INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392-I

H4.SMR/285 - 10

WINTER COLLEGE ON
LASER PHYSICS: SEMICONDUCTOR LASERS
AND INTEGRATED OPTICS

(22 February - 11 March 1988)

# OPTICAL WAVES IN CRYSTALS

H. Melchior
Swiss Federal Institute of Technology
Zurich, Switzerland

# Optical Waves in Crystals

## Propagation and Control of Laser Radiation

AMNON YARIV

California Institute of Technology

POCHI YEH

Rockwell International Science Center

*A Wiley-Interscience Publication*

*John Wiley & Sons*

*New York / Chichester / Brisbane / Toronto / Singapore*

# 4

# Electromagnetic Propagation in Anisotropic Media

There are many materials whose optical properties depend on the direction of propagation as well as polarization of the light waves. These optically anisotropic materials include crystals such as calcite, quartz, and KDP, as well as liquid crystals. They exhibit many peculiar optical phenomena, including double refraction, optical rotation, polarization effects, conical refraction, and electro-optical and acousto-optical effects. Many optical devices are made of anisotropic crystals, for example, prism polarizers, sheet polarizers, and birefringent filters. Anisotropic nonlinear materials are also used for phase-matched second-harmonic generation. A thorough understanding of light propagation in anisotropic media is thus important if these phenomena are to be used for practical applications. The present chapter is devoted entirely to the study of the propagation of electromagnetic radiation in these media.

## 4.1. THE DIELECTRIC TENSOR OF AN ANISOTROPIC MEDIUM

In an isotropic medium, the induced polarization is always parallel to the electric field and is related to it by a scalar factor (the susceptibility) that is independent of the direction along which the field is applied. This is no longer true in anisotropic media, except for certain particular directions. Since the crystal is made up of a regular periodic array of atoms (or molecules) with certain symmetry, we may expect that the induced polarization will depend, both in its magnitude and direction, on the direction of applied field. Instead of a simple scalar relation linking **P** and **E**, we have

$$\begin{aligned} P_x &= \epsilon_0(\chi_{11}E_x + \chi_{12}E_y + \chi_{13}E_z), \\ P_y &= \epsilon_0(\chi_{21}E_x + \chi_{22}E_y + \chi_{23}E_z), \\ P_z &= \epsilon_0(\chi_{31}E_x + \chi_{32}E_y + \chi_{33}E_z), \end{aligned} \qquad (4.1\text{-}1)$$

where the capital letters denote the complex amplitudes of the corresponding time-harmonic quantities. The 3 × 3 array of the coefficients $\chi_{ij}$ is called the electric susceptibility tensor. The magnitudes of the $\chi_{ij}$ depend, of course, on the choice of the $x$, $y$, and $z$ axes relative to the crystal structure. It is always possible to choose $x$, $y$, and $z$ in such a way that the off-diagonal elements vanish, leaving

$$\begin{aligned} P_x &= \epsilon_0 \chi_{11} E_x, \\ P_y &= \epsilon_0 \chi_{22} E_y, \\ P_z &= \epsilon_0 \chi_{33} E_z. \end{aligned} \tag{4.1-2}$$

These directions are called the principal dielectric axes of the crystal.

We can, instead of using Eq. (4.1-1), describe the dielectric response of the crystal by means of the dielectric permittivity tensor $\epsilon_{ij}$, defined by

$$\begin{aligned} D_x &= \epsilon_{11} E_x + \epsilon_{12} E_y + \epsilon_{13} E_z, \\ D_y &= \epsilon_{21} E_x + \epsilon_{22} E_y + \epsilon_{23} E_z, \\ D_z &= \epsilon_{31} E_x + \epsilon_{32} E_y + \epsilon_{33} E_z. \end{aligned} \tag{4.1-3}$$

From Eq. (4.1-1) and the relation

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{4.1-4}$$

we have

$$\epsilon_{ij} = \epsilon_0 (1 + \chi_{ij}). \tag{4.1-5}$$

These nine quantities $\epsilon_{11}, \epsilon_{12}, \ldots$ are constants of the medium and constitute the dielectric tensor. Equation (4.1-3) is often written in tensor notation as

$$D_i = \epsilon_{ij} E_j \tag{4.1-6}$$

where the convention of summation over repeated indices is observed.

In the greater part of this chapter we assume that the medium is homogeneous, nonabsorbing, and magnetically isotropic. The energy density of the stored electric field in the anisotropic medium is

$$U_e = \tfrac{1}{2}\mathbf{E} \cdot \mathbf{D} = \tfrac{1}{2} E_i \epsilon_{ij} E_j. \tag{4.1-7}$$



When we differentiate Eq. (4.1-7) with respect to time, we obtain

$$\dot{U}_e = \tfrac{1}{2}\epsilon_{ij}\left(\dot{E}_i E_j + E_i \dot{E}_j\right). \tag{4.1-8}$$

According to the derivation of Poynting's theorem in Section 1.2, the net power flow into a unit volume in a lossless medium is

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}}, \tag{4.1-9}$$

which, by using Eq. (4.1.6) for $\mathbf{D}$, can be written as

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = E_i \epsilon_{ij} \dot{E}_j + \mathbf{H} \cdot \dot{\mathbf{B}}. \tag{4.1-10}$$

Since the Poynting vector corresponds to the energy flux in the medium, the first term on the right side of Eq. (4.1-10) must be equal to $\dot{U}_e$. Therefore, we have the following equation:

$$\tfrac{1}{2}\epsilon_{ij}\left(\dot{E}_i E_j + E_i \dot{E}_j\right) = \epsilon_{ij} E_i \dot{E}_j. \tag{4.1-11}$$

It follows immediately from Eq. (4.1-11) that

$$\epsilon_{ij} = \epsilon_{ji}. \tag{4.1-12}$$

This means that the dielectric tensor is symmetric and has, in general, only six independent elements. This symmetry is a direct consequence of the definition (4.1-6) and the assumption that $\epsilon$ is a real dielectric tensor. In the event that a lossless medium is described by a complex dielectric tensor (e.g., optical activity—see Section 4.9), a similar derivation shows that

$$\epsilon_{ij} = \epsilon_{ji}^*. \tag{4.1-13}$$

In other words, the conservation of electromagnetic field energy requires that the dielectric tensor be Hermitian. In the special case when the dielectric tensor becomes real, the Hermitian property (4.1-13) reduces to a symmetric property (4.1-12).

## 4.2. PLANE-WAVE PROPAGATION IN ANISOTROPIC MEDIA

In an anisotropic medium such as a crystal, the phase velocity of light depends on its state of polarization as well as its direction of propagation. Because of the anisotropy, the polarization state of a plane wave may vary as it propagates through the crystal. However, given a direction of propagation in the medium, there exist, in general, two eigenwaves with well-defined eigen-phase-velocities and polarization directions. A light wave with polarization parallel to one of these directions will remain in the same polarization state as it propagates through the anisotropic medium. These eigenpolarizations, as well as the corresponding eigen-phase-velocities (or, equivalently eigenindices of refraction), can be determined from Eqs. (1.1-1) and (1.1-2) and the dielectric tensor.

To derive these results, we assume a monochromatic plane wave of angular frequency $\omega$ propagating in the anisotropic medium with an electric field

$$\mathbf{E}\exp[i(\omega t - \mathbf{k}\cdot\mathbf{r})] \tag{4.2-1}$$

and a magnetic field

$$\mathbf{H}\exp[i(\omega t - \mathbf{k}\cdot\mathbf{r})], \tag{4.2-2}$$

where $\mathbf{k}$ is the wave vector $\mathbf{k} = (\omega/c)n\mathbf{s}$ with s a unit vector in the direction of propagation. $n$ is the refractive index to be determined. Substitution for $\mathbf{E}$ and $\mathbf{H}$ from Eqs. (4.2-1) and (4.2-2), respectively, into Maxwell's equations (1.1-1) and (1.1-2) gives

$$\mathbf{k}\times\mathbf{E} = \omega\mu\mathbf{H}, \tag{4.2-3}$$

$$\mathbf{k}\times\mathbf{H} = -\omega\varepsilon\mathbf{E}. \tag{4.2-4}$$

By eliminating $\mathbf{H}$ from Eqs. (4.2-3) and (4.2-4), we obtain

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) + \omega^2\mu\varepsilon\mathbf{E} = 0. \tag{4.2-5}$$

In the principal coordinate system, the dielectric tensor $\varepsilon$ is given by

$$\varepsilon = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_x \end{pmatrix}. \tag{4.2-6}$$



Equation (4.2-5) can be written as

$$\begin{pmatrix} \omega^2\mu\varepsilon_x - k_y^2 - k_z^2 & k_xk_y & k_xk_z \\ k_yk_x & \omega^2\mu\varepsilon_y - k_x^2 - k_z^2 & k_yk_z \\ k_zk_x & k_zk_y & \omega^2\mu\varepsilon_z - k_x^2 - k_y^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0. \tag{4.2-7}$$

For nontrivial solutions to exist, the determinant of the matrix in Eq. (4.2-7) must vanish. This leads to a relation between $\omega$ and $\mathbf{k}$,

$$\det\begin{vmatrix} \omega^2\mu\varepsilon_x - k_y^2 - k_z^2 & k_xk_y & k_xk_z \\ k_yk_x & \omega^2\mu\varepsilon_y - k_x^2 - k_z^2 & k_yk_z \\ k_zk_x & k_zk_y & \omega^2\mu\varepsilon_z - k_x^2 - k_y^2 \end{vmatrix} = 0. \tag{4.2-8}$$

The above equation can be represented by a three-dimensional surface in $\mathbf{k}$ space (momentum space). This surface is known as the normal surface and consists of two shells, which, in general, have four points in common (see Fig. 4.1). The two lines that go through the origin and these points are known as the optic axes. Figure 4.1 shows one of the optic axes. Given a direction of propagation, there are in general two $k$ values which are the intersections of the direction of propagation and the normal surface. These two $k$ values correspond to two different phase velocities $(\omega/k)$ of the waves propagating along the chosen direction. The directions of the electric field vector associated with these propagations can also be obtained from Eq. (4.2-7) and are given by

$$\begin{pmatrix} \dfrac{k_x}{k^2 - \omega^2\mu\varepsilon_x} \\ \dfrac{k_y}{k^2 - \omega^2\mu\varepsilon_y} \\ \dfrac{k_z}{k^2 - \omega^2\mu\varepsilon_z} \end{pmatrix}. \tag{4.2-9}$$

It will be shown in Section 4.3 that the two phase velocities always correspond to two mutually orthogonal polarizations (for displacement vector $\mathbf{D}$) (see also Problem 4.1).

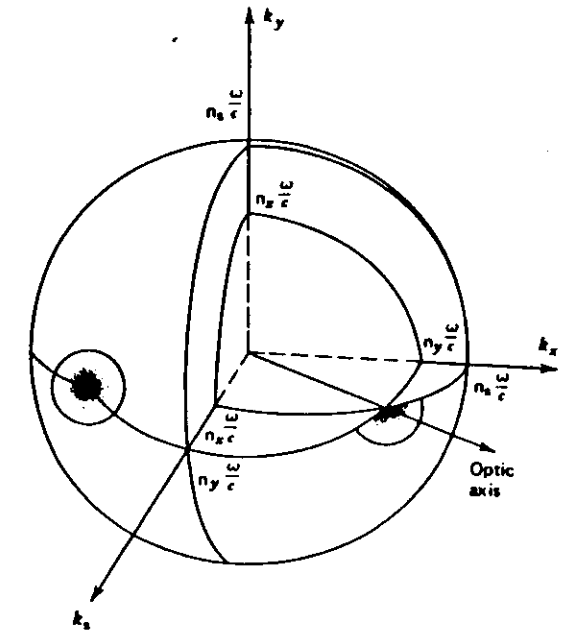



**Figure 4.1.** The normal surface.

For propagation in the direction of the optic axes, there is only one value of $k$ and, consequently, only one phase velocity. There are, however, two independent directions of polarization.

Equations (4.2-8) and (4.2-9) are often written in terms of the direction cosines of the wave vector. By using the relation $\mathbf{k} = (\omega/c)n\mathbf{s}$ for the plane wave given by Eq. (4.2-1), Eqs. (4.2-8) and (4.2-9) can be written as

$$\frac{s_x^2}{n^2 - \epsilon_x/\epsilon_0} + \frac{s_y^2}{n^2 - \epsilon_y/\epsilon_0} + \frac{s_z^2}{n^2 - \epsilon_z/\epsilon_0} = \frac{1}{n^2} \qquad (4.2\text{-}10)$$

and

$$\begin{pmatrix} \dfrac{s_x}{n^2 - \epsilon_x/\epsilon_0} \\ \dfrac{s_y}{n^2 - \epsilon_y/\epsilon_0} \\ \dfrac{s_z}{n^2 - \epsilon_z/\epsilon_0} \end{pmatrix} \qquad (4.2\text{-}11)$$

respectively.



Equation (4.2-10) is known as Fresnel's equation of wave normals and can be solved for the eigenindices of refraction, and Eq. (4.2-11) gives the directions of polarization. Equation (4.2-10) is a quadratic equation in $n^2$. Therefore, for each direction of propagation (set of $s_x$, $s_y$, $s_z$) it yields two solutions for $n^2$ (Problem 4.2). To complete the solution of the problem, we use the values of $n^2$, one at a time, in Eq. (4.2-11). This gives us the polarizations of these waves. It can be seen that in a nonabsorbing medium these eigenwaves are linearly polarized, since all the components are real in (4.2-11). Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be the electric field vectors and $\mathbf{D}_1, \mathbf{D}_2$ be the displacement vectors of the linearly polarized eigenwaves associated with $n_1^2$ and $n_2^2$, respectively. Maxwell's equation $\nabla \cdot \mathbf{D} = 0$ implies that $\mathbf{D}_1, \mathbf{D}_2$ are orthogonal to $\mathbf{s}$. Since $\mathbf{D}_1 \cdot \mathbf{D}_2 = 0$, the three vectors $\mathbf{D}_1$, $\mathbf{D}_2$, and $\mathbf{s}$ form an orthogonal triad and can be used as a coordinate system for the description of many physical phenomena, including optical activity. Maxwell's equations also imply that $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{H}$ are related by

$$\mathbf{D} = -\frac{n}{c}\mathbf{s} \times \mathbf{H} \qquad (4.2\text{-}12)$$

and

$$\mathbf{H} = \frac{n}{\mu c}\mathbf{s} \times \mathbf{E}. \qquad (4.2\text{-}13)$$

According to Eqs. (4.2-12) and (4.2-13), $\mathbf{D}$ and $\mathbf{H}$ are both perpendicular to the direction of propagation $\mathbf{s}$. Consequently, the direction of energy flow as given by the Poynting vector $\mathbf{E} \times \mathbf{H}$ is not, in general, collinear with the direction of propagation $\mathbf{s}$.

By substituting Eq. (4.2-13) for $\mathbf{H}$ in Eq. (4.2-12) and using the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$, we obtain the following expression:

$$\begin{aligned} \mathbf{D} &= -\frac{n^2}{c^2\mu}\mathbf{s} \times (\mathbf{s} \times \mathbf{E}) = \frac{n^2}{c^2\mu}[\mathbf{E} - \mathbf{s}(\mathbf{s} \cdot \mathbf{E})] \\ &= \frac{n^2}{c^2\mu}E_{\text{transverse}}, \end{aligned} \qquad (4.2\text{-}14)$$

and since $\mathbf{s} \cdot \mathbf{D} = 0$ and $n^2/c^2\mu = n^2\epsilon_0$,

$$D^2 = \frac{n^2}{c^2\mu}\mathbf{E} \cdot \mathbf{D} = n^2\epsilon_0 \mathbf{E} \cdot \mathbf{D}. \qquad (4.2\text{-}15)$$

In other words, **D**, **E**, and **s** all lie in the same plane. It can be shown that these field vectors satisfy the following relations (see Problem 4.1):

$$
\begin{aligned}
\mathbf{D}_1 \cdot \mathbf{D}_2 &= 0, \\
\mathbf{D}_1 \cdot \mathbf{E}_2 &= 0, \\
\mathbf{D}_2 \cdot \mathbf{E}_1 &= 0, \\
\mathbf{s} \cdot \mathbf{D}_1 = \mathbf{s} \cdot \mathbf{D}_2 &= 0.
\end{aligned}
\tag{4.2-16}
$$

$\mathbf{E}_1$ and $\mathbf{E}_2$ are in general not orthogonal. The orthogonality relation of the eigenmodes of propagation is often written as

$$\mathbf{s} \cdot (\mathbf{E}_1 \times \mathbf{H}_2) = 0. \tag{4.2-17}$$

This latter relation shows that the power flow in an anisotropic medium along the direction of propagation is the sum of the power carried by each mode individually.

### 4.2.1. Orthogonality Properties of the Eigenmodes

We now derive the orthogonality relation (4.2-17) between the two eigenmodes of propagation along a given direction **s**. Using Eqs. (4.2-1), (4.2-2) for the field vectors and the Lorentz reciprocity theorem, we obtain

$$\mathbf{s} \cdot (\mathbf{E}_1 \times \mathbf{H}_2) = \mathbf{s} \cdot (\mathbf{E}_2 \times \mathbf{H}_1). \tag{4.2-18}$$

If we substitute Eq. (4.2-13) for $\mathbf{H}_1$ and $\mathbf{H}_2$ in Eq. (4.2-18), it becomes

$$\frac{n_2}{\mu c} \mathbf{s} \cdot [\mathbf{E}_1 \times (\mathbf{s} \times \mathbf{E}_2)] = \frac{n_1}{\mu c} \mathbf{s} \cdot [\mathbf{E}_2 \times (\mathbf{s} \times \mathbf{E}_1)]. \tag{4.2-19}$$

This equation can be further simplified by using the identity

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}),$$

and becomes

$$\frac{n_2}{\mu c} (\mathbf{s} \times \mathbf{E}_1) \cdot (\mathbf{s} \times \mathbf{E}_2) = \frac{n_1}{\mu c} (\mathbf{s} \times \mathbf{E}_1) \cdot (\mathbf{s} \times \mathbf{E}_2). \tag{4.2-20}$$

Since this equation must hold for any arbitrary direction of propagation **s** with $n_1 \neq n_2$, it can be satisfied only when both sides vanish. This proves



$$\mathbf{s} \cdot (\mathbf{E}_1 \times \mathbf{H}_2) = \mathbf{s} \cdot (\mathbf{E}_2 \times \mathbf{H}_1) = 0. \tag{4.2-21}$$

To summarize: along an arbitrary direction of propagation **s**, there can exist two independent plane-wave, linearly polarized propagation modes. These modes have phase velocities $\pm c/n_1$ and $\pm c/n_2$, where $n_1^2$ and $n_2^2$ are the two solutions of Fresnel's equation (4.2-10).

In practice, the indices of refraction $n_1$, $n_2$ and the directions of **D**, **H**, and **E** are found, most often, not by the procedure outlined above but by using the formally equivalent method of the index ellipsoid. This method is discussed in the following section.

## 4.3. THE INDEX ELLIPSOID

The surfaces of constant energy density $U_e$ in **D** space given by Eq. (4.1-7) can be written as

$$\frac{D_x^2}{\varepsilon_x} + \frac{D_y^2}{\varepsilon_y} + \frac{D_z^2}{\varepsilon_z} = 2U_e,$$

where $\varepsilon_x$, $\varepsilon_y$, and $\varepsilon_z$ are the principal dielectric constants. If we replace $\mathbf{D}/\sqrt{2U_e}$ by **r** and define the principal indices of refraction $n_x$, $n_y$, and $n_z$ by $n_i^2 \equiv \varepsilon_i/\varepsilon_0$ $(i = x, y, z)$, the last equation can be written as

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1. \tag{4.3-1}$$

This is the equation of a general ellipsoid with major axes parallel to the $x$, $y$, and $z$ directions whose respective lengths are $2n_x$, $2n_y$, $2n_z$. The ellipsoid is known as the index ellipsoid or, sometimes, as the optical indicatrix. The index ellipsoid is used mainly to find the two indices of refraction and the two corresponding directions of **D** associated with the two independent plane waves that can propagate along an arbitrary direction **s** in a crystal. This is done by means of the following prescription: Find the intersection ellipse between a plane through the origin that is normal to the direction of propagation **s** and the index ellipsoid (4.3-1). The two axes of the intersection ellipse are equal in length to $2n_1$ and $2n_2$, where $n_1$ and $n_2$ are the two indices of refraction, that is, the solutions of (4.2-10). These axes are parallel, respectively, to the directions of the vectors $\mathbf{D}_{1,2}$ of the two allowed solutions (see Fig. 4.2).

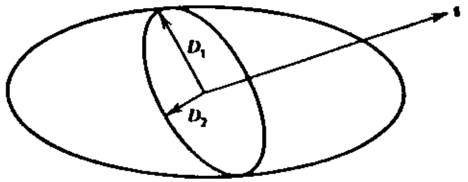



**Figure 4.2.** Method of the index ellipsoid. The inner ellipse is the intersection of the index ellipsoid with the plane normal to s.

To show that this procedure is formally equivalent to the method of the last section, we define the impermeability tensor $\eta_{ij}$ as

$$\eta_{ij} = \epsilon_0(\epsilon^{-1})_{ij}, \tag{4.3-2}$$

where $\epsilon^{-1}$ is the inverse of the dielectric tensor $\epsilon$. By using this definition, the relation between the field vectors **E** and **D** can be written

$$\mathbf{E} = \frac{1}{\epsilon_0}\eta\mathbf{D}. \tag{4.3-3}$$

Substitution of Eq. (4.3-3) for **E** in the wave equation (4.2-5), leads to

$$\mathbf{s} \times [\mathbf{s} \times \eta\mathbf{D}] + \frac{1}{n^2}\mathbf{D} = 0, \tag{4.3-4}$$

where we have used $\mathbf{k} = n(\omega/c)\mathbf{s}$, and **s** is a unit vector in the direction of propagation. Since **D** is always transverse to the direction of propagation ($\mathbf{s} \cdot \mathbf{D} = 0$), it is convenient to use a new coordinate system with one axis in the direction of propagation of the wave, and denote the two transverse axes by 1 and 2. In this coordinate system the unit vector **s** is given by

$$\mathbf{s} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{4.3-5}$$

and the wave equation (4.3-4) becomes

$$\begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ 0 & 0 & 0 \end{pmatrix}\mathbf{D} = \frac{1}{n^2}\mathbf{D}. \tag{4.3-6}$$

Since $\mathbf{s} \cdot \mathbf{D} = 0$, the third component of **D** is always zero. We can ignore



$\eta_{13}$, $\eta_{23}$ and define a transverse impermeability tensor $\eta_t$ as

$$\eta_t = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}. \tag{4.3-7}$$

The wave equation then becomes

$$\left(\eta_t - \frac{1}{n^2}\right)\mathbf{D} = 0 \tag{4.3-8}$$

where **D** is the displacement field vector.

The polarization vectors of the normal modes are eigenvectors of the transverse impermeability tensor with eigenvalues $1/n^2$. Since $\eta_t$ is a symmetric $2 \times 2$ tensor, there are two orthogonal eigenvectors. These two eigenvectors, $\mathbf{D}_1$ and $\mathbf{D}_2$, correspond to the two normal modes of propagation with refractive indices $n_1$ *and* $n_2$, respectively.

Let $\xi_1$, $\xi_2$, $\xi_3$ be the coordinates of an arbitrary point in the new coordinate system. The index ellipsoid in this coordinate system is expressed by

$$\eta_{\alpha\beta}\xi_\alpha\xi_\beta = 1, \tag{4.3-9}$$

where summation over repeated indices $\alpha, \beta$ (1, 2, 3) is assumed. The intersection ellipse between a plane ($\xi_3 = 0$) through the origin that is normal to the direction of propagation and the index ellipsoid is obtained by putting $\xi_3 = 0$ in Eq. (4.3-9). Thus we obtain the following equation for the intersection ellipse

$$\eta_{11}\xi_1^2 + \eta_{22}\xi_2^2 + 2\eta_{12}\xi_1\xi_2 = 1. \tag{4.3-10}$$

The coefficients of this ellipse form the transverse impermeability tensor $\eta_t$. The eigenvectors of this $2 \times 2$ tensor therefore are along the principal axes of this ellipse. The lengths of the principal axes determine the values of $n$ according to Eq. (4.3-8). This proves the equivalence of the method of the index ellipsoid and the method of the last section.

# Electro-optics

. we treated the propagation of electromagnetic radiation in anisotropic crystal media. It was shown that the normal modes of propagation can be determined from the index ellipsoid surface. In this chapter we consider the problem of propagation of optical radiation in crystals in the presence of an applied electric field. We find that, in certain types of crystals, the application of an electric field results in a change in both the dimensions and orientation of the index ellipsoid. This is referred to as the electro-optic effect. The electro-optic effect affords a convenient and widely used means of controlling the phase or intensity of the optical radiation. This modulation is used in an ever-expanding number of applications, including the impression of information onto optical beams, optical beam deflection, and spectral tunable filters. Some of these applications will be discussed further in the next chapter.

## 7.1. THE ELECTRO-OPTIC EFFECT

The propagation of optical radiation in a crystal can be described completely in terms of the impermeability tensor $\eta_{ij}$ (4.3-2). We recall that $\eta = \varepsilon_0\varepsilon^{-1}$. The two directions of polarization as well as the corresponding indices of refraction (i.e., velocity of propagation) of the normal modes are found most easily by using the index ellipsoid (4.3-9). The index ellipsoid assumes its simplest form in the principal coordinate system:

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1, \qquad (7.1\text{-}1)$$

where the directions $x$, $y$, and $z$ are the principal axes—that is, the directions in the crystal along which **D** and **E** are parallel. $1/n_x^2$, $1/n_y^2$, and $1/n_z^2$ are the principal values of the impermeability tensor $\eta_{ij}$.





According to the quantum theory of solids, the optical dielectric impermeability tensor depends on the distribution of charges in the crystal. The application of an electric field will result in a redistribution of the bond charges and possibly a slight deformation of the ion lattice. The net result is a change in the optical impermeability tensor. This is known as the electro-optic effect. The electro-optic coefficients are defined traditionally as

$$\eta_{ij}(\mathbf{E}) - \eta_{ij}(0) \equiv \Delta\eta_{ij} = r_{ijk}E_k + s_{ijkl}E_kE_l$$
$$= f_{ijk}P_k + g_{ijkl}P_kP_l, \qquad (7.1\text{-}2)$$

where **E** is the applied electric field and **P** is the polarization field vector. The constants $r_{ijk}$ and $f_{ijk}$ are the linear (or Pockels) electro-optic coefficients, and $s_{ijkl}$ and $g_{ijkl}$ are the quadratic (or Kerr) electro-optic coefficients. In the above expansion, terms higher than the quadratic are neglected because these higher-order effects are too small for most applications. The quadratic effect was first discovered by J. Kerr, in 1875, in optically isotropic media such as liquids and glasses. The Kerr electro-optic effect in liquids is associated mostly with the alignment of the anisometric molecules in the presence of an electric field. The substance then behaves optically as if it were a uniaxial crystal in which the electric field defines the optic axis. The linear electro-optic effect was first studied by F. Pockels in 1893.

## 7.2. THE LINEAR ELECTRO-OPTIC EFFECT

We have mentioned in the last section that the electro-optic effect results from the redistribution of charges due to the application of a dc electric field. It may be expected that the electro-optic effect will depend on the ratio of the applied electric field to the intraatomic electric field binding the charged particles such as electrons and ions. In most practical applications of the electro-optic effect, the applied electric field is small compared with the electric field inside the atom, which is typically of the order of $10^8$ V/cm. As a result, the quadratic effect is expected to be small compared to the linear effect and is often neglected when the linear effect is present. However, in crystals with centrosymmetric point groups, the linear electro-optic effect vanishes and the quadratic effect becomes the dominant phenomenon.

**Electro-Optic Modulators Using Cubic Crystals.** Cubic crystals are optically isotropic (no birefringence) and therefore offer a wide field of view in many optical systems. Here we consider the case of crystals of the $\bar{4}3m$ symmetry (zinc blende) group. Examples of this group are InAs, CuCl, GaAs, and CdTe. The last two are used for modulation in the infrared, since they remain transparent beyond 10 $\mu$m. These crystals are cubic and have axes of fourfold symmetry along the cube edges (i.e., ⟨100⟩, ⟨010⟩, ⟨001⟩ directions), and threefold axes of symmetry along the cube diagonals (i.e., ⟨111⟩, ⟨$\bar{1}$11⟩, ⟨1$\bar{1}$1⟩, ⟨1$\bar{1}$1⟩ directions).

According to Table 7.2 and Eq. (7.2-3), the index ellipsoid in the presence of the electric field is

$$\frac{x^2+y^2+z^2}{n^2}+2r_{41}(yzE_x+zxE_y+xyE_z)=1, \qquad (8.1\text{-}24)$$

where $E_x$, $E_y$, and $E_z$ are the components of the field along the crystal axes, and $r_{41}$ is the electro-optic coefficient. In this case, all the three variables $(x, y, z)$ are coupled. The general approach to transform Eq. (8.1-24) into its diagonal form is to solve the following eigenvalue problem:

$$\begin{pmatrix} \frac{1}{n^2} & r_{41}E_z & r_{41}E_y \\ r_{41}E_z & \frac{1}{n^2} & r_{41}E_x \\ r_{41}E_y & r_{41}E_x & \frac{1}{n^2} \end{pmatrix} \mathbf{V} = \frac{1}{n'^2}\mathbf{V}. \qquad (8.1\text{-}25)$$

The eigenvectors **V** are the new principal axes, and the eigenvalues $n'$ are the new principal indices of refraction. To be specific, we consider the case when the electric field is in the ⟨110⟩ direction. Taking the field magnitude as $E$, we have

$$E_x = E_y = \frac{1}{\sqrt{2}}E, \qquad E_z = 0. \qquad (8.1\text{-}26)$$

Substitution of Eq. (8.1-26) for $E_x$, $E_y$, and $E_z$ into Eq. (8.1-25) leads to the following secular equation:

$$\begin{vmatrix} \frac{1}{n^2}-\frac{1}{n'^2} & 0 & \frac{1}{\sqrt{2}}r_{41}E \\ 0 & \frac{1}{n^2}-\frac{1}{n'^2} & \frac{1}{\sqrt{2}}r_{41}E \\ \frac{1}{\sqrt{2}}r_{41}E & \frac{1}{\sqrt{2}}r_{41}E & \frac{1}{n^2}-\frac{1}{n'^2} \end{vmatrix} = 0. \qquad (8.1\text{-}27)$$

The roots of this equation are the principal indices of refraction and are given by

$$\begin{aligned} n_{x'} &= n + \tfrac{1}{2}n^3 r_{41}E, \\ n_{y'} &= n - \tfrac{1}{2}n^3 r_{41}E, \qquad (8.1\text{-}28) \\ n_{z'} &= n. \end{aligned}$$

The new principal axes are given by

$$\begin{aligned} x' &= \tfrac{1}{2}x + \tfrac{1}{2}y - \frac{1}{\sqrt{2}}z, \\ y' &= \tfrac{1}{2}x + \tfrac{1}{2}y + \frac{1}{\sqrt{2}}z, \qquad (8.1\text{-}29) \\ z' &= \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y. \end{aligned}$$

An amplitude modulator based on the foregoing situation is shown in Fig. 8.4. The phase retardation is

$$\Gamma = \frac{2\pi}{\lambda}n^3 r_{41}\left(\frac{L}{d}\right)V. \qquad (8.1\text{-}30)$$

Table 8.1 summarizes the phase-retardation and electro-optical properties of $\bar{4}3m$ crystals with the field along ⟨001⟩, ⟨110⟩, and ⟨111⟩ directions. It is seen that the maximum achievable phase retardation is given by Eq. (8.1-30). The half-wave voltage is given by

$$V_\pi = \frac{d}{L}\cdot\frac{\lambda}{2n^3 r_{41}}. \qquad (8.1\text{-}31)$$

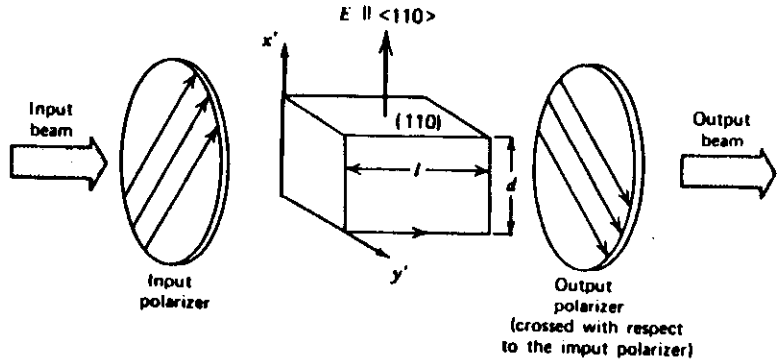
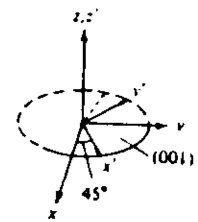
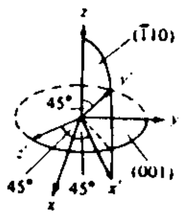
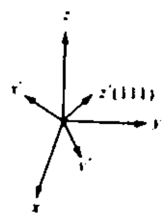
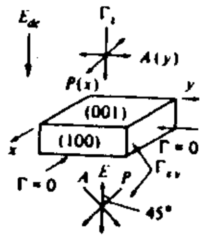
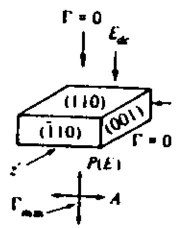
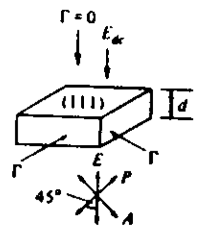



**Figure 8.4.** A transverse electro-optic modulator using a zinc-blende-type ($\bar{4}3m$) crystal with E parallel to a cube diagonal (⟨110⟩ direction).

**Table 8.1. Electro-optical Properties and Retardation in $\bar{4}3m$ (Zinc-Blende-Structure) Crystals for Three Directions of Applied Field**

| | $E \perp (001)$ plane | $E \perp (110)$ plane | $E \perp (111)$ plane |
|---|---|---|---|
| | $E_x = E_y = 0,\ E_z = E$ | $E_x = E_y = \dfrac{E}{\sqrt{2}},\ E_z = 0$ | $E_x = E_y = E_z = \dfrac{E}{\sqrt{3}}$ |
| Index ellipsoid | $\dfrac{x^2 + y^2 + z^2}{n_o^2} + 2 r_{41} E xy = 1$ | $\dfrac{x^2 + y^2 + z^2}{n_o^2} + \sqrt{2} r_{41} E(yz + zx) = 1$ | $\dfrac{x^2 + y^2 + z^2}{n_o^2} + \dfrac{2}{\sqrt{3}} r_{41} E(yz + zx + xy) = 1$ |
| $n'_x$ | $n_o + \frac{1}{2} n_o^3 r_{41} E$ | $n_o + \frac{1}{2} n_o^3 r_{41} E$ | $n_o + \dfrac{1}{2\sqrt{3}} n_o^3 r_{41} E$ |
| $n'_y$ | $n_o - \frac{1}{2} n_o^3 r_{41} E$ | $n_o - \frac{1}{2} n_o^3 r_{41} E$ | $n_o + \dfrac{1}{2\sqrt{3}} n_o^3 r_{41} E$ |
| $n'_z$ | $n_o$ | $n_o$ | $n_o - \dfrac{1}{\sqrt{3}} n_o^3 r_{41} E$ |
| Coordinates $x'y'z'$ |  |  |  |
| Directions of optical path and axes of crossed polarizer |  |  |  |
| Phase retardation $\Gamma\ (V = Ed)$ | $\Gamma_z = \dfrac{2\pi}{\lambda} n_o^3 r_{41} V$; $\Gamma_{x'y'} = \dfrac{\pi}{\lambda} \dfrac{L}{d} n_o^3 r_{41} V$ | $\Gamma_{max} = \dfrac{2\pi}{\lambda} \dfrac{L}{d} n_o^3 r_{41} V$ | $\Gamma = \dfrac{\sqrt{3}\pi}{\lambda} \dfrac{L}{d} n_o^3 r_{41} V$ |