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SECOND SCHOOL ON ADVANCED TECHNIQUES
IN COMPUTATIONAL PHYSICS
(18 January - 12 February 1988)

SMR.282/ 32

# FINITE ELEMENTS FOR PDE (SLIDES)

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# Finite Elements for Partial Differential Equations: an Introductory Survey

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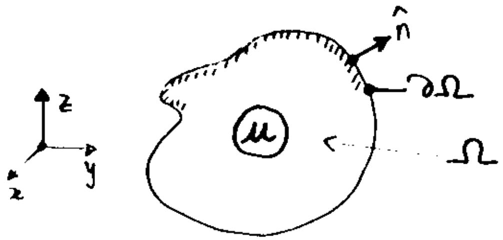
## Contents

# THE PDE problem

$$\boxed{\begin{aligned} \mathcal{D} u &\equiv \sum_{j,i=1}^{2} \partial_i A_{ij}(\vec{x}) \partial_j u = f \\ \mathcal{B} u &\equiv \lambda u + \mu \partial_n u = g \end{aligned}}$$

in $\Omega \in \mathbb{R}^D$

.c. in $\partial\Omega$

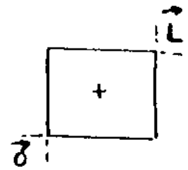



SOLVE ON A COMPUTER $\Leftrightarrow$ BRING IN ALGEBRAIC FORM

# GLOBAL SERIES EXPANSION

- Simple geometry
- Simple $\mathcal{D}$ operator ($A_{ij}$ = const.)

x. Poisson's
$$\begin{cases} u_{xx} + u_{yy} = f \\ u(\vec{0}) = u(\vec{L}) = 0 \end{cases}$$




) **Expand:**

$$u^N = \sum_{n=1}^{N} u_n^N \boxed{g_n(x,y)} \longrightarrow \text{global basis functions}$$

**Plug and Project:**

$$\sum_{m=1}^{N} \langle g_n | \mathcal{D} | g_m \rangle u_m^N = \langle g_n | f \rangle \qquad \text{Algebraic}$$

$$\mathcal{D}_{nm} u_m^N = f_n$$

**Solve $\mathcal{D}_{nm}$ and get $u_m^N$**

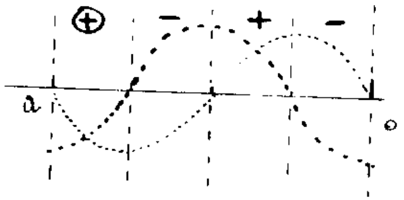
As $N \to \infty$ $u^N \to$ EXACT

## ORTHOGONALITY :

$$\langle g_n | g_m \rangle := \int_a^b g_n(x)\, g_m(x)\, dx = \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

The functions <u>must</u> oscillate !

$$\langle g_n | g_m \rangle = \sum_{i,j}^{n,m} g_{ni}\, g_{mj} \int_a^b x^{i+j}\, dx \;\simeq 0$$




As a result :

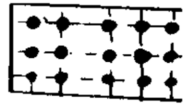
- good only if N is small (perturbative ... )
- Limited ( simple geometry ... )

# THE FINITE DIFFERENCE METHOD

## Local description.

$$\Omega \to \Lambda_N := \{ \vec{x}_1, \vec{x}_2 \dots\dots \vec{x}_N \}$$

$$u \to \vec{u}_N := \{ u_1, u_2 \dots\dots, u_N \}$$

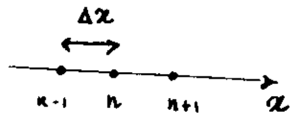


## FD discretization

$$u \longrightarrow u_n := u(\vec{x}_n)$$

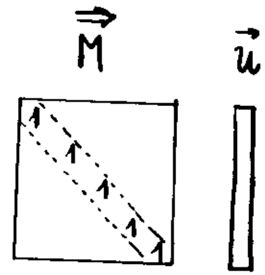
$$\frac{\partial u}{\partial x} \longrightarrow \Delta^+ u_n := \left( \frac{u_{n+1} - u_n}{\Delta x} \right)$$

$$\frac{\partial^2 u}{\partial x^2} \longrightarrow \Delta^- \Delta^+ u_n := \left( \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2} \right)$$

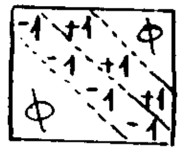



## FD Matrices

$$u \longrightarrow \sum_m \delta_{nm}\, u_m$$

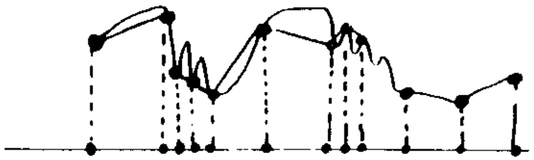



$$\frac{\partial u}{\partial x} \longrightarrow \sum_m \frac{1}{\Delta x} \left( \left( \delta_{n,m-1} - \delta_{n,m} \right) \right) u_m$$




$$\frac{\partial^2 u}{\partial x^2} \longrightarrow \sum_m \frac{1}{\Delta x^2} \left( \left( \delta_{n,m-1} - 2\,\delta_{n,m} + \delta_{n,m+1} \right) \right) u_m$$

## A problem: steep variations



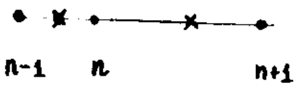
Resolve wavelength $\lambda = \frac{2\pi}{k} \Rightarrow \underline{\Delta x < \lambda}$

Mesh clustering $\Rightarrow$ Loss of Accuracy

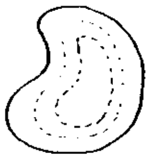
$$\frac{u}{x} \to \frac{\left(\frac{u_{n+1}-u_n}{x_{n+1}-x_n}\right)+\left(\frac{u_n-u_{n-1}}{x_n-x_{n-1}}\right)}{2} = \frac{du}{dx} + \cancel{\varepsilon_1 h} + \varepsilon_2 h^2$$

only if

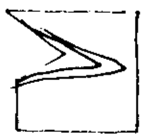
$x_{n+1}-x_n = x_n - x_{n-1}$




## A second problem: Non-conformal boundaries



o.k.



PROBLEM!

## THE FINITE ELEMENT METHOD

...tain both LOCALITY and GLOBALITY:

GLOBALITY: Variational Principles

...nctional: $I(u) = \int_\Omega \mathcal{L}(u, u_x, u_y, x, y)\, dx\, dy$ (F)

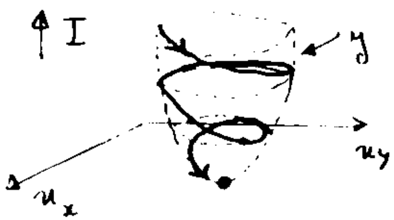
$\min_{u \in \mathcal{U}} \{I(u)\} \leftrightarrow \boxed{\frac{\partial \mathcal{L}}{\partial u} - \partial_x\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \partial_y\left(\frac{\partial \mathcal{L}}{\partial u_y}\right) = 0}$ (EL)

$\mathcal{L} = \frac{1}{2}(u_x^2 + u_y^2) \leftrightarrow u_{xx} + u_{yy} = 0$

: Requires 2° order derivatives

: Requires only 1° order

## ...TZ METHOD ( $I \geq 0$ )




$\delta \mathcal{J}_h = 0$

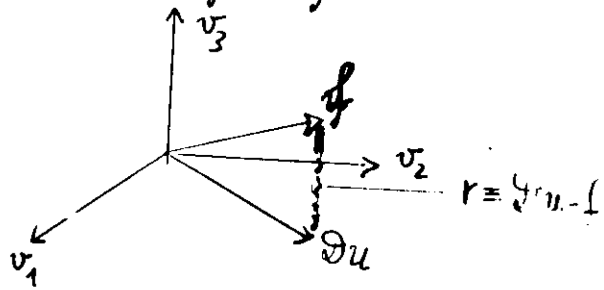
$u \to u_h(\alpha_1, \alpha_2 \dots \alpha_N) \Rightarrow \mathcal{J}_h(\alpha_1 \dots \alpha_N)$

trong form : $\mathcal{D}u = f$ (S)

eak form : "Find $u \in \mathcal{H}_B$ such that, for any $v \in \mathcal{H}_T$

$$\boxed{b(u,v) := \langle v \,|\, \mathcal{D}u - f \rangle = 0}\text{"} \qquad (W)$$

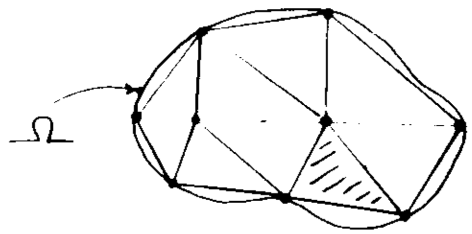
uality $\longrightarrow$ Orthogonality




$$\mathcal{H}_B = \mathcal{H}_T = \mathbb{R}^2 = \text{Span}\{v_1, v_2\}$$

timal : $r \equiv \mathcal{D}u - f$ is $\perp$ to $u$

$$\Omega \to \Omega_h = \bigcup_i^{NE} \Omega_i$$




EMENT REPRE.

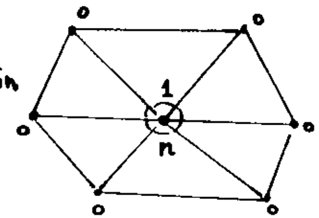
$$u = \sum_{i=1}^{NE} u_i(\vec{x})$$

NODAL REPRE.

$$u = \sum_{n=1}^{NN} u_n \psi_n(\vec{x})$$

↑ FINITE ELEM

$\psi_n(x) = \begin{cases} \pi(\vec{x}) \text{ in} \\ 0 \quad \text{elsewhere} \end{cases}$




$\equiv$ SUPPORT OF $\psi_n$

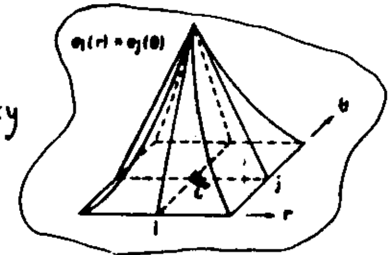
BASIS FUNCTION

NCTIONAL SPACE : $\mathcal{Y} = \text{Span}\{\psi_1, \psi_2 \dots \psi_{NN}\}$

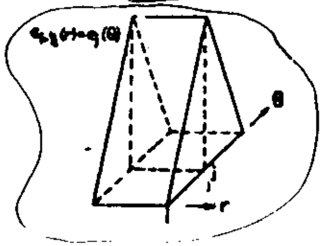
- Geometrical
- Functional

approximations MARCH TOGETHER !!!

$\psi(\vec{x}) = a_0 + a_1 x + a_2 y + a_3 xy$




$\psi(\vec{x}) = (a_0 + a_1 x)$



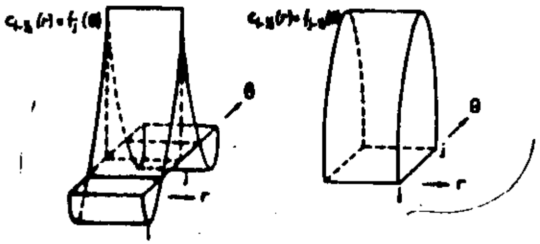
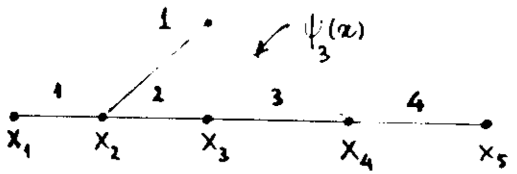




Fig. 16. Shape of the basis functions $e_i(r)\cdot e_j(\theta)$, $c_{i-1/2}(r)\cdot e_j(\theta)$, $c_{i-1/2}(r)\cdot f_j(\theta)$ and $c_{i-1/2}(r)\cdot f_{j-1/2}(\theta)$.

R. Gruber et Al. (Ref. 1, part II)

## NODAL vs ELEMENT REPRESENTATION



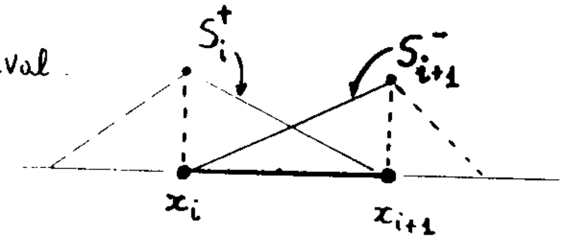

N = 5 nodes ⟶ 5 * 1

I = 4 intervals ⟶ 4 * 2 − 3 * 1 ⟶ 5

$$u = u_1\psi_1(x) + u_2\psi_2(x) + \dots\ u_5\psi_5(x)$$

$$\underbrace{u_1 S_1^+ + u_2 (S_2^-}_{u_1(x)} + \underbrace{S_2^+) + u_3 (S_3^-}_{u_2(x)} + \underbrace{S_3^+) + \dots}_{}\qquad u_4(x)$$

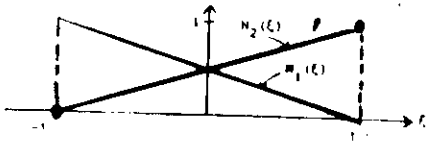
in the interval




$$u \to u_i(x) = u_i S_i^+ + u_{i+1} S_{i+1}^-$$

Theoretical Form. : <u>NODAL</u>

Practical Impl. : <u>ELEMENT</u>



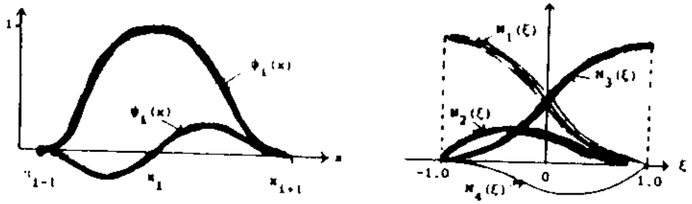




Fig. 4



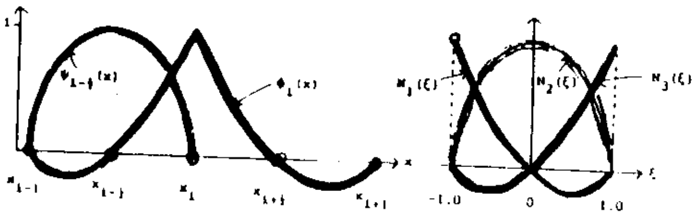
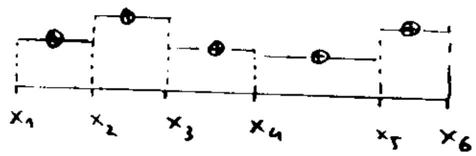


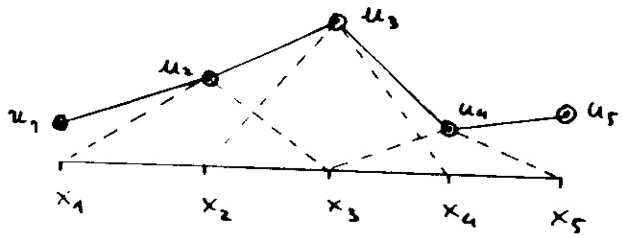

Fig. 3



IECEWISE NSTANTS : $p=0$

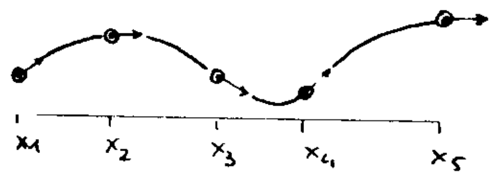



IECEWISE LINEAR : $p=1$ ('HAT' ≡ 'Chapeau')




$$u(x) \to \sum_{i=1}^{5} u_i \psi_i(x) \qquad \boxed{u_i \equiv u(x_i)}$$

RMITE CUBIC : $p=3$



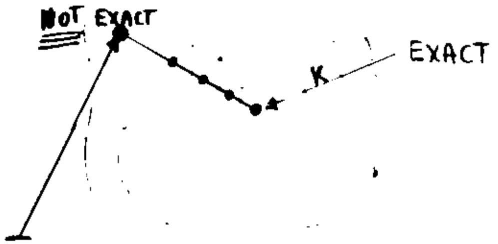

$$u(x) = \sum_{i=1}^{5} \left( u_i \phi_i(x) + v_i \psi_i(x) \right)$$

function derivative

$$\boxed{\begin{aligned} u_i &\equiv u(x_i) \\ v_i &\equiv \frac{du}{dx}(x=x_i) \end{aligned}}$$

# CONVERGENCE

SOBOLEV space $\mathcal{H}_k$




B. : $h$ is a "typical" length.


→ $\mathcal{E}_k$ is INDEPENDENT ON CLUSTERING ! ←

# MESH CLUSTERING



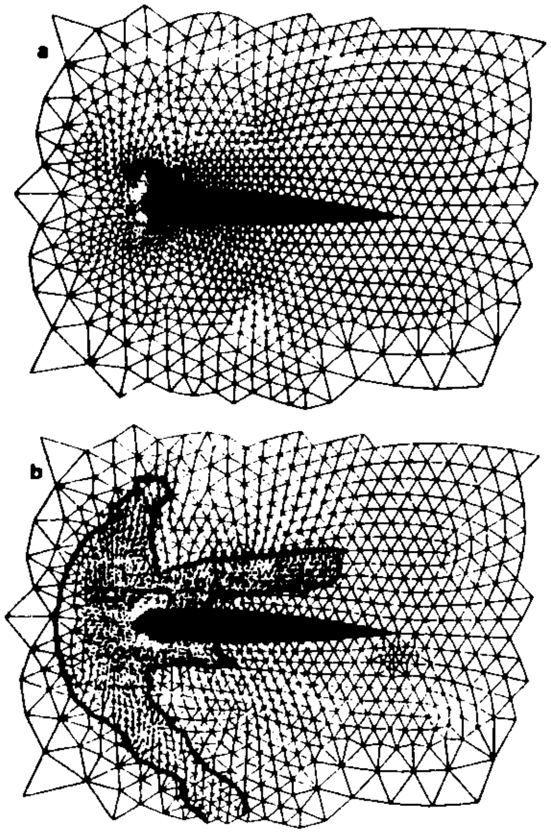
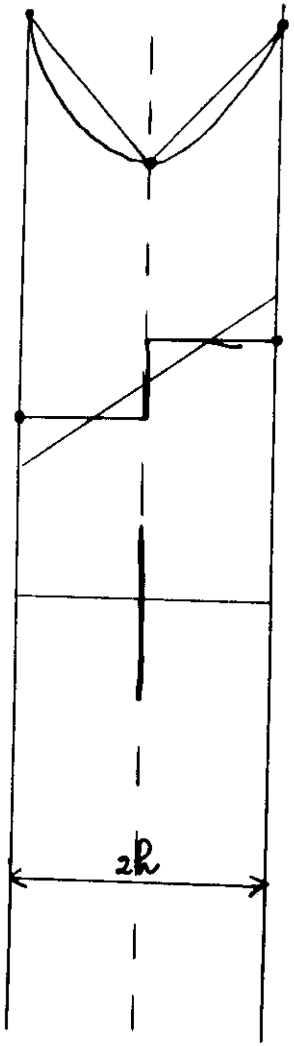



Fig. 16.13. Adapted meshes. Criterion $C_1 + C_4$. $M_\infty = 2$; Re = 106; $\alpha = 10°$. (a) Nodes: 1797, elements: 3474; (b) nodes: 2646, elements: 5172.

enriched using criterion $C_4$, an exceeded threshold of the vorticity generates finer elements in the vicinity of the leading and trailing edges of the air intake. After two successive enrichments, the adapted mesh depicted by figs. 16.15b–16.17b is obtained. The pressure and Mach contours of the solution computed on the initial coarse mesh are shown in fig. 16.18a, 16.19a, while the same iso-lines of the solution on the refined mesh are presented in figs. 16.18b, 16.19b. A significant improvement of the quality of pressure and Mach lines can be observed specially in the vicinity

Bristeau et al (Ref. 1, part II)




$k=0$ $\qquad \varepsilon_0 \sim h^2 \qquad u$

$k=1$ $\qquad \varepsilon_1 \sim h \qquad \dfrac{du}{dx}$

$k=2$ $\qquad \varepsilon_2 \sim h^0 \qquad \dfrac{d^2u}{dx^2}$

$$\varepsilon_0 \sim \left[\int |u - u_N|^2 dx\right]^{1/2} \sim h^2$$

$$\varepsilon_1 \sim \left[\int \left(|u-u_N|^2 + \left|\frac{du}{dx} - \frac{du_N}{dx}\right|^2\right) dx\right]^{\frac{1}{2}} \sim h$$

$\vdots$

$\downarrow$

## BOUNDARY CONDITIONS

- Natural
- Essential

x:

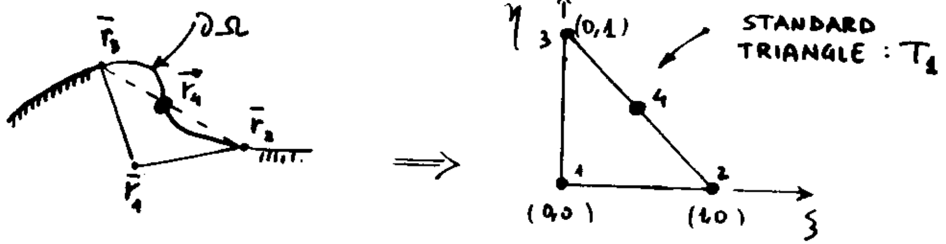
$$y'' + q(x)\,y = 0 \quad + \begin{cases} y'(0) = y'(1) = 0 & \text{NEU} \\ y(0) = a \\ y(1) = b & \text{DIRI} \end{cases}$$

:

$$,y'') \equiv \int_0^1 g\,y''\,dx = g y'\Big|_0^1 - \int_0^1 g' y' dx$$

$\Rightarrow$ NO RESTRICTION ON $g$ with NEUMANN

$g(0)\,y'(0) - g(1)\,y'(1) = 0$ with DIRICHL.




## Change from global to local coordinates

$$\bar r = \vec r_1 N_1(\xi,\eta) + \vec r_2 N_2(\xi,\eta) + \vec r_3 N_3(\xi,\eta) + \boxed{q_4 N_4(\xi,\eta)} \leftarrow \text{EXTRA TERM}$$

| | | |
|---|---|---|
| $\vec r_1 \rightarrow$ | $0,0$ | $N_1 = 1-\xi-\eta$ |
| $\vec r_2 \rightarrow$ | $1,0$ | $N_2 = \xi$ |
| $\bar r_3 \rightarrow$ | $0,1$ | $N_3 = \eta$ . |
| $\vec r_4 \rightarrow$ | $\frac12,\frac12$ | $N_4 = 4\xi\eta$ ←-- EXTRA-TERM |

## Matrix Elements

$$\int_{\Omega_i} \psi_i \cdots \psi_j \, dx\,dy \rightarrow \int_{T_1} \psi_i(\xi,\eta) \cdots \psi_j(\xi,\eta)\, J \, d\xi\, d\eta$$

where

$$J = \begin{cases} 2A_{123} & \text{interior} \\ 2A_{123} + \boxed{4(2A_{124} - A_{123})\xi + 4(2A_{134} - A_{123})\eta} \end{cases}$$

EXTRA-TERM

E CORRECTION is SIMPLE AND HIGHLY SYSTEMATIC !

## SOLVING PROCEDURE

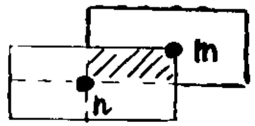
ODAL : $u = \sum_{n=1}^{NN} u_n \psi_n(x)$

lug in weak form, and project systematically onto $\psi_1, \psi_2 \quad \cdots \quad \psi_{NN}$ :

$$\sum_{m=1}^{NN} \langle \psi_n | \mathcal{D} | \psi_m \rangle u_m = \langle f, \psi_n \rangle$$

like any other projection method, but :

- $\mathcal{D}_{nm}$ is **SPARSE** but no oscillations !




- Theoretical convergence **CAN** be achieved !

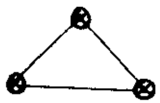
On each element $\Omega_i$ :

$$u_i(x,y) = \pi_i^{(p)}(x,y) = \sum_{q+r=0}^{p} u^i_{qr}\, x^q y^r$$

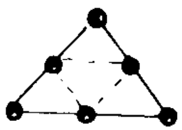
$u^i_{qr}$ ; $\frac{1}{2}(1+p)(p+2)$ d.o.f.

$u^i_{qr}$
- $u$ at nodal and int. nodal (Lagrange)
- $u$ and its deriv. at nodal (Hermite)

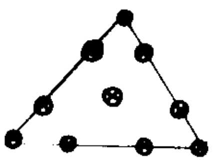
p = 1 ; d.o.f. = 3



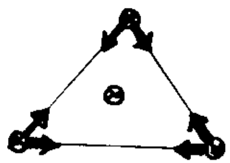
p = 2 ; d.o.f. = 6



p = 3 ; d.o.f. = 10



Lagrange



(Hermite)

## SUMMARY

- **GLOBALITY** : Retained through variational principles (weak form)
- **LOCALITY** : Ensured by definition !
- **MAIN MERITS**
  - Handle Difficult Geometries
  - Systematic Formulation & Implementation
  - " " " Boundary Conditions

APPLICATIONS

# PERFORMANCES DATA

Performances in CPU seconds/step/iteration

All data refer to the use of one processor; and
(80 × 40) × (10 × 20) grid

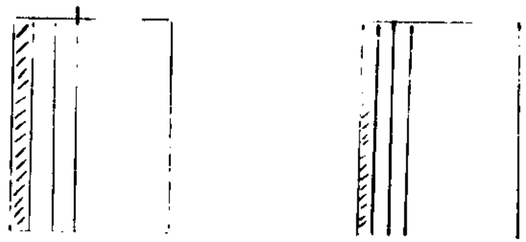
| | | | | | |
|---|---|---|---|---|---|
| 1) | IBM 4381 | Direct | : | 22.2 |
| 2) | IBM 3090 | Direct | : | 2.04 |
| 3) | IBM 3090/VF | Direct | : | 1.37 |
| 4) | IBM 3090/VF | Iterative | : | 0.67 |

ESSL

With the iterative solvers: (G. Radicati, M. Vitaletti, Y. Robert)

- ILUCG , ILUCGN, ILUCGS, ILUGMRES
- MILUCG , " " "
- ...

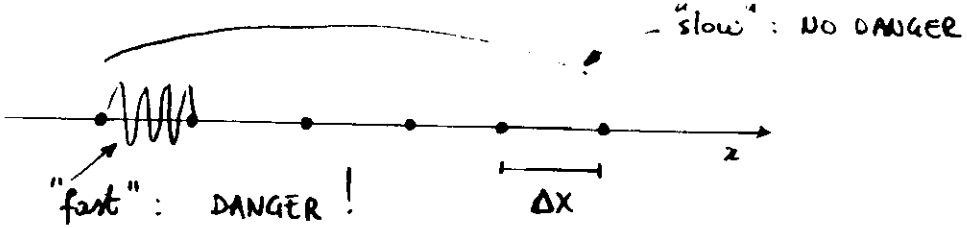
Make use of *Indirect Adressing* offered by the 3090 Vector Facility.



$A(i,j) \neq 0$ $(i,j)$

# REMEDY: ARTIFICIAL DIFFUSION

ngerous modes:

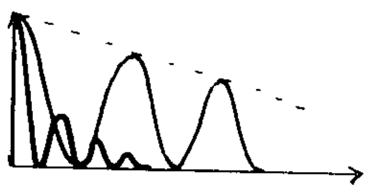



mping the dangerous modes

$$u_t \quad c\, u_z + \delta u_{xx} = 0$$

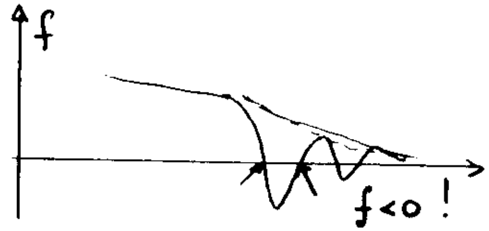
damping!

$$\ldots + i \delta k^2 = 0$$



# PECULIAR NUMERICAL PROBLEMS

vershoots




lvection: $\frac{\partial}{\partial x} \leftrightarrow (-1 \; \boxed{0} \; 1) \Rightarrow$ Imaginary Eigenvalues

perbolic problem: $\boxed{\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0}$

spersion Relation:

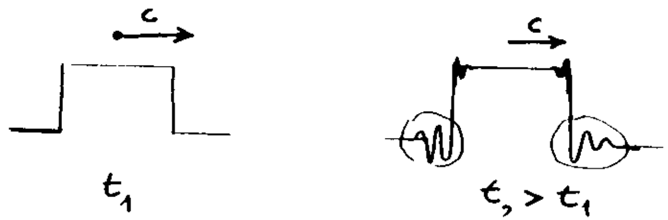
$$\boxed{D(\omega, k) \equiv \omega - ck = 0}$$

l the modes travel with the same speed: NO DISPERSION

iscrete Dispersion Relation

$$D(\omega, k; \underline{\Delta x}, \underline{\Delta t}) \equiv \boxed{(e^{i\omega\Delta t} - 1) - c\frac{\Delta t}{\Delta x}(e^{ik\Delta x} - 1) = 0}$$

he grid breaks Galileian Invariance $\Rightarrow$ DISPERSION ≡ DISTOR

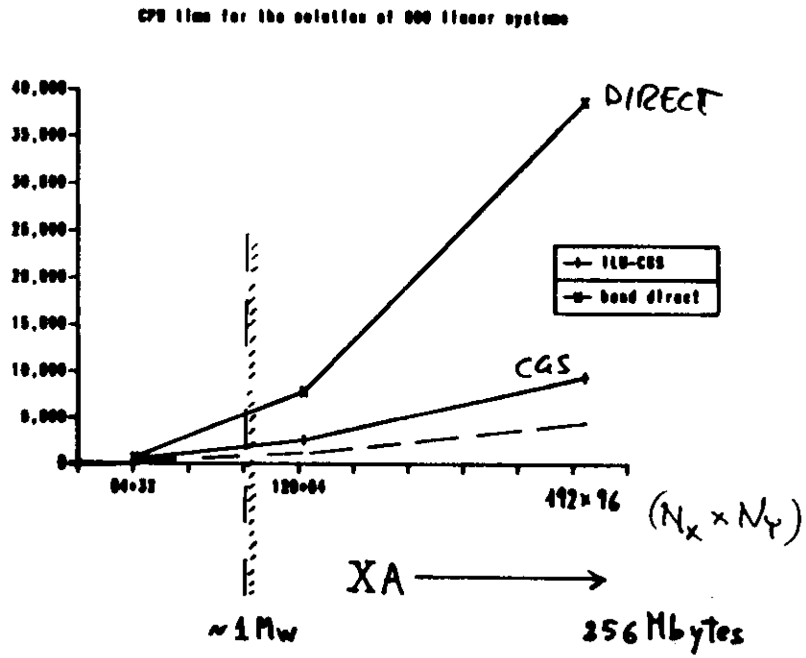


# SOLUTION METHODS

iRECT (Gauss elimination . . .)

TERATIVE ( . . . . . . Conjugate Gradient . . . )

| | DIRECT | ITE |
|---|---|---|
| STORAGE : | $N_x N_y^{2}$ | $N_x N_y$ |
| PU : | $N_x N_y^{3}$ | $N_x N_y$ |
| ONVERGENCE : | always | |




Use of 3090/XA (Extended Architecture) up to 2 Gigabyte (VM)

## EXPLICIT OR IMPLICIT :

**Explicit**

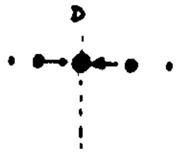
Small $\Delta t$ but no matrix inversion if A is diagonal

**Implicit**

High $\Delta t$ but always matrix inversion

Mass Lumping

$A \rightarrow D$




## EXPLICIT versus IMPLICIT schemes

Let $\boxed{x \equiv B\Delta t/A}$, consider

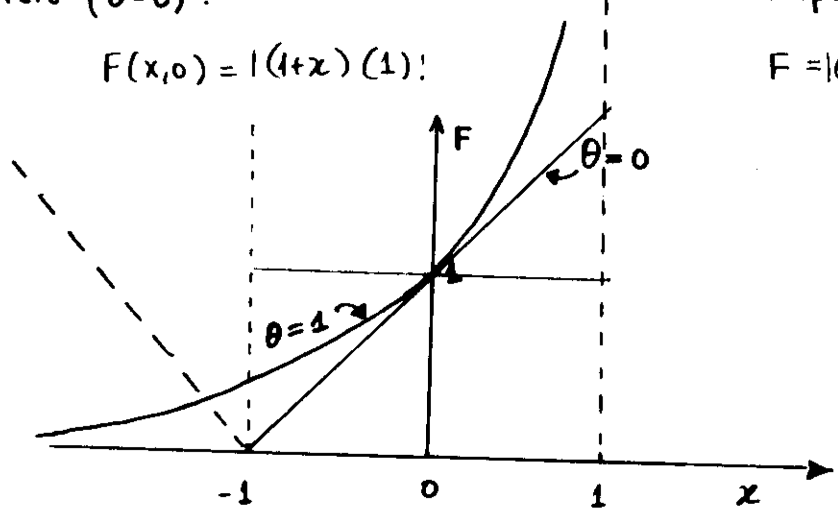
$$\boxed{F(x,\theta) = \left|\frac{1 + x(1-\theta)}{1 - x\theta}\right|} \simeq e^{x}$$

xplicit $(\theta = 0)$ :

$F(x,0) = |(1+x)(1)|$

Implicit (

$F = |(1-x)^{-}$




Physically : STABLE (B<0) UNSTABLE (B>0)

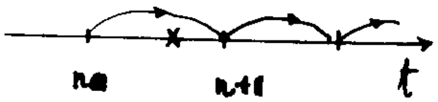
The implicit scheme is much better (closer to $e^{x}$) than explicit for large $x \equiv$ large $\Delta t$.

So Courant-Levy restriction : $B\Delta t/A < 1$ !

## TIME DIFFERENCING

$\dot{f} \longrightarrow \left(\frac{f_{n+1} - f_n}{\Delta t}\right)$ forward $\leftrightarrow$ dissipation

$f \longrightarrow \theta f_{n+1} + (1-\theta) f_n$




$$\theta = \begin{cases} 0 & \text{EXPLICIT} \\ 1/2 & \text{CRANK-NICOLSON} \\ 1 & \text{IMPLICIT} \end{cases}$$

## Algebraic Equations

$$\{A - B\Delta t\theta\} f_{n+1} = \{A + B\Delta t(1-\theta)\} \cdot f_n \equiv b_n$$

$$\mathcal{A} \cdot X = b$$

$$f_{n+1} = (A - B\Delta t\theta)^{-1} (A + B\Delta t(1-\theta)) f_n \equiv P^{n,n+1}(\theta) \cdot f_n$$

$$P^{n,n+1}(\theta) \sim \exp(A^{-1} B \Delta t)$$

EXACT PROPAGATOR

## PROPERTIES OF THE PROPAGATOR

**CONSISTENCY**

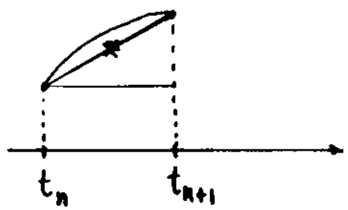
$P^{n,n+1} \longrightarrow 1$ as $\Delta t \to 0$

**ACCURACY**

$$\dot{f} = Bf \Rightarrow f_{n+1} - f_n = B\int_{t_n}^{t_{n+1}} f\,dt \equiv B\,\mathcal{I}_{n,n+1}$$

der 0 : $\int_{t_n}^{t_{n+1}} f\,dt = f_n \Delta t$

der 1 : $\int_{t_n}^{t_{n+1}} [f_n + \dot{f}_n (t - t_n)]\,dt = (f_n + f_{n+1}) \frac{\Delta t}{2}$




**STABILITY**

$\| P^{n,n+1} \| < 1$ : ( contracting map )

rror $\varepsilon_n := f_n - f^{EXACT}$ : $\varepsilon_{n+1} = \| P^{n,n+1} \| \, \varepsilon_n$

## THE EQ. OF MOTION

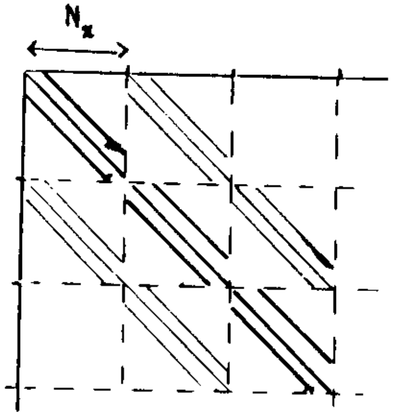
$$\boxed{\sum_{j=1}^{N^2} M_{ij}\dot{f}_j = \sum_{j=1}^{N^2} F_{ij} f_j} \qquad \left( g \equiv \psi_i \quad i = 1, 2 \dots N^2 \right)$$

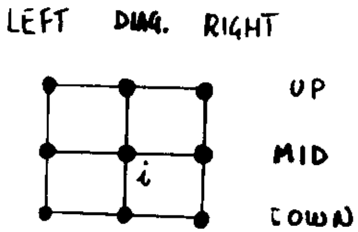
with

$$M_{ij} = \int \psi_i \psi_j \, d_2 v$$

$$F_{ij} = \int \vec{\partial}\psi_i \cdot \vec{R}\, \psi_j \, d_2 v + \int \vec{\partial}\psi_i \cdot \vec{\vec{D}} \cdot \vec{\partial}\psi_j \, d_2 v$$

These are <u>block-tridiagonal</u>



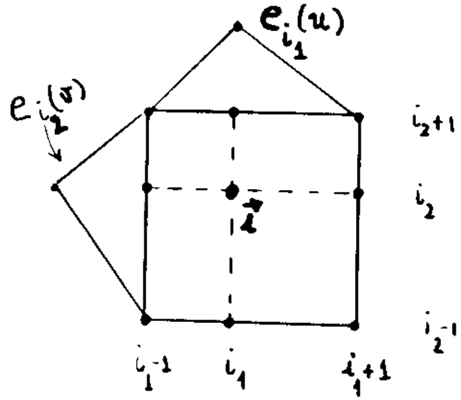



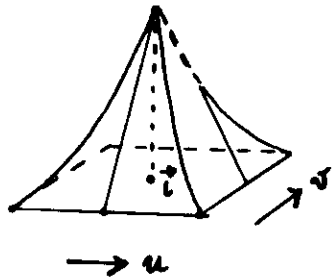

## APPROXIMATING SUBSPACE

Since $\Omega$ is a rectangle $\Rightarrow$ Bilinear F.E.

4 d.o.f. / element

$$\psi_{\vec{i}}(u, v) = e_{i_1}(u)\, e_{i_2}(v)$$



# The WEAK-FORMULATION

or any $g \in \mathcal{H}_T$, find $f$ such that

$$\boxed{\langle g \mid \partial_t f + \operatorname{div} \vec{J} \rangle = 0}$$

pon integr. by parts:

$$\langle g, \partial_t f \rangle + \cancel{\oint_{\partial\Omega} g \vec{J} \cdot \vec{n}\, ds} - \langle \vec{\partial} g, \vec{J} \rangle = 0$$

plicitely:

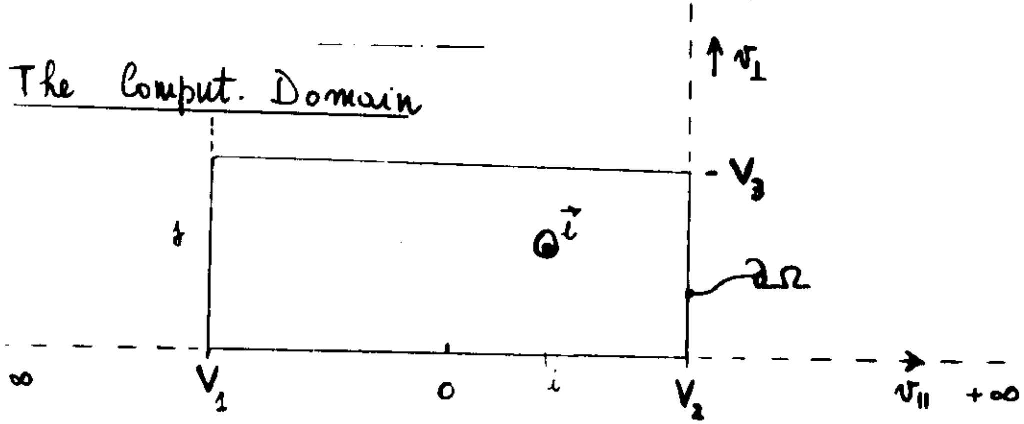
$$\boxed{\mathcal{J}(g,f) := \int_{\Omega} \left\{ g\, \partial_t f + \vec{\partial} g \cdot \left( \vec{R} f + \vec{\vec{D}} \cdot \vec{\partial} f \right) \right\} du\, dv = 0}$$

- $\mathcal{J}(g,g)$ is $\boxed{\text{NOT}}$ positive definite
- $\mathcal{J}$ requires only $\vec{\partial} f$ square-integr. $\Rightarrow (\mathcal{H} \equiv \mathcal{H}_1)$

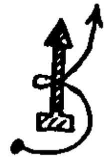
## Main Difficulties:

- TIME DEPENDENCE $(\partial_t \neq 0)$
- TWO-DIMENSIONS $\vec{v} = (u, v)$
- NON-SELF-ADJOINT

## The Comput. Domain




$$\mathbb{R}^2 \to \Omega = \left\{ V_1 \le v_\parallel \le V_2 \;;\; 0 \le v_\perp \le V_3 \right\}$$



## F.E.M. Expansion

$$f(u,v,t) = \sum_{\vec{i}=1}^{N^2} f_{\vec{i}}(t)\, \psi_{\vec{i}}(u,v)$$

(Kantarovich semi-discrete)

# THE FOKKER-PLANCK EQUATION

$$\partial_t f + \mathrm{div}\,\vec{J} = \quad \ldots \quad \vec{J} = \vec{R} f + \vec{\vec{D}}\cdot\vec{\partial} f$$

$$\oint_{\partial\Omega} \vec{J}\cdot \vec{n}\, d\sigma = \quad \ldots \quad \Omega$$

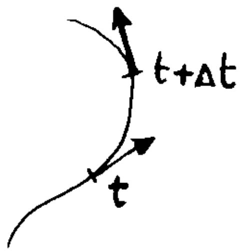
$$f(\vec{v}, t \ldots) = f_0 \ldots$$

$f$ : DISTRIB. FUNCTION

$\vec{R} f$ : ADVECTION

$\vec{\vec{D}}\vec{\partial} f$ : DIFFUSION

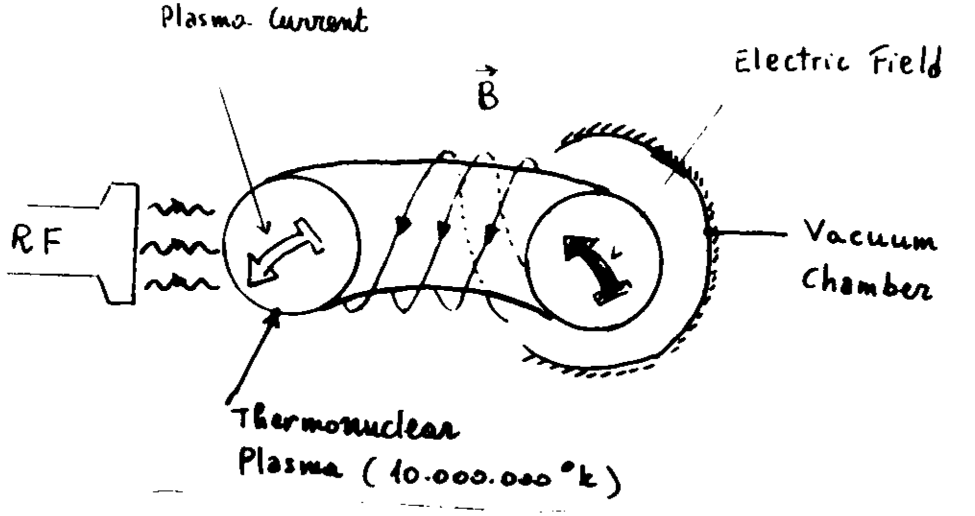
Electrons ... plasma

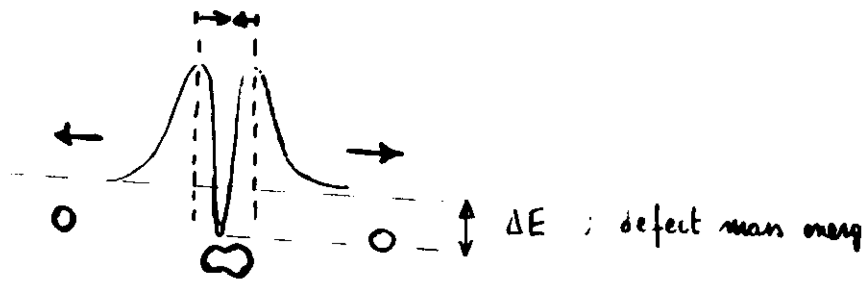



$\vec{R} = \langle \Delta\vec{V}/\Delta t \rangle$

$\vec{\vec{D}} = \langle \Delta\vec{V}\Delta\vec{V}/\Delta t \rangle$

# THE TOKAMAK




$$\boxed{D^+ + D^+ \rightarrow He^{++} + n}$$

OAL.: Produce bound states of light nuclei To extract their d.m.e.

## COMPARE WITH FINITE DIFF

t $p(x) = P_0$ ; $q(x) = q_0$ const. + UNIFORM mesh

| | FEM | FD |
|---|---|---|
| $(l_i, d_i, r_i)(P)$ = | $\frac{1}{2h}(-1, 2, 1)$ | $\frac{1}{2h^2}(-1, 2, 1) \leftrightarrow y''$ |
| $(l_i, d_i, r_i)(Q)$ = | $\frac{h}{6}(1, 4, 1)$ | $(0, 1, 0) \leftrightarrow y$ |

$$P_{ij}^{FEM} \equiv P_{ij}^{FD} \quad \text{(centered)}$$

$$Q_{ij}^{FEM} \neq Q_{ij}^{FD} \quad ; \quad \sum_{j=1}^{3} Q_{ij}^{FEM} = \sum_{j=1}^{3} Q_{ij}^{FD}$$

$$\text{FEM}\left\{ q(x)\, y(x) \right\} = \text{FD}\left\{ \int_{x-h/2}^{x+h/2} q(x')\, y(x')\, dx' \right\}_{\text{SIMPSON}}$$

y point : LOCAL AVERAGING !

## SOLVING PROCEDURE

1. CHOOSE THE APPROXIMATING SUBSPACE : $\psi$
2. CONSTRUCT THE MATRICES IN $\boxed{\text{SPARSE}}$ FORMAT

$$\begin{matrix} l_i \\ d_i \\ r_i \end{matrix} \leftarrow \int_{a_i} \psi \cdots \psi \, dx$$

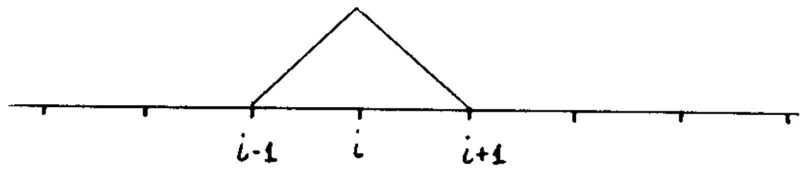
3. ASSEMBLE THE MATRICES ( SPARSE → FULL )

$$l_i \rightarrow A_{i,i-1}$$
$$d_i \rightarrow A_{i,i}$$
$$r_i \rightarrow A_{i,i+1}$$

4. SOLVE THE ALGEBRA

$$Ax = b \quad ; \quad x = A^{-1} b$$

5. OUTPUT THE RESULTS




$$Q_{ij} := \int_0^1 e_i(x)\, q(x)\, e_j(x)\, dx \sim \sum_{g=1}^{G} e_i(x_g)\, p(x_g)\, e_j(x_g)\, w_g$$

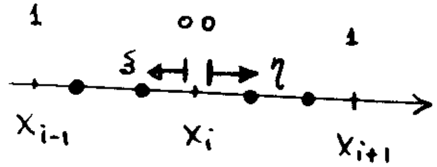
NODES WEIG

<u>Best way : on INTERVALS</u>

<u>$x_{i-1} \le x \le x_i$</u>

$i-1, i-1$
$i-1, i$
$i, i-1$
$i, i$

|  | L | D | R |
|---|---|---|---|
| $i-1$ |  | X | X |
| $i$ | X | XX | X |
| $i+1$ | X | X |  |




$$\begin{cases} \xi = (x_i - x)/(x_i \ldots \\ \eta = (x - x_i)/(x_i \ldots \end{cases}$$

<u>$x_i \le x \le x_{i+1}$</u>

$i, i$
$i, i+1$
$i+1, i$
$i+1, i+1$

$$l_i(q) = d_i^L + d_i^r = \left[\int_0^1 (1-\xi)\, q(x_i+\xi)(1-\xi)\, d\xi\right] * (x_i - x_{i-1})$$
$$+ \left[\int_0^1 (1-\eta)\, q(x_i+\eta)(1-\eta)\, d\eta\right] * (x_{i+1} - x_i)$$

CONSTRUCT MATRIX IN <u>SPARSE</u> FORM : $M_{ij} \Leftrightarrow (l, d, r)_i$

$$\left.\begin{array}{l} \text{Do } i = 1, N-1 \\ l_{i+1} \leftarrow l_{i+1} + J(i+1, i) \\ d_i \leftarrow d_i + J(i,i) \\ d_{i+1} \leftarrow d_{i+1} + J(i+1, i+1) \\ r_i \leftarrow r_i + J(i, i+1) \end{array}\right\} \text{FROM NUM. INTEGR.}$$

LEFT

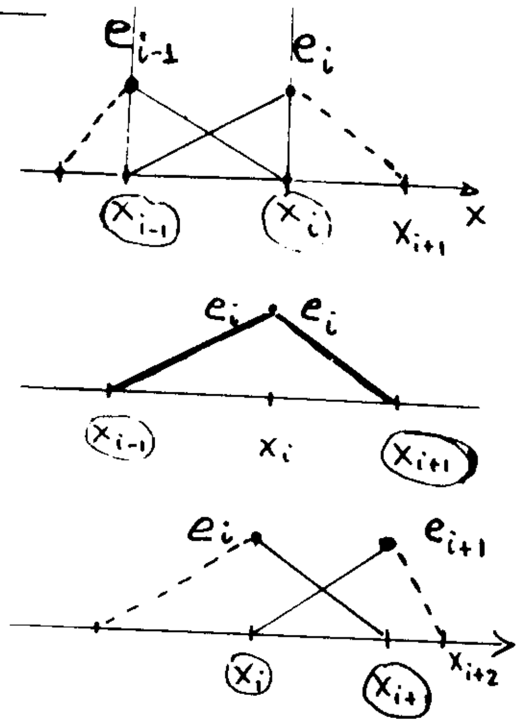
$_{i}(Q) \equiv Q_{i,i-1}$ ;

DIAGONAL

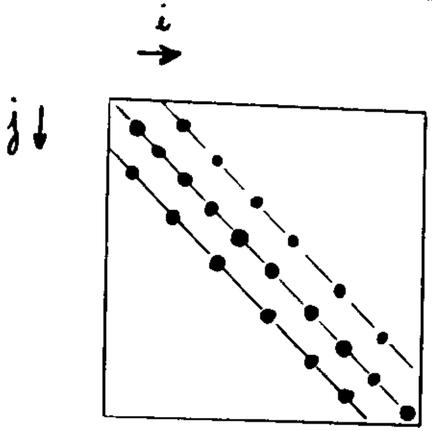
$(Q) \equiv Q_{ii}$

RIGHT

$(Q) \equiv Q_{i,i+1}$



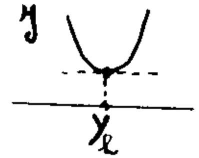

MATRIX-STRUCTURE : TRIDIAGONAL




Computational Molecule ;


$$J(y) \longrightarrow J(y_k) \equiv J_k(y_2, y_3 \cdots y_{N-1})$$

$$\frac{\delta J}{\delta y} = 0 \longrightarrow \frac{\partial J_k}{\partial y_i} = 0$$




$$\boxed{\sum_{j=2}^{N-1} (P_{ij} + Q_{ij}) y_j = f_i} \qquad i = 2, 3 \dots N-1$$

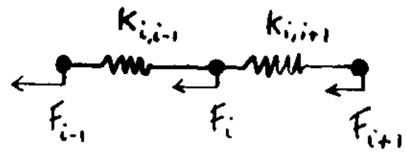
$$P_{ij} := \int_0^1 e_i'(x) p(x) e_j'(x)\, dx = \langle p \rangle K_{ij} \quad \text{STIFFNESS}$$

$$Q_{ij} := \int_0^1 e_i(x) q(x) e_j(x)\, dx = \langle q \rangle M_{ij} \quad \text{MASS}$$

$$f_i := \int_0^1 e_i(x) f(x)\, dx + (a\delta_{10} + b\delta_{NN-1})(P_{ij} + Q_{ij}) \quad \text{LOAD}$$

## General Properties

- Symmetry
- Positive def.
- Sparse



# STURM - LIOUVILLE

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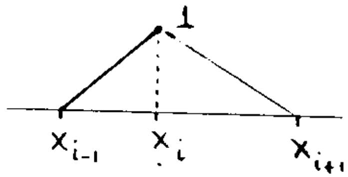
## RITZ Method

[illegible]

## Piecewise LINEAR ( "HAT", "CHAPEAU" )

$e_i(x)$ :

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