INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
CABLE: CENTRATOM - TELEX 46392 ICTP

SMR-30/ 23

WINTER COLLEGE ON ATOMIC AND MOLECULAR PHYSICS

26 January - 18 March 1977

APPENDIX TO PROF. ARRECHI'S LECTURE NOTES

F.T. ARECCHI

Istituto Nazionale di Ottica
Firenze

and

C.I.S.E.
Milano

Appendix - Markov (M.) processes & Fokker-Planck equation

Def. of M. process

def. of conditional probability

$$(A\text{-}1)\quad W_n(y_1\, y_2 \dots y_n) \equiv W_{n-1}(y_1 \dots y_{n-1})\, P_c(\cancel{\boxed{y_1 \dots}}\, y_{n-1} \,|\, y_n)$$

M. condition
≡ short memory

Hence

$$(A\text{-}2)\quad W_3(y_1\, y_2\, y_3) = W_2(y_1, y_2)\, P_c(y_2 \,|\, y_3)$$

$$= W_2(y_1, y_2)\, \frac{W_2(y_2, y_3)}{W_1(y_2)}$$

Properties of P_c

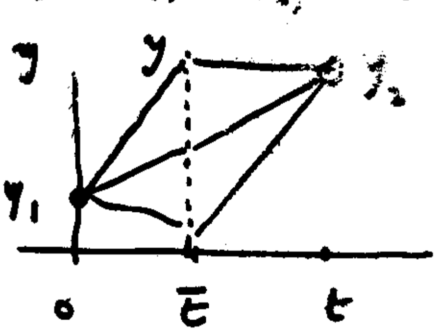
a) $P_c(y_1 \,|\, y_2) \geqslant 0$

b) $\int dy_2\, P_c = 1$

c) $\lim\limits_{t\to\infty} P_c(y_1 \,|\, y_2, t) \equiv W_1(y_2)$

d) $P(y_1, 0 \,|\, y_2\, t) = \int dy\, P(y_1 | y, \bar t)\, P(y, \bar t \,|\, y_2, t-\bar t)$

Smolukowsky eq.



Indeed

$$P(1|2) = \frac{W_2(1,2)}{W_1(1)} \equiv \frac{\int dy\, W_3(1, y, 2)}{W_1(1)}$$

$$\overset{\text{by } 2)}{=} \int \frac{W_2(1,y)}{W_1(1)} \frac{W_2(y,2)}{W_1(y)}\, dy = (d)$$

For arbitrary $R(y)$ $\quad \left(R(y) \xrightarrow[t\to\pm\infty]{} 0 \right)$ consider

$$\int dy\, R(y) \frac{\partial P(x|y\,t)}{\partial t} = \lim \frac{1}{\Delta t} \int dy\, R(y) \left[P(x|y, t+\Delta t) - P(x|y,t) \right]$$

d)

$$= \lim \frac{1}{\Delta t} \left\{ \int dy\, R(y) \int dz\, P(x|z,t) P(z|y,\Delta t) - \int dz\, R(z) P(x|z\,t) \right\}$$

write $\quad R(y) \simeq R(z) + R'(z)(z-y) + \frac{1}{2} R''(z)(z-y)^2$

Define transitional moments over the elementary interval Δt:

(A-3) $\quad A(z) \equiv \frac{1}{\Delta t} \langle (z-y) \rangle_{\Delta t} = \lim \frac{1}{\Delta t} \int dy\, (z-y) P(z|y,\Delta t)$

(A-4) $\quad B(z) \equiv \frac{1}{\Delta t} \langle (z-y)^2 \rangle_{\Delta t} = \lim \frac{1}{\Delta t} \int dy\, (z-y)^2 P(z|y\,\Delta t)$

<u>Assume</u>

$$\frac{1}{\Delta t} \langle (z-y)^{n \geq 3} \rangle_{\Delta t} = 0$$

Then (integrate by parts)

$$\int dy\, R(y) \Big(\underbrace{\frac{\partial P}{\partial t} + \frac{\partial}{\partial y} A P - \frac{1}{2} \frac{\partial^2}{\partial y^2} B P}_{\equiv 0} \Big) = 0$$

(A-5) Fokker - Planck eq.

Brownian motion of free particle

(A-6) $$\dot{y} + \gamma y = F(t)$$

(A-7) $\langle F \rangle = 0$

(A-8) $\langle F(t) F(t') \rangle = 2D \, \delta(t-t')$

Gaussian
$$\begin{cases} \langle \underbrace{F \cdots F}_{2n+1} \rangle = 0 \\ \langle \underbrace{F \cdots F}_{2n} \rangle = \sum\limits_{P} \langle FF \rangle \cdots \langle FF \rangle \end{cases}$$

Fluctuation-dissipation theorem (Kubo)

(A-9) $$\frac{2D}{\gamma} = k_B T$$

$$\Delta y = -\gamma y \, \Delta t + \int_t^{t+\Delta t} dt' \, F(t')$$

(A-10) $$A = \lim \frac{\langle \Delta y \rangle}{\Delta t} = -\gamma y$$

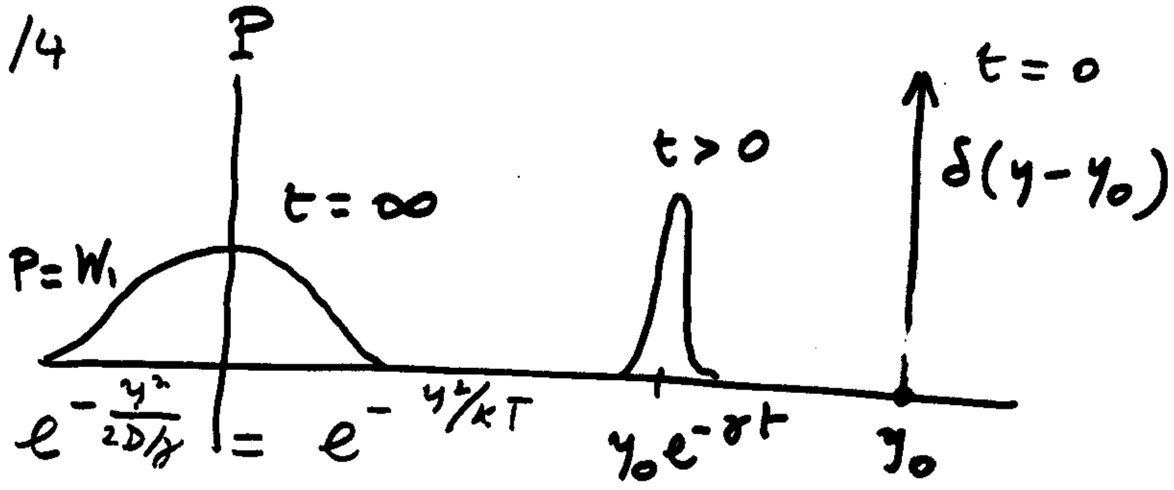
$$\langle \Delta y^2 \rangle = \gamma^2 y^2 \Delta t^2 + \iint dt' dt'' \langle F(t') F(t'') \rangle$$

(A-11) $$B = \lim \frac{\langle \Delta y^2 \rangle}{\Delta t} = 2D$$

(A-12) $$\partial P / \partial t = \gamma \frac{\partial}{\partial y} (y P) + D \frac{\partial^2 P}{\partial y^2}$$ + $P(y_0 | y, t) = \delta(y - y_0)$

(A-13) $$P(y_0 | y t) = \left[2\pi \frac{D}{\gamma} (1 - e^{-2\gamma t}) \right]^{-1} e^{-\frac{(y - y_0 e^{-\gamma t})^2}{\frac{2D}{\gamma}(1 - e^{-2\gamma t})}}$$

F. P. /4



<u>Harmonic oscillator</u>

$$\begin{cases} \dot q = p \\ \dot p = -\omega^2 q \underbrace{-\gamma p + \vec{F}} \end{cases} \qquad \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} = \frac{\omega q \pm i p}{\sqrt{2\omega}} e^{\mp i\omega t}$$

(A.14)

$$\begin{cases} \dot\alpha + \gamma\alpha = \Gamma \\ \dot\alpha^* + \gamma\alpha^* = \Gamma^* \end{cases}$$

$$\langle \Gamma(t)\Gamma^*(t')\rangle = 4q\,\delta(t-t') \qquad \boxed{\frac{2q}{\gamma} = kT} \equiv \langle n\rangle$$

(A.15) $$\frac{\partial P}{\partial t} = \gamma\, \mathrm{div}_\alpha(\alpha P) + q\,\nabla_\alpha^2 P$$

(A.16) $$F(\alpha_0|\alpha t) = \frac{1}{\pi\langle n\rangle(1-e^{-2\gamma t})}\, e^{-\frac{|\alpha-\alpha_0 e^{-\gamma t}|^2}{\langle n\rangle(1-e^{-2\gamma t})}}$$