMULATION!

→ [illegible] streng [illegible] Radiation d + [illegible] Rückstoß

At the interaction point:

$$z_{n+1} = z_n$$

$$z'_{n+1} = z'_n + f(z_n + \phi s_n)$$

$$\left(\frac{\Delta E}{E}\right)_{n+1} = \left(\frac{\Delta E}{E}\right)_n + \phi\, f(z_n + \phi s_n)$$

$$s_{n+1} = s_n$$

Between interaction points:

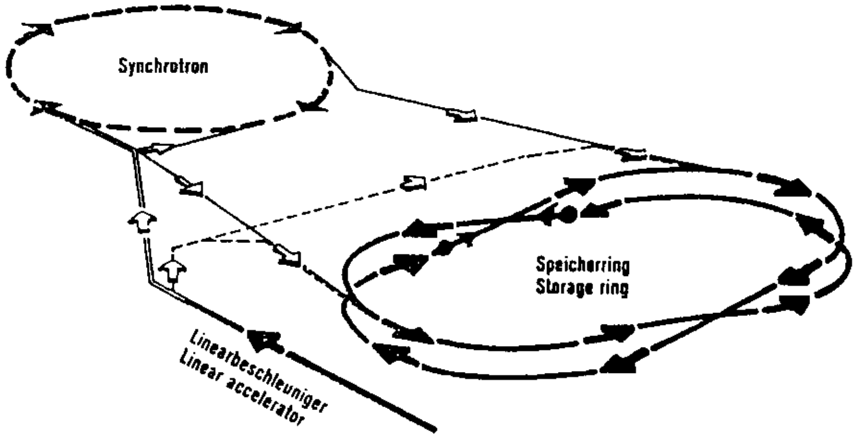
$$z_{n+2} = z_{n+1}\cos\mu + z'_{n+1}\beta\sin\mu$$

$$z'_{n+2} = -z_{n+1}\frac{\sin\mu}{\beta} + z'_{n+1}\cos\mu$$

$$\left(\frac{\Delta E}{E}\right)_{n+2} = \left(\frac{\Delta E}{E}\right)_{n+1} + \frac{eU}{E}\left[\sin\left(\Psi_s + \frac{2\pi}{\lambda_{RF}} s_{n+1}\right) - \sin\Psi_s\right]$$

$$s_{n+2} = s_{n+1} - \alpha_c C_0 \left(\frac{\Delta E}{E}\right)_{n+1}$$

DOUBLE RING STORAGE



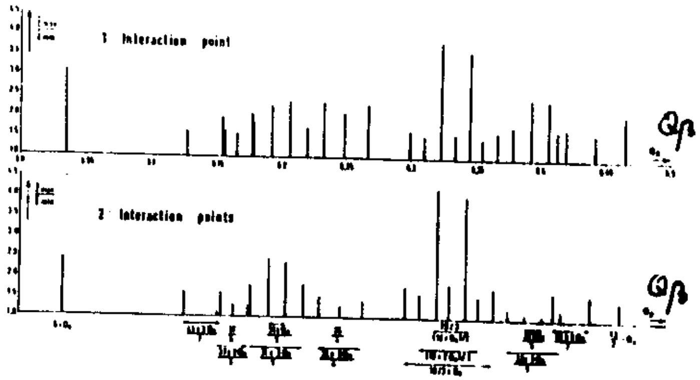
DORIS I: $\phi = 30$ mrad ... crossing angle

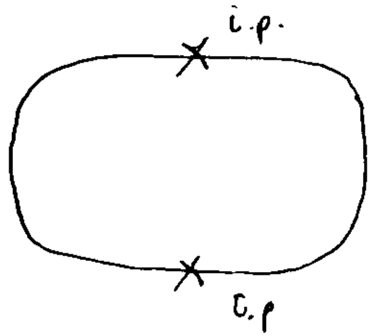
⇒ CURRENT LIMITING EFFECT WITH COLLIDING BEAMS

SSC: 2 p-rings has also a small crossing angle

Resonances due to collision with crossing angle

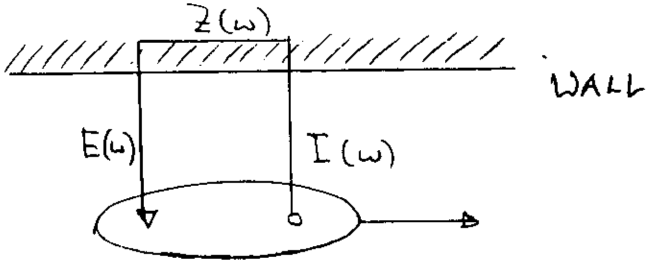
interaction point





beams separated during injection and ramping!

(2) Interaction of beam with environment



$$E_{\parallel} \sim Z_{\parallel} \cdot I_{beam}$$ → takes energy out of the beam

$$F_{\perp} \sim Z_{\perp}\, d_{beam}$$ → deflects the beam

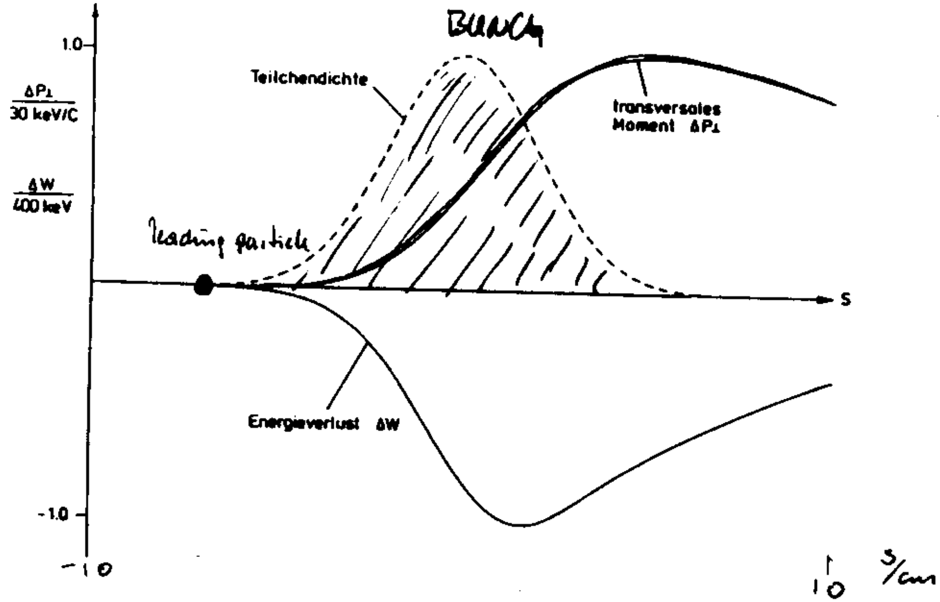
To evaluate the effect of the beam via the environment on itself →

Maxwell's equations must be solved with boundaries as given by the beam environment

Numerical solution of Maxwell's equation leads to

→ WAKE Potentials

→ inside a Gaussian bunch ($q = 1\mu C$, $\sigma = 2$ cm) after passage of a single cell 500 MHz cavity, 0.5 cm off axis



(ΔW) energy loss due to axis-symmetric (monopole fields)

($W_{\perp}$). transverse change in momentum due to dipole fields

Another case seem made is include WAKE fields in tracking codes: to simulate the FAST TRANSVERSE BLOW UP

Principle of simulation:

Define initial distribution of particles

↓

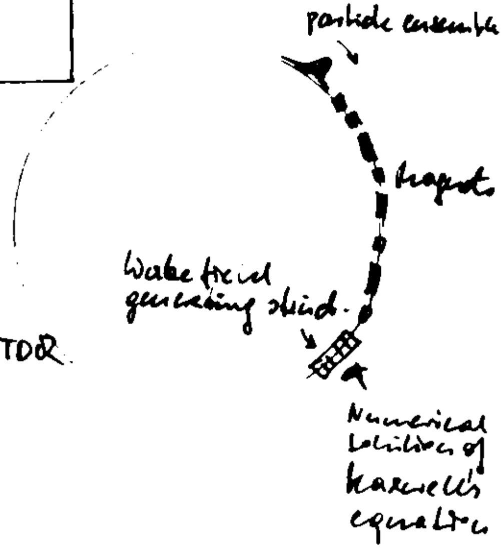
Transforme ensemble of particles through the nonlinear magnet structure

↓

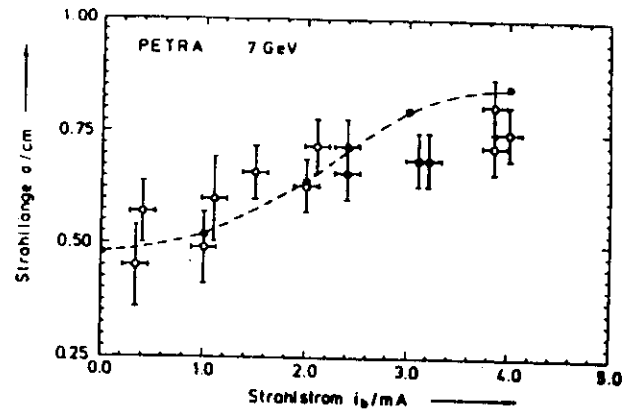
Calculate numerically wakefields of a bunch passing an inhomogeneous part of the vacuum chamber

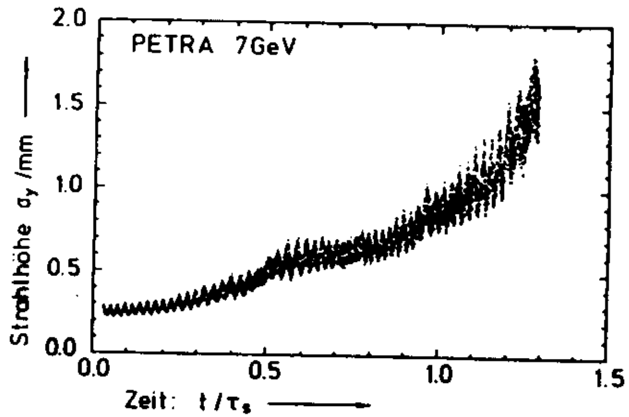
↓

end



CIRCULAR ACCELERATOR

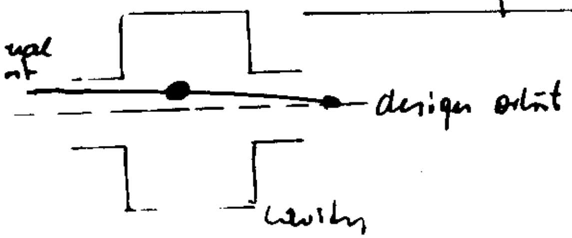




SIMULATION OF FAST TRANSVERSE BLOW UP !

Collective Effect generating satellite resonances:

→ needs a transverse field with a longitudinal gradient, which is produced by the **BUNCH CURRENT** in the accelerating structure due to displacement of the beam in the cavity



Transverse Kick (z):

$$\delta z' = \frac{e}{p}\int (E_z + vB_x)\,dt$$

$$\delta z' = s\cdot\frac{e}{p}\int\left(\frac{\partial E_z}{\partial s} + v\frac{\partial B_x}{\partial s}\right)dt$$

$$\delta z' = A\cdot s$$ linear in s

Energy gain:

$$\frac{\delta E}{E} = \frac{e}{E}\int E_s v\,dt$$

$$= z\,\frac{e}{E}\int\frac{\partial E_s}{\partial z}v\,dt$$

$$\frac{\delta E}{E} = A\cdot z$$

$$\nabla\times E = -\frac{\partial B}{\partial t}$$

Can equal!

In general the solution can only be found numerically →

NUMERICAL SOLUTION OF MAXWELL'S EQUATION

T. WEILAND

$$\boxed{\begin{aligned} \mathrm{curl}\,\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \mathrm{curl}\,\vec{H} &= \vec{J} + \frac{\partial\vec{D}}{\partial t} + \rho\vec{v} \\ \mathrm{div}\,\vec{D} &= \rho \\ \mathrm{div}\,\vec{B} &= 0 \end{aligned}}$$

rewrite in integral representation →

$\vec{E},\vec{H}$ ... fields
$\vec{D},\vec{B}$ .. flux densities

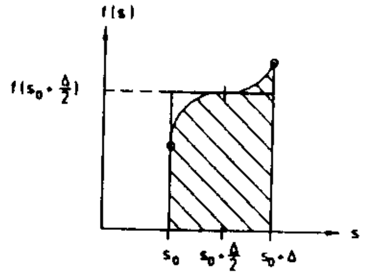
$$\oint_{(A)}\vec{E}\,d\vec{s} = -\iint_A\frac{\partial\vec{B}}{\partial t}\,d\vec{A}$$

$$\oint_{(A)}\vec{H}\,d\vec{s} = \iint_A\left(\vec{J} + \frac{\partial\vec{D}}{\partial t} + \rho\vec{v}\right)d\vec{A}$$

$$\iint_{(V)}\vec{B}\,d\vec{A} = 0$$

$$\iint_{(V)}\left(\frac{\partial\vec{D}}{\partial t} + \vec{J}\right)d\vec{A} = 0$$

Integral approximation in lowest order:

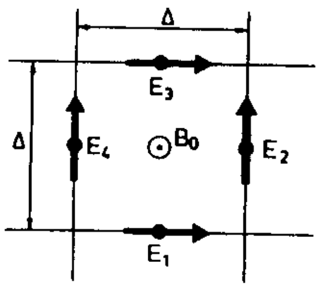


$$\int_{s_0}^{s_0+\Delta} f(s)\,ds = \Delta \cdot f\left(s_0+\frac{\Delta}{2}\right) + \mathcal{O}(\Delta^2)$$

gives for e.g. →

$$\int_{(A)} \vec{E}\,ds = -\int_A \frac{\partial \vec{B}}{\partial t}\,d\vec{A} \quad \Rightarrow$$

$$\Delta\left(E_1+E_2-E_3-E_4\right) = \dot{B}_0 \cdot \Delta^2$$



2nd Maxwell equation →

$$\oint_{(A)} \vec{H}\,ds = \iint_A \left(\frac{\partial D}{\partial t} + \vec{J} + \rho\vec{v}\right) d\vec{A}$$

→

$$\boxed{\frac{1}{\mu}\left(B_1 + B_2 + B_3 + B_4\right) = \Delta^2\left(\varepsilon \dot{E}_0 + \gamma E_0 + v_0 \rho\right)}$$

$\vec{D} = \varepsilon \vec{E}$ ε permittivity

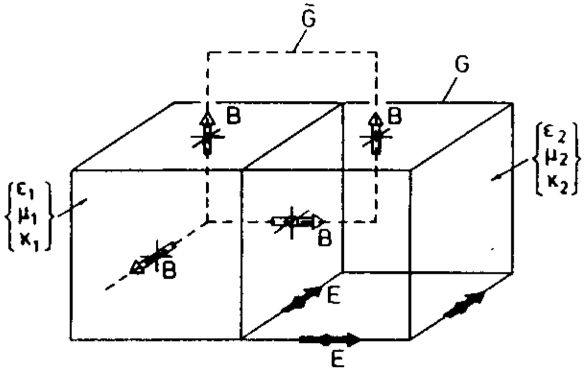
$\vec{B} = \mu \vec{H}$ μ permeability

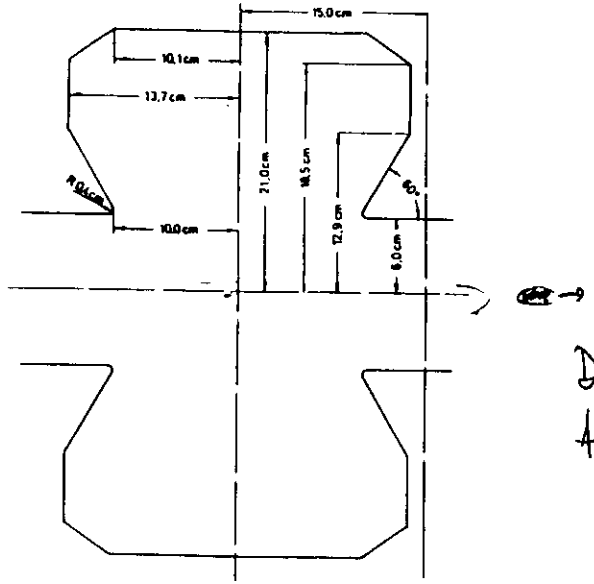
$\vec{J} = \gamma \vec{E}$ γ conductivity

(here ε, μ, γ = constant!)

The choice of a DUAL GRID implicitly satisfy Maxwell's Equation 3 and 4.

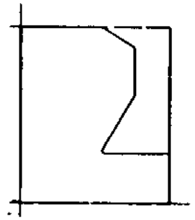
DUAL MESH →

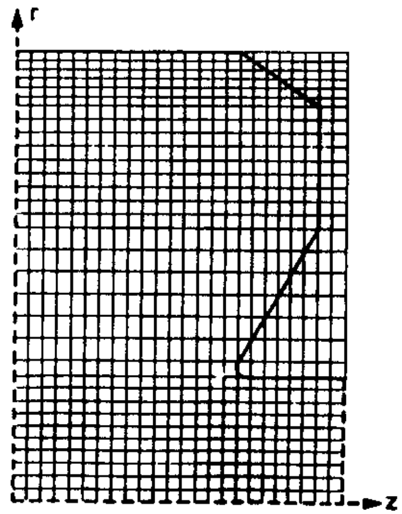




Mesh generator!

Dynamic MESH for cavity

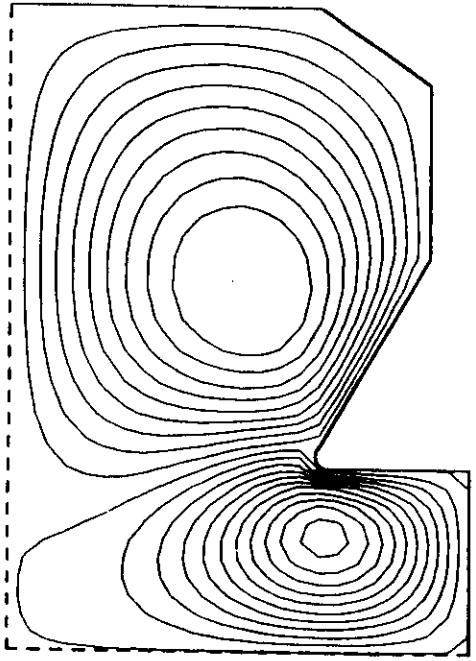




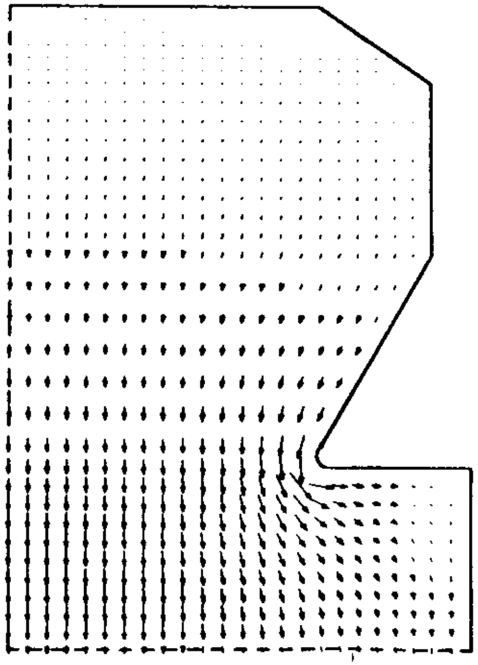
With time discretization we get for the Maxwell's equations in the mesh →

$$\boxed{\begin{aligned} B_0\left(t=T+\tfrac{\Delta t}{2}\right) &= B_0\left(t=T-\tfrac{\Delta t}{2}\right) - \frac{\Delta t}{\Delta}\left(E_1+E_2-E_3-E_4\right)(t=T) \\ E_0\left(t=T+\Delta t\right) &= E_0(t=T) + \frac{1}{\mu\varepsilon}\frac{\Delta t}{\Delta}\left(B_1+B_2-B_3-B_4\right)\left(t=T+\tfrac{\Delta t}{2}\right) \end{aligned}}$$

(here $\vec{J}, \rho = 0$)

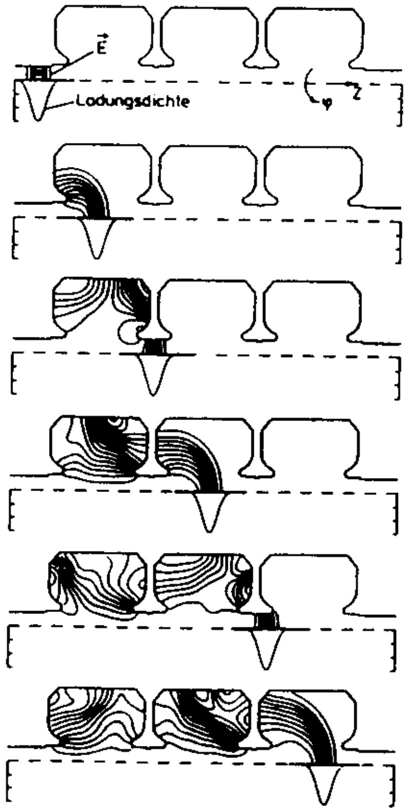


Contour plot
rE_φ (at $\varphi = 90°$)
– for the lowest dipole mode in a 500 MHz PETRA cavity



Magnetic field (at $\varphi = 90°$) of the lowest dipole mode in a 500 MHz PETRA cavity

WAKEFIELDS DUE TO MOVING CHARGES



Bunch passing a 3-cell accelerating structure

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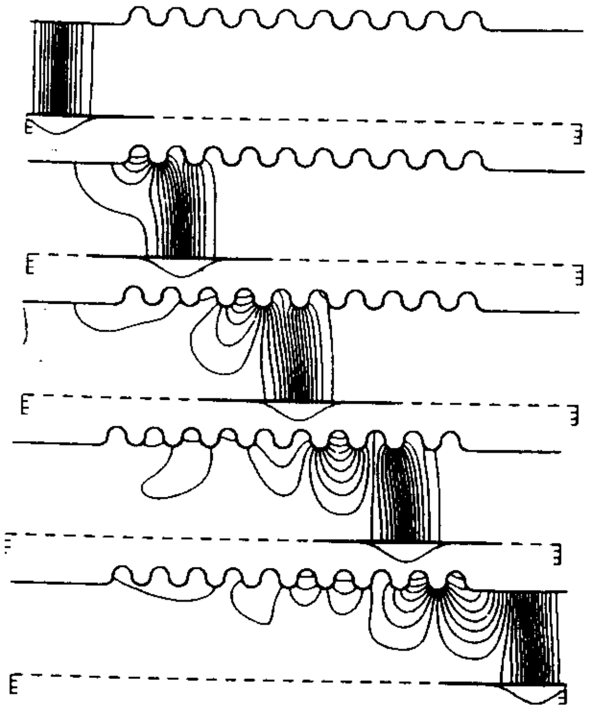
Leads to coupled bunch instabilities
→ dangerous for Radiation Sources!)



Electromagnetic field of a Gaussian bunch of charged particles traversing a bellow.

⇒ they must be shielded!

To find the modes of a cavity, Maxwell's equations can be solved in frequency domain:

$$\frac{\partial}{\partial t} \to i\omega \qquad \text{(SUPERFISH)}$$

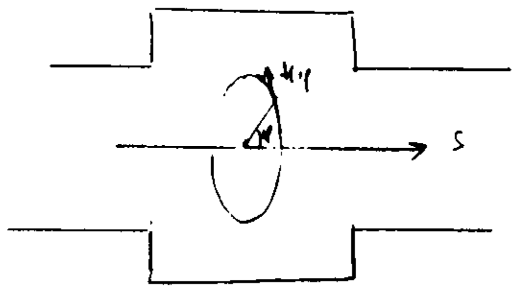
$$\left.\begin{array}{l} \nabla \times \vec{H} = k\vec{E} \\ \nabla \times \vec{E} = k\vec{H} \end{array}\right\} \to \boxed{\nabla \times (\nabla \times \vec{H}) = k^2 \vec{H}}$$

$$k = \frac{\omega}{c}$$

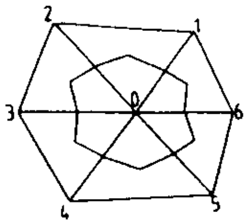
Difference equations

$$\int \nabla \times (\nabla \times \vec{H})\, d\vec{A} = k^2 \int \vec{H}\, d\vec{A}$$

For objects with cylindrical symmetry the only nonvanishing component is H_ϕ



Triangular MESH:



Assuming: H_ϕ = linear inside each triangle ⇒

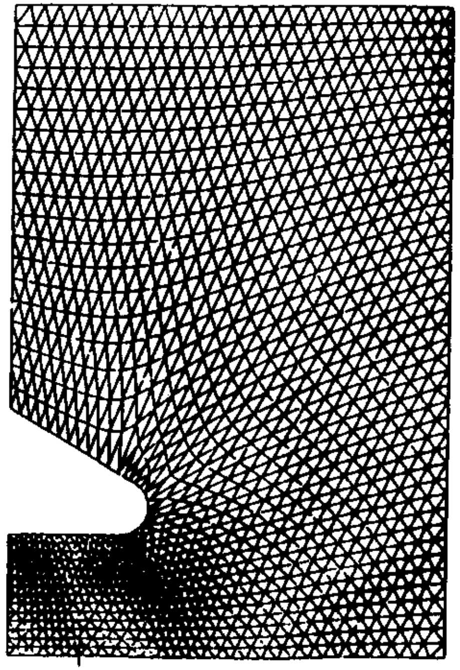
H_ϕ is uniquely determined everywhere in the triangle by its values at the corners.

Equation for each interior mesh point:

$$\boxed{\sum_{n=0}^{6} H_{\phi n}\left(a_n + k^2 b_n\right) = 0}$$

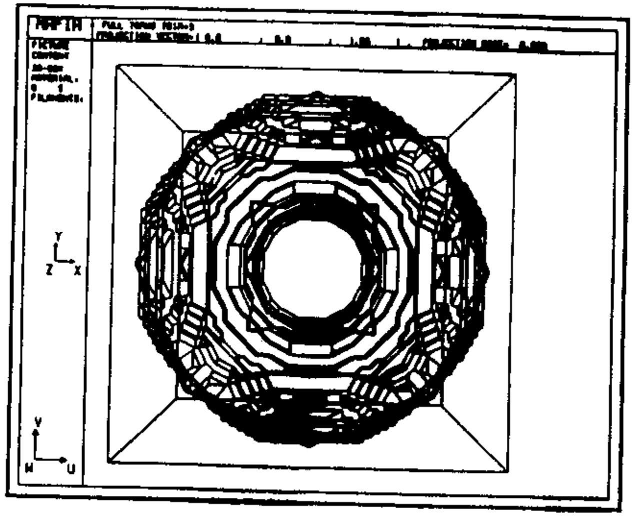
a_n, b_n … depend on meshpoint coordinates only

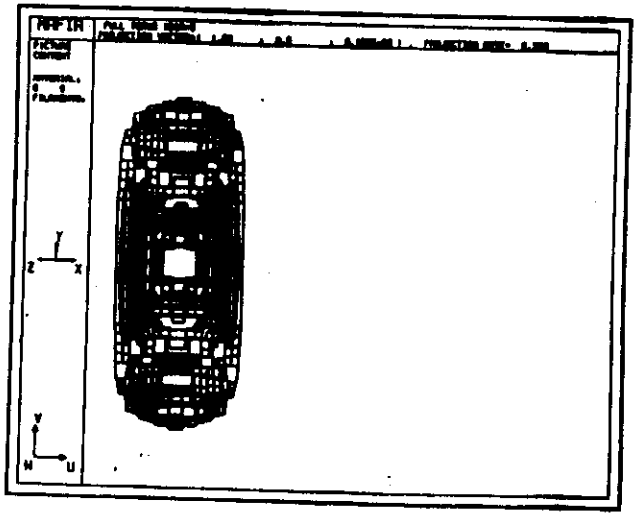
triangular mesh for cavity →



3-dimensional field calculation
(MAFIA)

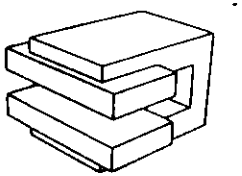
Mesh generation for Torus →

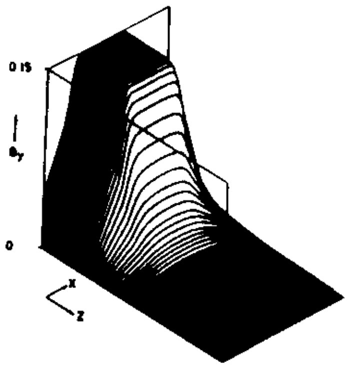




Correction dipole magnet field (vertical) in the midplane:

(Pgm. PROFI) W.R. NOVENDER





Median plane field!

W.R. NOVENDER

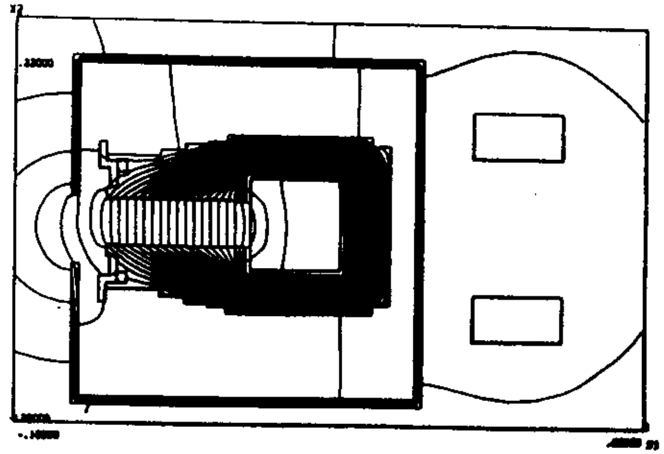


Fig. 3.6. HERA e⁻ arc bending magnet

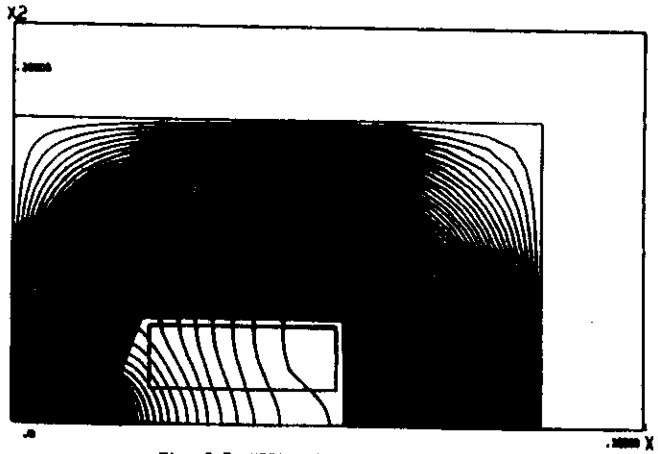


Fig. 3.7. HERA injection magnet