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WINTER COLLEGE ON ATOMIC AND MOLECULAR PHYSICS

26 January - 18 March 1977

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BACKGROUND TAKEN FROM Physics of Quantum Electronics, 1965)

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Quantum Theory of an Optical Maser*

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ABSTRACT

The semiclassical theory of an optical maser as given by Lamb[1] is here generalized by quantizing the radiation field. The density matrix for a laser oscillator is obtained; the resulting photon statistics are discussed and the decay times for all degrees of correlation given.

1. INTRODUCTION

Introductory Remarks

To set the stage for this work we quote from a recent article:[2] "The only reliable method we have of constructing density operators, in general, is to devise theoretical models of the system under study and to integrate [the] corresponding Schrödinger equation, or equivalently to solve the equation of motion for the density operator. These assignments are formidable ones for the case of the laser oscillator and have not been carried out to date in quantum mechanical terms. The greatest part of the difficulty lies in the mathematical complications associated with the nonlinearity of the device. The nonlinearity plays an essential role in stabilizing the field gen-

*This research has been supported in part by the U.S. Air Force Office of Scientific Research and by the National Aeronautics and Space Administration.

erated by the laser. It seems unlikely, therefore, that we shall have a quantum mechanically consistent picture of the frequency bandwidth of the laser or of the fluctuations of its output until further progress is made with these problems.'' It is this task to which we wish to address ourselves.

A recently introduced model[1] of an optical maser treats the atoms quantum-mechanically while considering the radiation as a classical electromagnetic field damped by ohmic currents. This theory gives a realistic account of the behavior of gas lasers but ignores quantum electrodynamic fluctuation phenomena. It is the purpose of the present work to generalize the theory by quantizing the radiation field. The density-matrix approach is well suited to many-body problems and in fact is absolutely essential for a complete description of the problem at hand.

There are several obvious extensions of this work which space does not allow us to include; these are contained in a more detailed publication to follow.

The Model and Formalism

To describe laser oscillation, the theory must include a nonlinear active medium and a damping mechanism. To obtain laser pumping action we introduce two-level atoms in their upper level ψ_a at random times t_i, decaying to a far-removed ground state by radiative decay. To simplify the presentation we consider static atoms interacting with a single mode of the field; a later publication will include atomic motion and multimode operation.

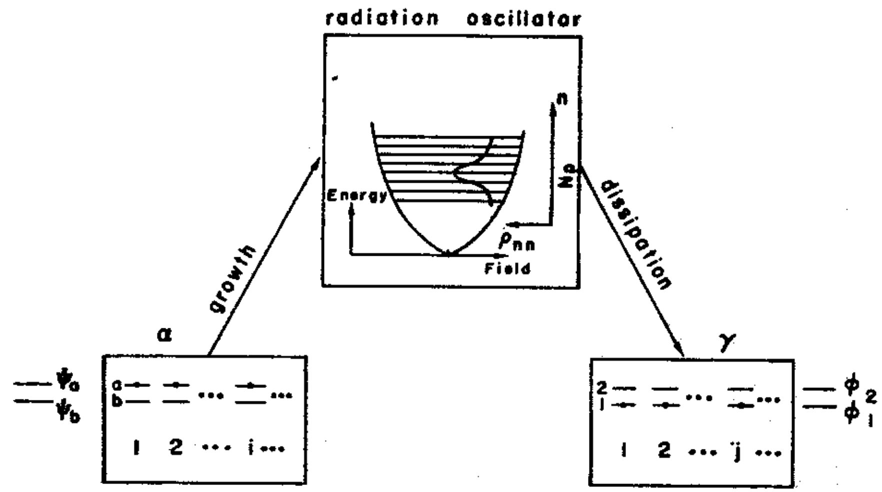
In the semiclassical theory[1] one is not very interested in the details of the dissipation mechanism, which is represented by ohmic currents; likewise in the present work we are uninterested in the details of the dissipation mechanism. The damping is included in our equations by coupling the field to an ensemble of atoms in their ground state (γ subsystem of Fig. 1). The γ atoms may be considered as atoms entering in the lower state φ_1 of a two-level system.

The coupling between the subsystems is a dipole interaction between the single mode of the field and each of the active and dissipative atoms.

$$H = \nu_k a_k^\dagger a_k + \sum_{\alpha=a,b} \sum_i \epsilon_\alpha b_\alpha^{i\dagger} b_\alpha^i + \sum_{\gamma=1,2} \sum_j \epsilon_\gamma c_\gamma^{j\dagger} c_\gamma^j \tag{1a}$$

$$+ (-i)\sum_i g b_b^{i\dagger} b_a^i a_k^\dagger - g b_a^{i\dagger} b_b^i a_k + (-i)\sum_j g' c_1^{j\dagger} c_2^j a_k^\dagger - g'^* c_2^{j\dagger} c_1^j a_k$$

$$= H_0 + V \tag{1b}$$



$[b_\alpha^{\,i}, b_{\alpha'}^{i'+}] = \delta_{\alpha\alpha'}\delta_{ii'}$ $\qquad$ $[c_\gamma^{\,j}\; c_{\gamma'}^{j'+}] = \delta_{\gamma\gamma'}\delta_{jj'}$

Fig. 1. Model.

where a_k, b_α^i, c_γ^j are, respectively, the annihilation operators of the radiation in the kth mode of the cavity, the ith active atom in its α state, and the jth damping atom in its γ state.

After contracting over the atomic states, the equation of motion for the radiation field density matrix is

$$\dot{\rho}_{n,n'}(t) = -\sum_{\{\alpha\}} \sum_{\{\gamma\}} [V(t), \rho(t)]_{\{\alpha\}\{\gamma\}n,\{\alpha\}\{\gamma\}n'} \tag{2}$$

where $V(t) = e^{iH_0t}\left(\sum_i V_\alpha^i + \sum_j V_\gamma^j\right)e^{-iH_0t}$

$$= \sum_i V_\alpha^i(t) + \sum_j V_\gamma^j(t)$$

and $\{\alpha\} = \alpha_1\alpha_2 \ldots \alpha_N \qquad \{\gamma\} = \gamma_1\gamma_2 \ldots \gamma_N$

Equation (2) may be written as the sum of interactions with individual atoms

$$\dot{\rho}_{n,n'}(t) = -i \sum_{\alpha_i=a,b} \sum_i [V_\alpha^i(t), \rho^i(t)]_{\alpha_i n,\alpha_i n'} +$$

$$-i \sum_{\gamma_j=1,2} \sum_j [V_\gamma^j(t), \rho^j(t)]_{\gamma_j n,\gamma_j n'} \tag{3}$$

where, for example, $\rho^i = \sum'_{\{\alpha\}'} \sum_{\{\gamma\}} \rho_{\{\gamma\}\{\alpha\}',\{\gamma\}\{\alpha\}'}$ is the projection onto the space of the ith atom and the field.

2. EQUATIONS OF MOTION

Spirit of the Calculation

As this work is the quantized version of the semiclassical theory of Ref. 1, the spirit of the present calculation may best be conveyed by pointing out the quantum analog of the essential steps in the earlier theory. The equation of motion for the radiation field density matrix, Eq. (3), is the quantum analog of the forced, damped harmonic oscillator (Eq. 7, Ref. 1)

$$\ddot{A}_n + (\sigma/\epsilon_0)\dot{A}_n + \Omega_n^2 A_n = (\nu^2/\epsilon_0) P_n$$

In the semiclassical theory one found P_n by following the evolution of the density matrix for the atoms interacting with the classical field; the quantum analog of this is to calculate ρ^i (atom, field) in the same spirit. In Ref. 1 we used the condition of slowly varying amplitudes and phases; here we note that the atomic lifetimes are much smaller than the times characteristic of the radiation density matrix (in the interaction picture); that is to say, the effect of one atom acting on the field will be small, and only when many atoms act on the field will macroscopic effects be observed. Finally, we do the sums over the atoms ($\sum_i \sum_j$) and obtain an equation for the radiation density matrix which depends on lumped constants but contains no reference to individual atoms.

Of the approximations that are commonly made when dealing with density matrices we wish to emphasize that we are not making the following:

1. Assume the random phase approximation (R.P.A.) $\rho_{n,n'} = 0$ if $n \neq n'$.

2. Factor the density matrix $\rho(\text{atom, field}) = \rho(\text{atom}) \otimes \rho(\text{field})$.

Obviously, the R.P.A. will not be valid. We need more than just the diagonal elements since the average value of the E operator is

$$\langle E(t) \rangle = -i(\nu/2\epsilon_0)^{1/2} \sum_n [\rho_{n,n+1}(t)\, a^{\dagger}_{n+1,n} - \rho_{n,n-1}(t)\, a_{n-1,n}] \tag{4}$$

As we shall see, a theory which inevitably leads to a vanishing value for $\langle E \rangle$ is missing a great deal of the physics of the problem. Many workers make the approximation that the density matrix factors. We have shown that this factorization[3] leads to the results of the semiclassical treatment.

Laser Dynamics

Beginning with Eq. (3) and making approximations consistent with the spirit of this study (as sketched in Sec. 2), one finds after a rather lengthy calculation an equation of motion for the radiation density matrix (given below only for the case $\omega = \epsilon_a - \epsilon_b = \nu$, i.e., at resonance)[4]

$$\begin{aligned} \dot{\rho}_{n,n'}(t) = &-[C_{n,n'}(n+1) + C_{n',n}(n'+1)]\rho_{n,n'} \\ &+ (C_{n-1,n'-1}\sqrt{nn'} + C_{n'-1,n-1}\sqrt{n'n})\rho_{n-1,n'-1} \\ &-(\gamma/2)(n+n')\rho_{n,n'} + \gamma\sqrt{(n+1)(n'+1)}\,\rho_{n+1,n'+1} \end{aligned} \tag{5}$$

where $C_{n,n'} = \alpha/2 - (\beta/8)[(n+1+n'+1) + (n'+1+n'+1)]$

and the parameters α, β, and γ are given by the equations

$$\alpha = \nu\bar{N}\frac{\wp^2}{\epsilon_0}\frac{1}{\gamma_a} \qquad \beta = \frac{3}{2}\bar{N}\left(\frac{\nu}{\epsilon_0}\right)^2 \frac{\wp^4}{\gamma_a^3} \qquad \text{and} \qquad \gamma = \frac{\nu}{Q}$$

which are written in terms of the notation of the earlier theory (with $\hbar = 1$) for the dipole matrix element $\wp$, excitation density $\bar{N}$, and decay constant (for simplicity we take $\gamma_a = \gamma_b$).

An important point is that only terms of the density matrix $\rho_{n,n'}$ of equal degree of "off diagonality" (i.e., of equal $k = n - n'$) are coupled.

Expecially interesting are the diagonal equations implied by Eq. (5)

$$\begin{aligned} \dot{\rho}_{n,n} = &-\underbrace{[\alpha - \beta(n+1)](n+1)\rho_{n,n} + [\alpha - \beta n]\, n\rho_{n-1,n-1}}_{\text{pumping}} \\ &\underbrace{-\gamma n\rho_{n,n} + \gamma(n+1)\rho_{n+1,n+1}}_{\text{damping}} \end{aligned} \tag{6}$$

The terms have been grouped to make the physical interpretation obvious. The time rates of change of probability for the nth level of the radiation oscillator are further depicted in Fig. 2.

The equations for the off-diagonal elements are similar but less intuitively obvious.

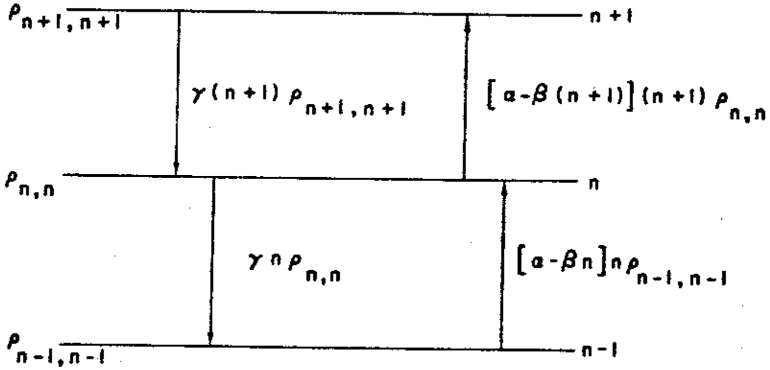


Fig. 2.

3. DISCUSSION OF EQUATIONS OF MOTION

Diagonal Equations

These equations have transient solutions; which would describe, for example, the buildup from vacuum. Here we shall limit the discussion to the steady-state situation.

It appears from Fig. 2 that the steady-state equations are equivalent to the two identical systems of equations implied by detailed balance

$$[\alpha - \beta^n]n\rho_{n-1,n-1} - \gamma n\rho_{n,n} = 0 \tag{7a}$$

$$-[\alpha - \beta(n + 1)]\,(n + 1)\rho_{n,n} + \gamma(n + 1)\rho_{n+1,n+1} = 0 \tag{7b}$$

The solution to these difference equations is clearly[5]

$$\rho_{n,n} = \frac{\gamma}{\alpha}\rho_{0,0}\prod_{m=0}^{n}\frac{\alpha - \beta m}{\gamma} \tag{8}$$

From the normalization condition, $\sum_n \rho_{n,n} = 1$, we find

$$\rho_{0,0} = \left(\frac{\gamma}{\alpha}\sum_{n=0}^{\infty}\prod_{m=0}^{n}\frac{\alpha - \beta m}{\gamma}\right)^{-1} \tag{9}$$

There are two cases of interest implied by Eq. (8):

1. $\alpha/\gamma < 1$, the distribution is peaked at n = 1 (below threshold).
2. $\alpha/\gamma > 1$, the distribution is peaked[6] at $n = n_p = (\alpha - \gamma)/\beta$ (above threshold).

For large n_p the dependence of $\rho_{n,n}$ on n (photon statistics) is indicated by the laser distribution in Fig. 3. The width of this distribution σ is given by the equation

$$\sigma = 2\sqrt{n_p}\left(\frac{\gamma}{\alpha - \gamma}\right)^{1/2} \tag{10}$$

Assuming that one is 10 per cent above threshold, this is about three times the width of a Poisson distribution peaked at n_p.[7]

We would like to emphasize that the effects of saturation are crucial in obtaining the proper photon statistics for a laser oscillator. The distribution characterizing a linear theory is found from our results by letting $\beta \to 0$

$$\rho_{n,n} = \lim_{\beta\to 0}\rho_{0,0}\prod_{m=0}^{n}\frac{\alpha - \beta m}{\gamma} = \left(1 - \frac{\alpha}{\gamma}\right)\left(\frac{\alpha}{\gamma}\right)^n \tag{11}$$

If we define $(\alpha/\gamma) = \exp - \hbar\nu/kT$ the above density matrix is seen to describe single-frequency blackbody radiation. We compare this blackbody (linear) distribution with the laser (nonlinear) distribution in Fig. 3.

Off-Diagonal Elements

Equations (5) for $n = n'$ describe the time dependence of the photon distribution function, which for our case is time-independent (steady-state). For the off-diagonal equations, however, one finds that the only steady-state solution is $\rho_{n,n'} = 0$, $n \neq n'$. At first sight this would appear to imply that the ensemble average electric field is zero. This may well be desirable for certain ensembles, but if a measurement is made on a single laser, one would find a nonvanishing electric field. To the extent that a measurement prepares a system, the difficulty associated with a vanishing ensemble average field may be avoided by taking a density matrix which begins with

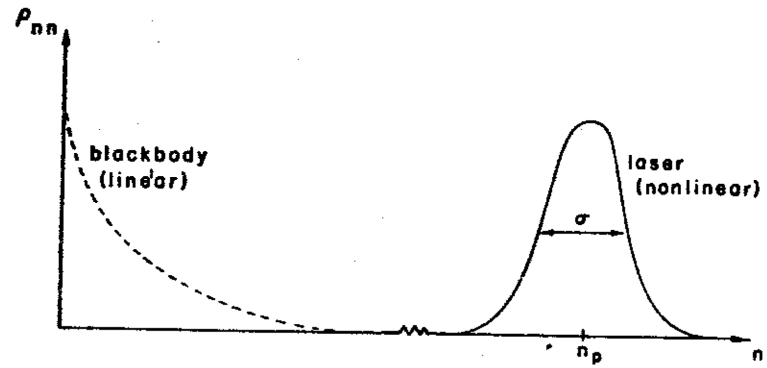


Fig. 3. Photon statistics.

nonvanishing off-diagonal elements. An analogous situation is found in the field of ferromagnetism; since in the absence of an external magnetic field the problem is invariant under inversion, statistical mechanics predicts that there will be no spontaneous ferromagnets in existence—a most unappealing state of affairs. This is resolved by considering a more selective ensemble such that the off-diagonal elements of the density matrix (and therefore the vector magnetization) are nonzero.

Equations (5) have solutions which decay very slowly (neglecting terms of order $1/n_p^2$); these solutions are

$$\rho_{n,n'}(t) = \rho_{n,n+k}(t) = C_k \sqrt{\prod_{m=0}^{n} \frac{\alpha - \beta m}{\gamma} \prod_{\ell=0}^{n'} \frac{\alpha - \beta \ell}{\gamma}} \exp - k^2\left(\frac{\gamma}{4n_p} + \frac{\beta}{8}\right)t \tag{12}$$

the constants C_k being specified by the initial conditions. Equations (12) may be written[8]

$$\rho_{n,n'}(t) = \rho_{n,n'}(0)\, e^{-k^2\mu t} \qquad \mu \equiv (\gamma/4n_p + \beta/8) \tag{13}$$

From (4), (10), and (11) it is easily shown that the ensemble electric field is

$$E(t) = \operatorname{tr} \rho(t)E = (\nu/2L)^{1/2}\left[\sum_n \rho_{n,n+1}(0)\sqrt{n} + 0(1/\sqrt{n_p})\right] e^{-\mu t} \sin \nu t \tag{14a}$$

$$= \langle E(0) \rangle\, e^{-\mu t} \sin \nu t \tag{14b}$$

One interpretation of decaying $\langle E \rangle$ is that the amplitude is decaying; here, however, the energy of the system is time-independent; rather this decay is to be viewed as a diffusion in phase, resulting in a diminishing vector sum.

It is an essential feature of the model that, even if we begin with a pure case, the "openness" of the system (because atoms are added and removed at random times) leads to a mixture in the course of time. Again the analogous situation is found in ferromagnetism; the magnetization of an open system will experience the same kind of decay, albeit on a geological time scale. Measured in terms of reciprocal frequencies the decay of $\langle E \rangle$ is also very slow.

Line Shape

To obtain the line shape we take the Fourier transform (in the rotating-wave approximation) of the average value of the E field from Eq. (14b):[9]

$$E(\omega) = \int_0^\infty e^{-i\omega t} \langle E(0) \rangle\, e^{-\mu t} \sin \nu t\, dt \tag{15a}$$

$$= \langle E(0) \rangle \frac{1}{i(\omega - \nu) + \mu} \tag{15b}$$

and the spectral profile is

$$|E(\omega)|^2 = |\langle E(0) \rangle|^2 \frac{1}{(\omega - \nu)^2 + \mu^2} \tag{16}$$

Thus the full width at half height in circular frequency units is

$$D = 2\mu = 2(\gamma/4n_p + \beta/8) \tag{17}$$

When this is written in terms of power (neglecting saturation) we have

$$D \cong (\Delta\nu_c)^2 h\nu/2P \qquad \Delta\nu_c = \nu/Q \tag{18}$$

It has been shown by Lamb[10] that vacuum fluctuations lead to one-half the above line width. The obvious interpretation is that spontaneous emission (the other source of quantum noise) accounts for the remaining half. In fact, a detailed study of the origin of our line width substantiates this conjecture.

Space does not allow us to compare our result with all earlier workers; however, we note

1. Our result is twice that recently obtained by Korenman.[11]
2. The additional factor $N_a/(N_a - N_b)$ as given by Shimoda et al.[12] does not appear because, for simplicity, we take our excitation to populate only the upper level.

4. CONCLUSION

We have constructed a model of quantized laser oscillation in generalization of the semiclassical theory. The equations of motion for the radiation were solved; the resulting distribution is similar to, but not identical with, a Poisson distribution. It was further noted that a linear laser theory leads to qualitatively different results. The off-diagonal elements describe the effects of phase diffusion in general and provide the spectral profile $|E(\omega)|^2$ as a special case.

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2. R. J. Glauber, "Quantum Optics and Electronics," p. 162, 1964, Les Houches Summer Lectures, C. DeWitt, A. Blandin, and C. Cohen-Tannoudji, eds.
3. More specifically, we assume

$$\rho(\text{atom, field}) = \rho(\text{atom}) \otimes \rho(\text{field})$$

where $\rho(\text{field}) = |E\rangle\langle E|$; i.e., the field is in a purely coherent state.

4. As one would expect, we obtain similar results by considering the quantum field density matrix generated by stochastic classical currents.
5. It would appear that $\rho_{n,n}$ can go negative for $n > n^* = \alpha/\beta$. At least three ways out of this apparent difficulty have occurred to us: (a) n^* is such a large number that we "don't care much" what happens out there. (b) $\rho_{n,n}$ cuts off if n^* is an integer and therefore never goes negative. (c) This is a result of going only to third order in the treatment of the polarization: when we solve the problem exactly, the diagonal elements remain positive definite. We have here presented the treatment to third order in the polarization, as this contains the essential physics.
6. The expression $\frac{\alpha - \gamma}{\beta} \frac{\nu}{2\epsilon_0} = E^2$ in the semiclassical theory.
7. We would like to point out that Eq. (8) is essentially a broad Poisson distribution; it is peaked at n_p but is wider by the factor $\left(\frac{\gamma}{\alpha - \gamma}\right)^{1/2}$
8. We note that the higher values of k (larger degree of correlation) decay more rapidly. $\rho_{n,n+k}$ is related to the k-point Green's functions which have been considered by Glauber.
9. This calculation can be made under circumstances where the off-diagonal elements of the density matrix have decayed to zero. This involves a much more tedious calculation. It is far easier to obtain the line shape by considering the exponentially decaying $\langle E \rangle$; this has the considerable advantage of corresponding closely to the semiclassical situation.
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