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COOPERATIVE PHENOMENA IN QUANTUM OPTICS

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1. Introduction

Cooperative phenomena occur in macroscopic systems,i.e.in systems composed of several identical constituents (atoms,molecules). One has a cooperative effect when the behaviour of the systems cannot be reduced to the simple superposition of the contributions of the single constituents(1). In such a situation the free particle description fails completely and does not even qualitatively describe the behaviour of the system.

One meets cooperative phenomena both in systems in thermodynamic equilibrium (e.g.ferromagnets, superconductors) and in systems far from thermal equilibrium(e.g.lasers, superradiating systems).

In these lectures we shall illustrate some cooperative effects which occur in the resonant interaction of an atomic system with a monochromatic (or quasi-monochromatic) electromagnetic field , namely

a) singlemode laser dynamics ,

b) cooperative spontaneos emission of radiation (superradiance) ,

c) cooperative effects in resonance fluorescence .

2) Theory of open systems

Cooperative effects in far from equilibrium situations concern open systems, i.e. systems which are in condition to exchange energy

and possibly matter with their surroundings. Such systems have in general a source of gain as well as of loss, and just the interplay between gain and loss originates the variety of situations that we find in this type of cooperative phenomena. Hence one has to deal with the problem of describing the dynamics of open systems. Our first lecture is devoted to this problem. Namely we shall illustrate a new method to treat open systems (2), that we shall later apply both to the description of the laser and to superradiance. This method is more powerful than the well known projection operator technique, (3), since it avoids the introduction of perturbative expansions in the coupling constant of the open system with its reservoir.

Let us first properly define the problem. We consider a quantum system Q composed by two interacting subsystems S and R. The Hamiltonian of Q is accordingly given by

$$(2.1) \qquad H = H_S + H_R + H_{SR} ,$$

where H_S and H_R are the Hamiltonians of S and R respectively and H_{SR} is the interaction Hamiltonian. The statistical operator $W(t)$ of Q obeys the von Neumann equation

$$(2.2) \qquad \frac{dW(t)}{dt} = -i L W(t) ,$$

where the Liouville operator (or Liouvillian) L is given by

$$(2.3) \qquad L = \hbar^{-1} [H, \ ] .$$

From equation (2.I) we have, with obvious definitions

$$(2.4) \qquad L = L_S + L_R + L_{SR}$$

Now let us assume that S is the system in which we are interested ,

while R is the reservoir. Hence it is not necessary to know the full statistical operator $W(t)$ of the system Q, but it is enough to know the reduced statistical operator $\rho(t)$ of the subsystem S alone, which is defined as

$$(2.5) \qquad \rho(t) = Tr_R W(t) ,$$

where Tr_R means partial trace over the Hilbert space of the reservoir. Then it naturally arises the problem of deducing from the von Neumann equation (2.2) for $W(t)$ a closed time evolution equation for $\rho(t)$. Usually this problem is solved by means of Zwanzig's projection technique with the projection of Argyres and Kelley (3). Let us briefly sketch this method. One introduces first a statistical operator B_{ref} of R alone, which is termed " reference state " of the reservoir. For the moment we leave B_{ref} arbitrary, since the following formal steps are indipendent of the choice of B_{ref} . In the final discussion we shall specify on physical grounds how B_{ref} must be chosen. Now one considers the following operation P on the full statistical operator W (t):

$$(2.6) \qquad PW(t) = B_{ref} Tr_R W(t) = B_{ref} \rho(t) .$$

Clearly the operation P is linear; furthermore since $Tr_R B_{ref} = 1$ one has that $P^2 = P$. Hence P is a projection operator (Argyres and Kelley's projection). Accordingly W (t) can be subdivided into two parts by means of P:

$$(2.7) \qquad W(t) = \Phi(t) + \Gamma(t) , \quad \Phi(t) = PW(t) , \quad \Gamma(t) = (1-P)W(t) .$$

Clearly $\Phi(t)$ is the "relevant" part of the statistical operator W (t), since by (2.6) and (2.7) the knowledge of $\Phi(t)$ is equivalent to the knowledge of $\rho(t)$. Hence the problem of deducing a closed equation for $\rho(t)$ is equivalent to the problem of deducing a closed equation for $\Phi(t)$. The latter problem is formally solved using the following

procedure, due to Zwazig. Multiplying eq. (2.2) times P on the left and taking (2.7) into account, we get

$$(2.8)\qquad \frac{d\Phi(t)}{dt} = -i\,PL\,\Phi(t) - i\,PL\,\Gamma(t)\,.$$

Similary multiplying eq. (2.2) times $1-P$ we get the equation

$$(2.8b)\qquad \frac{d\Gamma(t)}{dt} = -i\,(1-P)L\,\Phi(t) - i\,(1-P)L\,\Gamma(t)\,.$$

Eqs. (2.8a,b) are two coupled linear equations for $\Phi(t)$ and $\Gamma(t)$; to get a closed equation for $\Phi(t)$ one must simply eliminate $\Gamma(t)$. In such a way one obtains

$$(2.9)\qquad \frac{d\Phi(t)}{dt} = -i\,PLP\,\Phi(t) + \int_0^t d\tau\, K_P(\tau)\,\Phi(t-\tau) + I_P(t),$$

where the kernel $K_P(t)$ is built by the projection P and the Liouvillian L as follows

$$(2.10)\qquad K_P(t) = -PL(1-P)\,\mathcal{G}(t)\,(1-P)LP,$$
$$\mathcal{G}(t) = \exp\{-i\,(1-P)L\,(1-P)\,t\}\,,$$

and the inhomogeneous term $I_P(t)$, which depends on the initial value of $\Gamma(t)$, is given by

$$(2.11)\qquad I_P(t) = PL(1-P)\,\mathcal{G}(t)\,\Gamma(0).$$

Finally taking the partial trace Tr_R of eq. (2.9) and using eq. (2.6) one obtains a closed equation for $\rho(t)$:

$$(2.10')\qquad \frac{d\rho(t)}{dt} = -i\,Tr_R\{L\,B_{ref}\}\,\rho(t) + \int_0^t d\tau\, K_S(\tau)\,\rho(t-\tau) + I_S(t)\,,$$

where

$$(2.11')\qquad K_S(t) = -Tr_R\{L(1-P)\,\mathcal{G}(t)\,(1-P)\,L\,B_{ref}\}\,,$$
$$I_S(t) = Tr_R\{L\,(1-P)\,\mathcal{G}(t)\,\Gamma(0)\}\,.$$

The integro_differential equation (2.10') is called "generalized master equation" (G.M.E.) or "nonmarkoffian master equation", since it clearly exhibits a memory (or retardation) effect. In fact, due to the presence of the integral term in eq. (2.10'), the value of $d\rho/dt$ at time t is not fixed by the value of ρ at time t only but it is determined by the whole history of ρ from time 0 to time t. We notice further that the inhomogeneous term vanishes when $PW(0) = W(0)$. Let us now comment the projection method we have outlined. Clearly this procedure is very elegant and provides a very straightforward elimination of the bath variables leading to the closed equation (2.10') for $\rho(t)$. However such elimination produces a very formal expression of the kernel. In fact the semigroup $\mathcal{G}(t)$ cannot be calculated exactly in any nontrivial model. Pratically one can calculate $K_S(t)$ only in a perturbative way. Namely, by (2.4) one writes

$$(2.12)\qquad L = L_0 + L_{SR}\,, \qquad L_0 = L_S + L_R$$

and one expands $K_S(t)$ into a perturbative series (Born expansion) in the interaction L_{SR}:

$$(2.13)\qquad K_S(t) = g^2 K^{(2)}(t) + g^3 K^{(3)}(t) + \cdots\cdots$$

where g is the bath-system coupling constant. In practice one can explicitly evaluate only the first few terms of the Born expansion. Hence the G.M.E. (2.10') is meaninfuf only if one can truncate the expansion (2.13) at a very low order: preferably at lowest order performing the so-called Born approximation (B.A.), which consists in replacing $\mathcal{G}(t)$ by $\exp\{-iL_0 t\}$.
Then we are naturally led to consider the problem of the limit of validity of the B.A. Clearly the B.A. is a weak coupling approximation; however the smallness of g is not sufficient to guarantee its validity. In fact, as we shall explicitly show in the following, the B.A. holds only if the state of the reservoir does never

deviate appreciably from the reference state B_{ref}. This result immediately suggests how B_{ref} must be chosen. In fact, the reservoir R is a macroscopic system which has well defined equilibrium or stationary states. Initially (i.e. at t=0) R is in one of these states. Of course the interaction with S makes the reservoir deviate from the initial equilibrium state; however R has definite relaxation mechanisms which sistematically and rapidly lead R back to that equilibrium state. Hence if we choose B_{ref} as the equilibrium state in which R is initially we have that actually R does never deviate appreciably from B_{ref}, and the B.A. holds.

However there are situations in which the state of the reservoir deviates from the reference state in a way which is relevant for the dynamics of S. Examples of such situations are the laser above threshold and oscillatory superradiance. In these cases one must take into account several or ever infinite terms of the Born expansion. In general one has to deal with the problem of making partial resummations of the Born expansion, which is of course a very difficult task. We want to illustrate now a different method ([2]), which circumvents this problem providing approximate integrodifferential equations for $\rho(t)$ which automatically contain a partial resummation of the Born expansion.

This method starts from an integrodifferential formulation of the von Neumann equation (2.2), which is obtained in the following way. Let us rewrite eq. (2.2) taking into account eq. (2.I2):

$$\frac{dW(t)}{dt} = -i L_0 W(t) - i L_{SR} W(t) \qquad (2.I4)$$

and let us solve formally eq. (2.I4) treating the second term in the r.h.s. as a known term. We get

$$W(t) = e^{-iL_0 t} W(0) - i \int_0^t d\tau \; e^{-iL_0 \tau} L_{SR} W(\tau). \qquad (2.I5)$$

Finally we substitute the expression (2.I5) of W (t) back into the second term in the r.h.s. of eq. (2.I4) obtaining

$$\frac{dW(t)}{dt} = -i L_0 W(t) - i L_{SR} e^{-iL_0 t} W(0) - \int_0^t d\tau \; L_{SR} e^{-iL_0 \tau} L_{SR} W(\tau). \qquad (2.I6)$$

The integrodifferential equation (2.I6) has the nice feature to ren-semble a generalized master equation. Its kernel is exactly of second order in g. However, eq. (2.I6) is still an equation for the full statistical operator W(t). A preliminary step to obtain a closed equation for $\rho(t)$ is to take the partial trace Tr_R of eq. (2.I6). In doing that, we take into account the identities

$$Tr_R L_R = 0 \quad , \quad Tr_R L_S = L_S Tr_R \; . \qquad (2.I7)$$

Furthermore, we assume that the initial condition W(0) is such that the partial trace of the inhomogeneous term of eq. (2.I6) vanishes. This assumption is not at all vital for the following considerations but we introduce it for reasons of simplicity. Hence we obtain the equation

$$\frac{d\rho(t)}{dt} = -i L_S \rho(t) - \int_0^t d\tau \; Tr_R L_{SR} e^{-iL_0 \tau} L_{SR} W(t-\tau). \qquad (2.I8)$$

The first term in the r.h.s. of eq. (2.I8) gives the free motion of the subsystem S, while the second term describes the influence of the reservoir R on S. Clearly eq. (2.I8) is not yet a closed equation for $\rho(t)$, since it still contains the full statistical operator W(t) in the integral term. To see precisely in what sense

eq.(2.I8) is nonclosed, we consider hereafter the special class of interaction Hamiltonians of the type

$$H_{SR} = g\,(A B^{+} + A^{+} B)\,, \qquad (2.I9)$$

where A is a variable of S, andB is a variable of R . We introduce here the restriction (2.I9) in order to illustrate the method in the simplest situation. Incidentally, the interaction Hamiltonian of the singlemode laser model - to which we shall apply the present method - has precisely the structure (2.I9). Inserting (2.I9) into eq. (2.I8) one can explicitly calculate the integral term of eq. (2.I8). It is not necessary to report here this expression, which can be found in ref. 2. We only mention the relevant point, i.e. that eq.(2.I8) is nonclosed because the integral term contains, beside the quantity $\rho(t) = Tr_R\, W(t)$, also the new quantities

$$W_1(t) = Tr_R\{B^{+}B\,W(t)\}\,, \quad W_2(t) = Tr_R\{B\,B^{+}W(t)\}$$

$$W_3(t) = Tr_R\{B\,B\,W(t)\}\,, \quad W_4(t) = Tr_R\{B^{+}B^{+}W(t)\} \qquad (2.20)$$

Let us briefly examine the quantities (2.20) to understand their meaning. First of all, the quantities $W_i(t)$ $(i=1,2,3,4)$ are operators on the Hilbert space of S alone as well as $\rho(t)$. However, contrary to $\rho(t)$, they are not normalized to one; in fact

$$Tr_S\, W_1(t) = Tr\,(B^{+}B\,W(t)) \overset{def}{=} \langle B^{+}B\rangle(t)\,, \qquad (2.2I)$$
$$Tr_S\, W_2(t) = \langle B\,B^{+}\rangle(t)\,, \quad etc.$$

Furthermore the quantities W_i can be used to calculate suitable bath-system correlation functions. E. g;, if O_S is any observable of S we have from (2.20) that

$$\langle O_S\, B^{+}B\rangle(t) = Tr_S\,(O_S\, W_1(t))\,. \qquad (2.22)$$

Hence we shall call the quantities $W_i(t)$ "bath-system correlation functions".

Let us now come back to eq. (2.I8). As we said, it is nonclosed because of the presence of the $W_i(t)$. On the other hand, following the same procedure which has led from eq. (2.I6) to eq. (2.I8), we can deduce from eq. (2.I6) linear integrodifferential equations for the correlation functios $W_i(t)$. Clearly such equatios will contain in general higher order bath-system correlation functions, as e.g. $Tr_R\{B^{+}B^{+}B\,B\,W(t)\}$. Hence proceeding in this way we see that eq. (2.I6) generates a hierarchy of linear integro-differential equations for $\rho(t)$ and the bath-system correlation functions of all orders. This hierarchy is the main tool of the present method.

When one deals with a hierarchy, one has the problem of truncating it. Of course there are several different truncation procedures, which gives flexibility to the method. The choice of the most suitable truncation depends ultimately on each individual problem. A trivial truncation is the following one: let us introduce the ansatz

$$W(t) = B_{ref}\,\rho(t)\,, \qquad (2.23)$$

i.e. let us assume that S and R are uncorrelated and that R is always in the reference state. By (2.23) all the bath- system correlation functions factorize into the product of $\rho(t)$ times a suitable expectation value in the reference state; e.g.

$$W_1(t) = \langle B^{+}B\rangle_0\,\rho(t)\,, \qquad (2.24)$$
$$\langle B^{+}B\rangle_0 = Tr_R\,(B^{+}B\,B_{ref})\,.$$

Hence inserting (2.23) into (2.28) one gets a closed equation for $\rho(t)$. One can show ([2]) that this equation coincides with the generalized master equation in Born approximation. Incidentally, this

proves what anticipated above, i.e. that the B.A. holds when the reservoir is always in the reference state. On the other hand, when the state of R deviates from B_{ref} we must use truncation procedures more refined than (2.23). A most natural procedure, which will be used in the following treatment of the singlemode laser (section 3), consists in neglecting the bath-system correlation functions from a certain order upwards. Then the hierarchy becomes a system of integrodifferential equations for $\rho(t)$ and a suitable finite set of bath-system correlation functions. To get a closed equation for $\rho(t)$, one must eliminate the correlation functions. As a result we obtain an equation of the type([2])

$$(2.25)\qquad \frac{d\rho(t)}{dt} = -i L_S \rho(t) + \int_0^t d\tau\, K(\tau)\rho(t-\tau) + I(t),$$

i.e. an equation with the same structure of the G.M.E. (2.10').
In connection with the comparison of eqs. (2.10') and (2.25) we stress the following features.
i) The kernel K(t) is explicitly obtained in closed form, whereas $K_S(t)$ is given by the formal expression (2.11') or by the expansion (2.13).
ii) K(t) contains the coupling constant g at all orders ([2]), which shows that the present procedure is equivalent to a partial resummation of the Born expansion.
These features will be explicitly shown in the analysis of the singlemode laser (section 4).
To end this section we remark that the method we have illustrated can be used also in the case that the composed system Q=S+R is not isolated, but undergoes a markoffian time evolution under the action of external reservoirs or pump mechanisms. This is the case of the singlemode laser model that we shall illustrate below. In such a situation, eq.(2.2) is no longer the von Neumann equation but is a markoffian master equation, and the Liouvillian L is given by

$$(2.26)\qquad L = \hbar^{-1}[H, \ ] + i\Lambda,$$

where Λ describes the influence of the external reservoirs and pumps on Q.

3. The singlemode laser model

The following analyses will start from the so-called singlemode laser model of the Stuttgart school, which has been deduced from first principles by Weidlich and Haake ([4]).
We consider a pencil-shaped laser cavity of length L and cross section d^2. This cavity contains N identical atoms which interact with the radiation field, or more precisely with the modes of the cavity. Since this interaction is resonant, we can consider a single frequency of the cavity and picture the atoms as two- level atoms ([5]). Then the ingredients of the model are
a) One running axial mode of frequency ω, associated to a destruction operator a, such that $[a, a^+] = 1$.
b) N two-level atoms of frequency ω. The i-th atom $(i = 1, \dots, N)$ is associated to the raising and lowering operators r_i^+, r_i^- and to the operator r_{3i}, which is $\frac{1}{2}$ of the population inversion of the i-th atom. r_i^+, r_i^-, r_{3i} are spin $\frac{1}{2}$ operators, namely:

$$(3.1)\qquad [r_i^+, r_j^-] = 2\delta_{ij} r_{3i}, \quad [r_{3i}, r_j^\pm] = \pm \delta_{ij} r_i^\pm, \quad (r_i^+)^2 = 0.$$

Hence we consider a homogeneously broadened atomic system in perfect resonance with the radiation mode. This is the simplest version of singlemode laser model. Let us consider the statistical operator $W(t)$ of the system atoms+field in the interaction picture. It obeys the master equation

$$(3.2)\qquad \frac{dW(t)}{dt} = -i L W(t), \quad L = L_{AF} + i\Lambda_F + i\Lambda_A.$$

The Liouvillian L contains the coherent interaction term L_{AF} and the two incoherent damping terms Λ_F and Λ_A. L_{AF} describes

the interaction of the atoms with the resonant running mode in the dipole and rotating wave approximation:

$$\Lambda_{AF} X = \hbar^{-1} \left[H_{AF}, X \right] ,$$

$$(3.3)\quad H_{AF} = i\hbar g \sum_{i=1}^{N} \left\{ e^{i\vec{\alpha}\cdot\vec{x}_i} a r_i^+ - h.c. \right\} ,$$

$$g = \sqrt{\frac{\omega}{\hbar V}}\, \mu ,$$

where $\vec{\alpha}$ is the wave vector of the mode, $\vec{x}_i$ is the position of the ith atom, $V = Ld^2$ is the volume of the cavity and μ is the modulus of the dipole moment of the atoms. The field Liouvillian Λ_F takes into account that the mean lifetime of the photons in the cavity is finite. In fact the photons escape from the cavity at the rate

$$(3.4)\quad K = \frac{c(1-R)}{L} ,$$

where R is the reflectivity coefficient of the mirrors. Λ_F is explicitly given by

$$(3.5)\quad \Lambda_F X = K \left\{ \left[a, X a^+ \right] + \left[a X, a^+ \right] \right\} .$$

On the other hand the atomic Liouvillian Λ_A takes into account the following facts.

i) The atoms interact not only with the resonant axial mode but also with all others modes, which makes the atoms decay. The lifetime of the excited state is further reduced by nonradiative processes (atomic collisions, etc). All these phenomena give rise to a finite transition probability per unit time $\gamma_\downarrow$ from the upper to the lower state.

ii) To have laser action we must pump the atoms, which originates a finite transition probability per unit time $\gamma_\uparrow$ from the lower to the upper level.

Λ_A is explicitly given by

$$(3.6)\quad \Lambda_A X = \sum_{i=1}^{N} \left\{ \frac{\gamma_\downarrow}{2} \left(\left[r_i^-, X r_i^+ \right] + \left[r_i^- X, r_i^+ \right] \right) + \right.$$
$$+ \frac{\gamma_\uparrow}{2} \left(\left[r_i^+, X r_i^- \right] + \left[r_i^+ X, r_i^- \right] \right) +$$
$$\left. + \frac{\eta}{2} \left(\left[r_{3i}, X r_{3i} \right] + \left[r_{3i} X, r_{3i} \right] \right) \right\} ,$$

where the last one is a dephasing term, i.e. a term which increases the relaxation rate of the nondiagonal part of the statistical operator of the atoms, while it leaves unchanged the relaxation rate of the diagonal part. The atomic damping term is characterized by three parameters:

- the longitudinal relaxation time (or relaxation time of the population inversion) T_1

$$(3.7)\quad T_1 = \left(\gamma_\uparrow + \gamma_\downarrow \right)^{-1} ,$$

- the transverse relaxation time (or relaxation time of the polarization) T_2

$$(3.8)\quad T_2 = 2 \left(\gamma_\uparrow + \gamma_\downarrow + \eta \right)^{-1} ,$$

- the unsaturated inversion per atom σ_0, i.e. the population inversion per atom which would be established by the pump if one could eliminate the interaction with the resonant mode:

$$(3.9)\quad \sigma_0 = \frac{\gamma_\uparrow - \gamma_\downarrow}{\gamma_\uparrow + \gamma_\downarrow} , \qquad -1 \leq \sigma_0 \leq +1$$

4. Dynamics of the singlemode laser

Let us put ourselves in conditions that are typical of a class of lasers:

$$(4.1) \qquad K \ll T_1^{-1}, T_2^{-1}$$

i.e. the lifetime of the photons in the cavity is much longer than the atomic relaxation times. Then we have two interacting systems - the atomic system and the radiation field - and the atomic system has the shortest relaxation times. Now the shortness of the relaxation times is one of the most typical features of a reservoir. Hence we see that in the laser the atomic system is a reservoir for the radiation field. Accordingly we are led to consider the problem of deducing from eq. (3.2) a closed time evolution equation for the reduced statistical operator of the field alone:

$$(4.2) \qquad \rho(t) = Tr_A W(t).$$

We shall treat this problem by means of the method illustrated in section 2. For major details we refer to ref. (6a,b). The strategy of section 2 consists in writing down a hierarchy of linear integro-differential equations for $\rho(t)$ and suitable atom- field correlation functions. The first two equations of this hierarchy are

$$(4.3) \qquad \frac{d\rho(t)}{dt} = \Lambda_F \rho(t) + g^2 N \int_0^t d\tau \; e^{-\frac{\tau}{T_2}} \left\{ \left[a^+, W_1(t-\tau)\, a \right] + \left[a, \left(\rho(t-\tau) - W_1(t-\tau) \right) a^+ \right] + h.c. \right\},$$

$$(4.4) \qquad \frac{dW_1(t)}{dt} = -\frac{1}{T_1} W_1(t) + \frac{1+\sigma_0}{2T_1} \rho(t) + g^2 \int_0^t d\tau \; e^{-\frac{\tau}{T_2}} \left\{ a \left(\rho(t-\tau) - W_1(t-\tau) \right) a^+ - a\, a^+ W_1(t-\tau) + h.c. \right\},$$

where the correlation function $W_1(t)$ is given by

$$(4.5) \qquad W_1(t) = Tr_A \left\{ n_i^+ n_i^- W(t) \right\};$$

we have omitted the index i in $W_1(t)$ because there is a complete simmetry between the atoms. $W_1(t)$ is normalized to the mean value of the population of the upper level of the single atom:

$$(4.6) \qquad Tr_F W_1(t) = \langle n_i^+ n_i^- \rangle (t).$$

What is important to stress is that in writing eqs. (4.3, (4.4) we have neglected multiatomic correlation functions as e.g. $Tr_A \{ n_i^+ n_j^- W(t$ with $i \neq j$. This means that we consider only the interaction processes in which the radiation field interacts with a single atom, neglecting all the interaction processes in which the field interacts first with one atom, then with a second atom, etc. This "one-atom" approximation is made in the ninety percent of laser literature, with few exceptions ([7]). The relevant point is that this approximation truncates the hierarchy; in fact eqs. (4.3), (4.4) are a closed system of linear equations for $\rho(t)$ and $W_1(t)$. To get a closed equation for $\rho(t)$ we proceed in three steps.

First step: eq. (4.3), (4.4) are abstract operator equations; it is convenient to translate them into c-number equations using a definite representation. The most suitable one is the Glauber diagonal representation.([8]), because in this representation abstract operator equations become classical-looking partial differential equations. Hence let us consider the coherent states $|\beta\rangle$:

$$a\,|\beta\rangle = \beta\,|\beta\rangle$$

and the distribution functions $P(\beta,\beta^*,t)$ and $W_1(\beta,\beta^*,t)$ corresponding to $\rho(t)$ and $W_1(t)$ respectively:

$$\rho(t) = \int d_2\beta \; P(\beta,\beta^*,t) \; |\beta><\beta| \,, \tag{4.7}$$

$$W_1(t) = \int d_2\beta \; W_1(\beta,\beta^*,t) \; |\beta><\beta|$$

$$d_2\beta = d\mathrm{Re}\beta \; d\mathrm{Im}\beta$$

Eqs. (4.3), (4.4) become

$$\frac{\partial P(\beta,\beta^*,t)}{\partial t} = \kappa \mathcal{D} P(\beta,\beta^*,t) +$$

$$+ g^2 N \int_0^t d\tau \; e^{-\frac{\tau}{T_2}} \left\{ \mathcal{D} P(\beta,\beta^*,t-\tau) \right. \tag{4.8}$$

$$\left. - 2\mathcal{D} W_1(\beta,\beta^*,t-\tau) + 2 \frac{\partial^2}{\partial\beta^*\partial\beta} W_1(\beta,\beta^*,t-\tau) \right\} ,$$

$$\frac{\partial W_1(\beta,\beta^*,t)}{\partial t} = -\frac{1}{T_1} W_1(\beta,\beta^*,t) + \frac{1+\sigma_0}{2T_1} P(\beta,\beta^*,t) \tag{4.9}$$

$$- g^2 \int_0^t d\tau \; e^{-\frac{\tau}{T_2}} \left\{ -2|\beta|^2 P(\beta,\beta^*,t-\tau) \right.$$

$$\left. - \left[\mathcal{D} - 2\left(1+2|\beta|^2\right) \right] W_1(\beta,\beta^*,t-\tau) \right\} ,$$

where $\mathcal{D}$ is the differential operator

$$\mathcal{D} = \frac{\partial}{\partial\beta}\beta + \frac{\partial}{\partial\beta^*}\beta^* . \tag{4.10}$$

Second step: we take advantage of the shortness of the atomic relaxation times T_1, T_2, formalized by (4.I), to make the Markoff approximation (which consists in replacing $\int_0^\infty d\tau \, e^{-\frac{\tau}{T_2}} F(t-\tau)$ by $T_2 F(t)$ in eqs. (4.8), (4.9) and the adiabatic approximation(which consists in putting $\frac{\partial W_1}{\partial t} = 0$)in eq. (4.9). We stress that the Markoff and adiabatic approximations become exact in the stationary state. Then we get for $t \gg T_1, T_2$

$$\frac{\partial P(\beta,\beta^*,t)}{\partial t} = \kappa \mathcal{D} P(\beta,\beta^*,t) + \tag{4.11}$$

$$+ g^2 N T_2 \left\{ \mathcal{D} P(\beta,\beta^*,t) - 2\left(\mathcal{D} - \frac{\partial^2}{\partial\beta^*\partial\beta}\right) W_1(\beta,\beta^*,t) \right\}$$

$$\left\{ 1 + 4g^2 T_1 T_2 |\beta|^2 - g^2 T_1 T_2 (\mathcal{D}-2) \right\} W_1(\beta,\beta^*,t) = \tag{4.12}$$

$$= \left\{ \frac{1+\sigma_0}{2} + 2g^2 T_1 T_2 |\beta|^2 \right\} P(\beta,\beta^*,t)$$

The final step consists in eliminating the atom-field correlation function $W_1(\beta,\beta^*,t)$. Hence we obtain the closed markoffian equation for $\rho(t)$ ([6a]) :

$$(4.I3)\qquad \frac{\partial P(\beta,\beta^*,t)}{\partial t} = \Big\{ (\kappa + g^2 N T_2)\mathcal{D} - 2g^2 N T_2 \Big(\mathcal{D} - \frac{\partial^2}{\partial\beta^*\partial\beta}\Big)\cdot \Big[1 + 4g^2 T_1 T_2 |\beta|^2 - g^2 T_1 T_2 (\mathcal{D}-2)\Big]^{-1}\cdot \Big[\frac{1+\sigma_0}{2} + 2g^2 T_1 T_2 |\beta|^2\Big]\Big\} P(\beta,\beta^*,t),$$

Let us now comment eq. (4.I3). First, it is not a Fokker-Planck equation, since it contain derivatives of all orders in β, β^*, as one sees from the presence of the differential operator $\mathcal{D}$ (cf. eq. (4.I0)) in the denominator appearing in the r.h.s. of eq. (4.I3). We recall that the terms with derivatives of first order give the mean classical motion of the system, while the terms with derivatives of order higher than the first one describe the quantum corrections (fluctuations). The Fokker-Planck approximation of eq.(4.I3), which is obtained neglecting the term $g^2 T_1 T_2 (\mathcal{D}-2)$ in the denominator, has been firstly deduced from eq.(3.2) by P. Mandel.([9])
The time evolution operator in eq.(4.I3) contains the coupling constant at all orders, as one sees from the presence of g in the denominator. This shows explicitly that the method of ref. I is equivalent to a partial resummation of the Born expansion of the G.M.E. in the coupling constant.
Eq.(4.I3) holds for all values of the pump parameter σ_0. It is presumably the most general kinetic eqation for the singlemode laser avaliable. As it is shown in refs.(6a,b) it contains as approximations well known equations as the Risken's equation ([10]) or the strong-signal Scully-Lamb's equation([11]). In fact, the present analysis allows to establish a clear connection between the singlemode laser model of the Stuttgart school and the master equation of Scully and Lamb.

In ref.(6b) eq.(4.I3) is solved in the statinary state (i.e. putting $\partial P/\partial t = 0$). This is interesting because it allows to study the quantum effects arising from the terms of eq .(4.I3) with derivatives of order higher than the second one, which are neglected in the Fokker-Planck appproximation of ref.(9). The stationary solution $P^{(st)}(\beta,\beta^*)$ is a function of the modulus of β only, i.e. of the field intensity

$$(4.I4)\qquad z = \beta^*\beta .$$

Namely, $P^{(st)}(z)$ is given by

$$(4.I5)\qquad P^{(st)}(z) = \begin{cases} \mathcal{N}\, e^{2z}\,(\bar{z}-z)^{2(\bar{z}-\hat{z})-1} & \text{for } 0 \le z \le \bar{z}, \\ 0 & \text{for } z \ge \bar{z}, \end{cases}$$

where $\mathcal{N}$ is the normalization constant and

$$(4.I6)\qquad \hat{z} = \frac{N}{4\kappa T_1}(\sigma_0 - \sigma_{thr}), \qquad \sigma_{thr} = \frac{\kappa}{N g^2 T_2},$$

$$\bar{z} = \frac{N}{4\kappa T_1}(1+\sigma_0),$$

and σ_{thr} is the threshold inversion per atom.
From eq. (4.I5) we see immediately an effect of the terms with derivatives of order higher than the second: they give rise to a nonanalytic stationary distribution, which takes the value zero for $z \ge \bar{z}$. The parameter $\bar{z}$ can be interpreted as the maximum number of photons present in the mode.

The function $P^{(st)}(z)$ has one maximum at $z=\hat{z}$, which is the value of the intensity given by the semiclassical theory([I2]). To get some information on the shape of $P^{(st)}(z)$, we approximate this distribution by a Gaussian centered on $\hat{z}$:

$$(4.I7)\quad P^{(st)}(z) = e^{F(z)} \simeq e^{F(\hat{z}) + \frac{1}{2}(z-\hat{z})^2 F''(z)} \stackrel{def}{=} \mathcal{N} \exp\left\{-\frac{(z-\hat{z})^2}{q^2}\right\}.$$

One gets

$$(4.I8)\quad q^2 = \frac{N}{4\kappa T_2}\left(1+\sigma_{th}\right).$$

Notice that the width q of the Gaussian does not depend on σ_0, i.e. is pump- independent. On the other hand, the gaussian approximation of the stationary solution of Mandel's equation is given by (4.I7) with

$$(4.19)\quad q^2 = \frac{N}{4\kappa T_2}\left(1+\sigma_{th}\right)\frac{\sigma_0}{\sigma_{th}}, \qquad \sigma_0 \gtrsim \sigma_{th}.$$

Hence this distribution broadens increasing the pump. We see therefore a second effect of the terms with derivatives of order higher than second: to destroy this broadening.
In ref.(6b) the steady-state solution (4.I5) is compared also with the stationary solution of ref.(7), of Risken's and Scully-Lamb's equations.

5. Superradiance

After the pioneering of Dicke ([I3]), superradiance has been the object of an increasing interest in recent years ([I4-22]). Let us first of all qualitatively describe this phenomenon.

We consider the usual active volume of length $L \gg \lambda$ (λ = wavelength of the radiation), but without mirrors (R=0) and without pump ($\gamma_\uparrow = 0$) because superradiance is a purely decay phenomenon, so that no feedback as that of mirrors and of pump is concerned. Hence the photon lifetime in the cavity $K^{-1} = L/c$, which now coincides with the transit time in the cavity, turns out to be much shorter than the atomic relaxation times:

$$(5.I)\quad K \gg T_1^{-1}, T_2^{-1}.$$

We stress that (5.I) is just the opposite of the situation we had in the laser.
Let us assume that initially all the atoms are in the excited state. In normal conditions (fluorescence) the atoms decay independently from one another following the same exponential law. The radiation emission is likewise exponential in time and is given by the simple superposition of the isotropic emissions of the single atoms. On the other hand, when the density of the atomic system exceeds a suitable critical value, to be specified below, a completely different time evolution occurs. Initially the atoms are uncorrelated, so that they begin to radiate by normal fluorescence. However very soon correlations between the atoms begin to arise and increase, increase until they become so strong that the atoms decay no longer independently but cooperatively. Cooperative decay is very fast with respect to single-atom decay, since it is characterized by a rate constant proportional to the number of atoms N. The radiation output takes the shape of a pulse whose peak is proportional to N^2 and whose width is proportional to N^{-1}. Simultaneously the radiation emission becomes strongly concentrated in the axial direction (Dicke calls this phenomenon

"coherence brightening"). We remark that

a) The total energy emission (i.e. the emission integrated in time) is of course the same in normal fluorescence and in superradiance and equals the initial energy of the atomic system $N\hbar\omega$. The difference lies in the fact that superradiant emission is very fast with respect to fluorescent emission.

b) The really striking feature of superradiance is not simply the fact that the emission is proportional to N^2 . In fact, preparing the atoms with an initial nonzero polarization and in phase with one another (which in principle can be obtained by a strong fast coherent exciting pulse), the radiation emissions of the single atoms add coherently, giving rise to a total radiation intensity proportional to N^2 . This phenomenon is not striking: it is simply the emission of an antenna array. On the contrary what is amazing in superradiance is that one gets radiation proportional to N^2 from a system in which initially the polarization is exactly zero. I.e. we have an atomic system which is prepared to emit normal fluorescence, but this system spontaneously creates the conditions which gives rise to the N^2- emission. For this reason we prefer to call this phenomenon superfluorescence, rather than superradiance.

There are two kinds of superfluorescence ([I8]): "pure" superfluorescence, which consists in a single radiation pulse with a typical hyperbolic secant time behaviour, and "oscillatory" superfluorescence, in which the first pulse is followed by few pulses of decreasing height.

Let us now turn to the theoretical description of superfluorescence. The starting point is again eq. (3.2). The singlemode laser model has been firstly used to describe superradiance in ref.(I5b); a justification of this procedure from first principles is given in refs.(I7) and (I8). In ref.(I7) one finds a many- mode theory of pure superfluorescence. In ref.(I8) it is given a complete treatment of singlemode pure and oscillatory superfluorescence, including

the effect of inhomogeneous broadening.

From (5.2) we have that the atomic relaxation times are much longer than the field relaxation time. More precisely, in superradiance homogeneous and inhomogeneous atomic relaxation times are the longest characteristic times in play. In fact single-atom incoherent atomic relaxation must involve a time scale that is much longer than that of superradiant emission; otherwise it would forbid the rise of atomic correlations which originate superradiance. Hence in eq. (3.2) we can neglect the part Λ_A of the Liouvillian, which describes single-atom decay; In fact Λ_A becomes relevant only when the decay process is over. Then we have

$$(5.2) \qquad L = L_{AF} + i\,\Lambda_F \,,$$

where L_{AF} and Λ_F are given by (3.3), §3.5) with R=0. Now the Liouvillian L can be totally expressed in terms of collective atomic operators, i.e. of operators which do not depend on a single atom but on the atomic system as a whole. We introduce the collective dipole operators $R^{\pm}$:

$$(5.3) \qquad R^{\pm} = \sum_{i=1}^{N} r_i^{\pm}\; e^{\pm i\,\vec{\alpha}\cdot\vec{x}_i}$$

and the $1/2$ total population inversion operator

$$(5.4) \qquad R_3 = \sum_{i=1}^{N} r_{3i} \,.$$

The operators $R^{\pm}, R_3$ obey angular momentum commutation relations:

$$(5.5) \qquad \left[R^{+}, R^{-}\right] = 2\,R_3 \,, \qquad \left[R_3, R^{\pm}\right] = \pm R^{\pm} \,.$$

However $R^{\pm}, R_3$ are not spin-$\frac{1}{2}$ operators; in fact, the "spin" of the atomic system is $J = \frac{N}{2}$. The interaction Hamiltonian H_{AF} given by (3.3) is expressed in terms of collective operators as follows:

$$(5.6)\quad H_{AF} = i\hbar g \left(a R^{+} + a^{+} R^{-} \right).$$

Furthermore, neglecting Λ_A one has that the total angular momentum is a constant of the motion, so that:

$$(5.7)\quad \langle R^{+} R^{-} \rangle(t) + \langle R_3^2 \rangle(t) - \langle R_3 \rangle(t) = \frac{N}{2}\left(\frac{N}{2} + 1\right).$$

The initial condition for eq. (3.2) in the case of superfluorescence is:

$$(5.8)\quad W(0) = |0\,\tfrac{N}{2}\rangle\langle 0\,\tfrac{N}{2}|$$

where

$$(5.9)\quad a^{+}a\,|0\,\tfrac{N}{2}\rangle = 0 \quad ; \quad R_3\,|0\,\tfrac{N}{2}\rangle = \tfrac{N}{2}\,|0\,\tfrac{N}{2}\rangle.$$

As we said, the relaxation time of the radiation field is much shorter than the relaxation times of the atomic system. Hence in the superradiance the field is a reservoir for the atomic system. Then we are led to consider the problem of deducing from eq. (3.2) a closed time evolution equation for the reduced statistical operator

of the atomic system alone:

$$(5.10)\quad W_A(t) = \mathrm{Tr}_F\, W(t).$$

Using the method illustrated in section 2, we write the first equations of the hierarchy:

$$(5.11)\quad \frac{dW_A(t)}{dt} = g^2 \int_0^t d\tau\, e^{-\kappa\tau} \Big\{ \left[R^{-} W_A(t-\tau), R^{+} \right] + \left[\left[R^{+}, N(t-\tau) \right], R^{-} \right] + \left[\left[R^{+}, L(t-\tau) \right], R^{+} \right] + \text{h.c.} \Big\},$$

$$(5.12)\quad \frac{dN(t)}{dt} = -2\kappa N(t) + g^2 \int_0^t d\tau\, e^{-\kappa\tau} \cdot \Big\{ R^{-} W_A(t-\tau) R^{+} + \left[R^{-}, N(t-\tau) \right] R^{+} + \left[R^{+}, L(t-\tau) R^{+} \right] + \text{h.c.} + \mathcal{F}\left(N_2(t-\tau), \ldots \right) \Big\},$$

where the atom field correlation functions $N(t)$, $L(t)$, $N_2(t)$ are given by

$$N(t) = Tr_F \{ a^+ a \, W(t) \},$$

$$L(t) = Tr_F \{ a \, a \, W(t) \},$$

$$N_2(t) = Tr_F \{ (a^+)^2 a^2 W(t) \},$$

(5.I3)

and $\mathcal{F}$ is a linear functional of the correlations N_2 etc., which it is not necessary to specify here. We notice that

$$\langle a^+ a \rangle (t) = Tr_A \, N(t). \qquad (5.I4)$$

We use eqs. (5.II), (5.I2) to deduce some macroscopic equations of motion, on which our physical discussion of superfluorescence will be based. Namely, we shall deduce two equations for $\langle R_3 \rangle (t)$ and $\langle a^+ a \rangle (t)$. These equations can be easily obtained from eqs.(5.II), (5.I2 taking into account that:

$$\langle R_3 \rangle (t) = Tr_A \{ R_3 W_A(t) \},$$

$$\langle \dot{R}_3 \rangle (t) = Tr_A \left\{ R_3 \frac{dW_A(t)}{dt} \right\},$$

$$\langle \dot{a^+ a} \rangle (t) = Tr_A \frac{dN(t)}{dt},$$

$$\langle R^+ R^- \rangle (t) = Tr_A \{ R^+ R^- W_A(t) \},$$

$$\langle a^+ a \, R_3 \rangle (t) = Tr_A \{ R_3 N(t) \},$$

(5.I5)

and eq. (5.I4). The term $\mathcal{F}$ of eq. (5.I2) does not give any contribution to these equations, which are explicitly given by

$$\langle \dot{R}_3 \rangle (t) = -2g^2 \int_0^t d\tau \, e^{-K\tau} \cdot \{ \langle R^+ R^- \rangle (t-\tau) + 2 \langle a^+ a \, R_3 \rangle (t-\tau) \},$$

(5.I6)

$$\langle \dot{a^+ a} \rangle (t) = -2K \langle a^+ a \rangle (t) + 2g^2 \int_0^t d\tau \, e^{-K\tau} \cdot \{ \langle R^+ R^- \rangle (t-\tau) + 2 \langle a^+ a \, R_3 \rangle (t-\tau) \}.$$

(5.I7)

Let us now comment the meaning of the terms in eqs. (5.I6), (5.I7). Let us begin by the term with $\langle R^+ R^- \rangle$, which does not depend on the photon number. Using (5.3), we have:

$$\langle R^+ R^- \rangle (t) = \sum_{i=1}^{N} \langle r_i^+ r_i^- \rangle (t) + \sum_{\substack{i,j=1 \\ i \neq j}}^{N}{}' \langle r_i^+ r_j^- \rangle \, e^{i\vec{\alpha} \cdot (\vec{x}_i - \vec{x}_j)}.$$

(5.I8)

The first term of (5.I8) is the total number of atoms in the excited state and is therefore the term which originates normal spontaneous emission. The second term is a sum of contributions each one of which arises from the correlation of a couple of atoms; this term is responsible of cooperative spontaneous emission. Finally the term with $\langle a^+ a \, R_3 \rangle$ describes stimulated emission and absorption.

Differentiating eq. (5.I6) with respect to time and using the conservation law (5.7) we obtain:

$$(5.19)\quad \langle \ddot{R}_3 \rangle(t) + \kappa \langle R_3 \rangle(t) = -2g^2 \left\{ \frac{N}{2}\left(\frac{N}{2}+1\right) - \langle R_3^2 \rangle(t) + \langle R_3 \rangle(t) + 2\langle a^+ a\, R_3 \rangle(t) \right\}.$$

On the other hand adding eqs. (5.I6), (5.I7) up we get the equation:

$$(5.20)\quad \frac{d}{dt}\left\{ \langle a^+ a \rangle(t) + \langle R_3 \rangle(t) \right\} = -2\kappa \langle a^+ a \rangle(t).$$

Eq. (5.20) is the equation of conservation of energy; in fact $\langle a^+ a \rangle$ is (in $\hbar\omega$ units) the energy of the field, $\langle R_3 \rangle$ is (still in $\hbar\omega$ units) the energy of the atomic system and $-2\kappa \langle a^+ a \rangle$ is the outgoing flux of energy.
Introducing the factorization ansatz

$$(5.21)\quad \langle R_3^2 \rangle(t) = \left(\langle R_3 \rangle(t)\right)^2,$$
$$\langle a^+ a\, R_3 \rangle(t) = \langle a^+ a \rangle(t)\, \langle R_3 \rangle(t)$$

eqs. (5.I9), (5.20) become a closed system of equations for $\langle R_3 \rangle(t)$, $\langle a^+ a \rangle(t)$. Eq.(5.I9) becomes nonlinear, and it is just the nonlinear term $\frac{N^2}{4} - \left(\langle R_3 \rangle\right)^2$ which gives rise to cooperative emission. Eqs. (5.I9), (5.20) with (5.2I) can be reduced to a single equation as follows. Taking (5.7) and (5.2I) into account we can put

$$(5.22)\quad \langle R_3 \rangle(t) - \frac{1}{2} = \left(\frac{N}{2}+\frac{1}{2}\right)\cos\vartheta(t),$$
$$\langle R^+ R^- \rangle(t) = \left(\frac{N}{2}+\frac{1}{2}\right)^2 \sin\vartheta(t).$$

Now one easily verifies that both eqs. (5.I9), (5.20) with (5.2I) are satisfied if the foolowing equations are satisfied:

$$(5.23)\quad \langle a^+ a \rangle(t) = \frac{1}{4g^2}\,\dot{\vartheta}(t),$$

$$(5.24)\quad \ddot{\vartheta}(t) + \kappa\,\dot{\vartheta}(t) = \frac{1}{\tau_c^2}\sin\vartheta(t)$$

Eqs.(5.24), (5.25) reduce the problem of superfluorescence to a simple damped pendulum equation. τ_c is a quantity with the dimensions of a time, which coincides with the Arecchi- Courtens cooperation time ([19, 5]). Using the expression of g given in (3.3), one can reformulate τ_c in terms of experimental quantities as follows

$$(5.25)\quad \tau_c = \sqrt{\frac{8\pi\tau_0}{\rho c \lambda^2}},$$

where τ_0 is the radiative lifetime of the atoms and $\rho = N/V$ is the density of the atomic system. From the initial conditions $\langle R_3 \rangle(0) = \frac{N}{2}$, $\langle a^+ a \rangle(0) = 0$ and from (5.22), (5.23) one finds the following initial conditions for the pendulum equation:

$$(5.26)\quad \vartheta(0) = \arccos\frac{N/2 - 1/2}{N/2 + 1/2} \simeq \frac{2}{\sqrt{N}},$$

$$\dot{\vartheta}(0) = 0$$

Hence for large N the pendulum starts very near to the unstable equilibrium point (north pole).
In the case of very large K, namely $K\tau_c \gg 1$, eq.(5.24) reduces to the overdamped pendulum equation

$$(5.27)\quad \dot{\vartheta}(t) = \frac{1}{\tau_R}\sin\vartheta(t),$$

where τ_R is a time constant firstly introduced by Eberly and Rehler([I4]) given by:

$$(5.28)\qquad \tau_R = K\tau_c^2 = \frac{8\pi\tau_0}{\rho L \lambda_0}\ .$$

To see what is the physical meaning of the overdamped pendulum approximation let us come back to eqs.(5.I6),(5.I7). For very large k the photons escape so fast from the active volume that stimulated processes (i.e. the term $\langle a^+ a R_3\rangle$) are negligible. Furthermore in this conditions one can make the markoff approximation in both equations (5.I6),(5.I7)(i.e.put $\int_0^t d\tau\ e^{-K\tau}\langle R^+R^-\rangle(t-\tau) = K^{-1}\langle R^+R^-\rangle(t)$, and the adiabatic approximation in eq.(5.I7)(i.e.put $\langle a^+ a\rangle(t) = 0$). Then eqs.(5.I6),(5.I7)reduce for $t \gg K^{-1}$ to

$$(5.29)\qquad \langle \dot{R}_3\rangle(t) = -\frac{2g^2}{K}\langle R^+R^-\rangle(t),$$

$$(5.30)\qquad \langle a^+a\rangle(t) = \frac{g^2}{K^2}\langle R^+R^-\rangle(t).$$

Eqs.(5.30) clearly illustrates that for very large K the field follows the motion of the atoms adiabatically, i.e. without retardation. By (5.22) equation (5.29) reduces to the overdamped pendulum equation (5.27).On the other hand, using (5.22) and(5.27)equation (5.30) reduces to eq. (5.23).

Let us now discuss superfluorescent emission on the basis of the pendulum equation,beginning with the overdamped case(5.27) $(K\tau_c \gg 1)$. By simple inspection of eqs.(5.23), (5.27) and (5.28)one sees that $\langle a^+a\rangle(t)$(and therefore the radiation intensity $2K\langle a^+a\rangle(t)$, cfr.eq.(5.20)) is proportional to N^2. In fact

$$(5.3I)\qquad \langle a^+a\rangle(t) \propto \dot{\vartheta}^2 \propto \tau_R^{-2} \propto \rho^2 .$$

Furthermore, from eqs. (5.30) and (5.I8) we see that the N^2-dependence of $\langle a^+a\rangle$ can arise only from the correlation term $\sum_{i\neq j}\langle r_i^+ r_j^-\rangle \exp\{i\vec{\alpha}\cdot(\vec{x}_i-\vec{x}_j)\}$ of eq.(5.I8). THis clearly shows that superfluorescence is characterized by relevant atom-atom correlations. The solution of eq. (5.27) with the initial condition (5.26) is

$$(5.32)\qquad \sin\vartheta(t) = \operatorname{sech}\left[\frac{1}{\tau_R}(t-t_M)\right],$$

where the time t_M , which corresponds to the peak of the radiation output, is given by

$$(5.33)\qquad t_M = \frac{1}{2}\tau_R \log N.$$

Hence for $K\tau_c \gg 1$ one has a single pulse of radiation with hyperbolic secant shape, i.e. pure superfluorescence.

Let us now consider the case $K\tau_c \sim 1$. In this case the pendulum does not fall monotonically to the equilibrium point (south pole) but oscillates, which by (5.23) gives rise to some (say 2,3,4,) radiation pulses of decreasing height. The full pendulum equation (5.24) is not analytically solvable; numerical solutions show that the peak of $\langle a^+a\rangle(t)$ is still proportional to N^2 . This oscillatory character is a memory (i.e. nonmarkoffian) effect, as we see from the deduction of eq.(5.24). The first pulse is broader and lower than the pulse (5.32).

Finally for $K\tau_c \ll 1$ the number of oscillations becomes very large and the height of the successive pulses decreases very slowly in time. The width of the pulses is no longer τ_R but τ_c (*)

(*) The time scale for cooperative emission is τ_R for $K\tau_c \gg 1$, is $\tau_R \sim \tau_c$ for $K\tau_c \sim 1$, is τ_c for $K\tau_c \ll 1$.

and their height is no longer proportional to N^2 but simply proportional to N. Hence superfluorescence disappears for $K\tau_c \ll 1$.
In conclusion, one has superfluorescence only for

(5.34) $K\tau_c \gtrsim 1$ or equivalently $K\tau_R \gtrsim 1$.

However (5.34) is not the only condition for superfluorescence.
In fact, as we said at the beginning of this section, superfluorescence can occur only if the atomic relaxation times are much longer than the time scale τ_R of superfluorescent emission. Then we have the further condition

(5.35) $\tau_R \ll T_a$,

where T_a is the shortest atomic relaxation time. Conditions (5.34), (5.35) can be phrased in terms of the density ρ or of the length L as follows:

(5.36) $$\frac{8\pi\tau_0}{T_a L\lambda^2} \ll \rho \lesssim \frac{8\pi\tau_0 c}{L^2\lambda^2} ,$$

(5.37) $$L_{abs} \ll L \lesssim L_c ,$$

where $L_c = c\tau_c$ and L_{abs} is the absorption length given by

(5.38) $$L_{abs} = \frac{8\pi\tau_0}{\rho T_a \lambda^2} .$$

The lefthand condition in (5.37) says that the sample must be optically thick; the righthand condition in (5.37) says that one has superfluorescence only if the atoms are contained in a cooperation length L_c.

One has pure superfluorescence if $L \ll L_c$, oscillatory superfluorescence if $L \sim L_c$. Clearly it is not easy to match both conditions (5.34),(5.35). This feature explains why until now only two experiments on superradiance have been performed ([20],[21]). Both experiments investigate the situation $K\tau_c \sim 1$ ($L \sim L_c$). A critical discussion on the comparison of theory and experiments in superfluorescence is given in ref. (22). We only mention here that, while (5.35) is universally recognized as a necessary condition for superradiance, there is not a general agreement on condition (5.34).
E.g. according to the authors of ref.(20) (5.35) is the only condition for superradiance, and the cooperation length L is not relevant in the discussion of superradiance.
We end this section with a critical remark concerning fluctuations. The analysis of ref. (15b) shows that there are relevant fluctuations in superfluorescence, and that these fluctuations do not become negligible for large N. Hence the fully quantum mechanical theory, which is based on eqs. (5.11), (5.12) gives quantitatively relevant corrections to the previous semiclassical treatment, based on the factorization ansatz (5.21). However from the qualitative viewpoint the semiclassical results remain basically correct.

6. Cooperative effects in resonance fluorescence.

In this section we shall establish a direct link between otherwise rather disconnected topics, namely:

a) superradiance ,

b) resonance fluorescence ,

c) optical bistability .

We have already described superradiance in the previous section. Let us now briefly illustrate topics b) and c). The theory of resonant fluorescence is due firstly to Mollow ([23]); the first experimental verification is due to Schuda et al ([24]). The theory has been fully illustrated in the lectures of Eberly. It treats the resonant interaction of an incident coherent monochromatic field and a single two-level atom having a natural linewidth γ . If the field intensity is low enough the spectrum of the fluorescence light is Lorentzian and has width γ . However, if one increases the field intensity so that the Rabi frequency ([25,5]) $\mathcal{E}$ becomes larger than γ , two sidebands begin to emerge in the spectral line. Then the spectrum becomes three-peaked; the main peak is centered on the resonant frequency ω and the two sidebands are simmetrically centered on the frequencies $\omega \pm \mathcal{E}$. Hence the distance of the satellites increases with the injected field intensity. The appearance of a three-peaked spectrum is usually termed "Dynamical Stark Shift".

The theory of optical bistability is due to McCall ([26]); the first experimental observation has been made very recently by Gibbs et al([27]). A coherent monochromatic field is injected into an optical cavity, say a Fabry-Perot interferometer, in the axial direction. If the cavity is void of atomic material the transmitted field E_T is simply proportional to the incident field amplitude E_I . Let us arrange the proportionality constant in such a way that the plot of E_T versus E_I is the straight line at 45° , which is indicated as a dotted line in Fig. (2a). If on the contrary the

cavity is filled of atomic material-say, sodium vapour as in the experiment of ref. (27)- in such a way that the sample is optically thick (i.e. $L > L_{abs}$) one finds a completely different dependence of E_T on E_I . As shown in Fig.(2a), for low enough values of E_I only a very small fraction of the incident light is transmitted. E_T increases very slowly with E_I , until at a certain value $E_I^{(+)}$ of E_I there is a sudden change of E_T , such that almost all the incident light is transmitted. Increasing E_I further, E_T increases continously approaching the empty cavity behaviour. On the other hand if one decreases E_I starting from high values, nothing spectacular happens when E crosses the transition value $E_I^{(+)}$. The transmitted light E_T decreases continously, until E_I crosses a second transition point $E_I^{(-)} < E_I^{(+)}$, when E_T suddenly jump downwards. Thus the plot of E_T versus E_I exhibits a first order phase transition with a hysteresis cycle. This can be contrasted with the behaviour of the laser, in which as it is well known one finds a second order phase transition ([28]). The first order phase transition arises from the fact that the atomic system driven by the external field is bistable. In fact, the two values of E_T corresponding to each value of E_I for $E_I^{(-)} < E_I < E_I^{(+)}$ are associated to two different stationary states of the system.

The treatment of optical bistability given in ref.([26]) is very complete taking into account boundary conditions, inhomogeneos broadening, etc.. However it is very complicated and provides only numerical results. On the contrary the very recent treatment given by the present authors ([29]), based on a simple generalization of the singlemode laser model, provides a completely analytical treatment of optical bistability . E.g., it gives explicit conditions for the appearance of hysteresis cycle. Furthermore, it gives an analysis of cooperative effects in resonance fluorescence, leading to new predictions, namely :

i) if in the same experimental situation of ref.(27) (i.e.optically thick active medium) one looks at the fluorescent light at 90°, one should find an hysteresis cycle also for the total fluorescent light(see Fig.(2b));

ii) if one looks at the spectrum of the fluorescent light, one should see a behaviour completely different from that predicted by the single atom theories of resonance fluorescence ([23]). In fact, according to these theories increasing E_I one should see a continous emerging of two sidebands as described above, which should begin at a value of E_I much smaller than $E_I^{(+)}$. On the contrary, according to ref. ([29]) the spectrum should remain single-peaked for all values $E_I < E_I^{(+)}$. Not only, but one should observe a line narrowing approaching $E_I^{(+)}$ from below, until crossing $E_I^{(+)}$ one should see an abrupt appearance of the three peaked spectrum, with two largely separated sidebands. Hence one should observe a first order phase transition also in the spectrum of the fluorescent light.

Let us now turn to the analytical treatment. We start from the singlemode laser model (3.2) with $\gamma_\uparrow = 0$, i.e. no pump. We take into account that the field a is composed by an injected classical field α and a reaction field A emitted by the atoms:

$$(6.1) \qquad a = A + \alpha \quad , \quad [A, A^+] = 1 \ ,$$

where α is a given c-number, which for definiteness we assume real and positive. In this section we shall make a purely semiclassical analysis. Hence we firstly write down the time evolution equations for the macroscopic quantities $\langle R^- \rangle, \langle R_3 \rangle, \langle A \rangle$. Taking into account that for any observable O

$$(6.2) \qquad \langle \dot{O} \rangle (t) = Tr \left\{ O \frac{dW(t)}{dt} \right\}$$

and using (6.1) we obtain from eq.(3.2) the following equations:

$$(6.3) \qquad \langle \dot{R}^- \rangle = 2g \langle (A+\alpha) R_3 \rangle - \gamma_\perp \langle R^- \rangle \ ,$$

$$\langle \dot{R}_3 \rangle = -g \langle (A+\alpha) R^- \rangle - g \langle (A^+ + \alpha) R^+ \rangle - \gamma_\parallel \left(\langle R_3 \rangle + \frac{N}{2} \right) \ ,$$

$$\langle \dot{A} \rangle = +g \langle R^- \rangle - \kappa \langle A \rangle \ ,$$

where

$$(6.4) \qquad \gamma_\parallel = T_1^{-1} \quad , \quad \gamma_\perp = T_2^{-1} \ .$$

Now we introduce the semiclassical approximation (or selfconsistent field approximation) which consists in factorizing all the expectation values of products of a field and an atomic operator into the product of the expectation values. Hence putting

$$(6.5) \qquad b = \langle A \rangle \quad , \quad S = -\langle R^- \rangle \ ,$$

$$\Delta = -\langle R_3 \rangle \ ,$$

we obtain the nonlinear coupled equations

$$(6.6a) \qquad \dot{S} = 2g (b+\alpha) \Delta - \gamma_\perp S$$

$$(6.6b) \qquad \dot{\Delta} = -2g (b+\alpha) S - \gamma_\parallel \left(\Delta - \frac{N}{2} \right) \ ,$$

$$(6.6c) \qquad \dot{b} = -gS - \kappa b \ .$$

The initial condition for eqs.(6.6) is $b=0$, $S=0$, $\Delta = \frac{N}{2}$ (i.e. all atoms in the ground state). Since all the coefficients of eqs.(6.6) are real also the quantities S and b are real (Δ is real by definition). The interpretation of the coherent part of eqs. (6.6)(i.e. of the part arising from L_{AF}) is clear: the coherent injected field α creates a macroscopic polarization S (eq.(6.6a)), which in turn originates reaction field b (eq.(6.6c)). Since the interaction is coherent, S and b have the same frequency and direction of the injected field. The total field $b+\alpha$ in turn acts on the atomic system (eqs.(6.a,b)). The injected and total fields α and $b+\alpha$ are related to the quantities E_I, E_T introduced in the previous discussion by the simple proportionality relations

(6.7) $\hbar\omega\alpha^2 = E_I^2 V \quad , \quad \hbar\omega (b_T\alpha)^2 = E_T^2 V .$

Neglecting the reaction field b (i.e. putting $b = 0$) eqs.(6.6a,b) becomes a closed system of linear equations which for $\gamma_{\parallel} = 2\gamma_{\perp}$ coincide with the equations used in ref.(23) to discuss resonance fluorescence in the case of exact resonance. As we said, in ref.(23) it is treated only the interaction of the field with a single atom, assuming that the atoms evolve independently of one another, so that S and Δ are simply proportional to N. On the contrary, in paper (29) we take into account the cooperative effects arising from radiation reaction via the nonlinear terms of eqs.(6.6).
Let us now analyze the stationary solutions of eq.(6.6)(i;e. $\dot{S} = \dot{\Delta} = \dot{b} = 0$). We introduce the Rabi frequencies $\mathcal{E}_I$ and $\mathcal{E}_T$ in the incident and in the transmitted field respectively:

(6.8) $$\mathcal{E}_I = 2g\alpha = \frac{2E_I\mu}{\hbar} \quad ,$$

$$\mathcal{E}_T = 2g(b_{st} + \alpha) = \frac{2E_T\mu}{\hbar} \quad ,$$

where we have used (6.7) and the expression of g given in eq.(3.3); b_{st} is the stationary value of the reaction field. Correspondingly, we consider the saturation parameters y and x of the incident and of the transmitted field respectively:

(6.9) $$y = \frac{\mathcal{E}_I}{\sqrt{\gamma_\perp\gamma_{\parallel}}} \quad , \quad x = \frac{\mathcal{E}_T}{\sqrt{\gamma_\perp\gamma_{\parallel}}} \quad .$$

Let us call S_{st}, Δ_{st} the stationary values of S,Δ. Using (6.6a,b) we get the following expression of S_{st}, Δ_{st} in terms of x :

(6.10a) $$S_{st} = \frac{N}{2}\sqrt{\frac{\gamma_{\parallel}}{\gamma_\perp}}\,\frac{x}{1+x^2} \quad ,$$

(6.10b) $$\Delta_{st} = \frac{N}{2}\,\frac{1}{1+x^2} \quad .$$

Finally substituting (6.10a) into eq.(6.6c) with $\dot{b}=0$ and taking (6.8,9) into account we get the equation which determines the value of x (i.e. of b_{st}):

(6.11) $$y = \frac{2cx}{1+x^2} + x \quad ,$$

where the parameter c is given by

(6.12) $$c = \frac{\gamma_R}{2\gamma_\perp} \quad , \quad \gamma_R = \frac{1}{\tau_R(1-R)} \quad ,$$

τ_R being the characteristic duration time of the hyperbolic secant pulse in pure superradiance. An equivalent expression of c is

(6.12') $$c = \frac{L}{2L_{abs}(1-R)} \quad ,$$

where L_{abs} is given by (5.38) with $T_a = \gamma_\perp^{-1}$. We notice that in the single atom theory ($b = 0$) one has by definition $x=y$, which is the solution of eq.(6.11) if one neglects the term with C . In this case, by (6.10a,b) S_{st} and Δ_{st} are proportional to N

as anticipated above. Hence the term with c in eq.(6.II) takes atomic cooperation into account, as it is clear also from the fact that C is proportional to the cooperative decay rate γ_R. Since γ_R is proportional to N, x is no longer independent of N so that S_{st}, Δ_{st} are no longer extensive quantities. The term with c reproduces the typical dependence of S_{st} on the saturation parameter x. In fact S=0 for x=0, S increases with x reaching its maximum $S_{max} = \frac{N}{4}\sqrt{\frac{\gamma_{\parallel}}{\gamma_{\perp}}}$ at x=1 and then saturates to zero. Hence in the saturation region $x \gg 1$ the cooperative term is negligible, whereas in the unsaturated region it is relevant and can dramatically change the physical picture given by the single atom theory.

Let us now discuss eq.(6.II). Since $E_I \propto y$ and $E_T \propto x$ (6.II) expresses the incident field as a function of the transmitted field. What we want is just the inverse function $x = x(y)$. To find this function one has to solve a cubic equation, so that $x = x(y)$ can be a multivalued function. As shown in fig.I, the function $y = y(x)$ defined by eq. (6.II) has a qualitatively different shape according to whether $C<4$ or $C>4$. In fact, for $C<4$ y is a monotonic function of x, so that the inverse function $x = x(y)$ is single-valued. In such conditions, the transmitted light increases monotonically and continuously with the incident light, so that no phase transition occurs. On the contrary, for $C>4$ the function(6.II) has a maximum and a minimum. For $C \gg 1$ the coordinates of the maximum are $x_M \simeq 1$, $y_M \simeq C$ and the coordinates of the minimum are $x_m \simeq \sqrt{2C}$, $y_m \simeq \sqrt{8C}$. Hence for $C>4$ the inverse function $x = x(y)$ is multivalued. In fact eq.(6.II) has one solution x_1 for $y < y_m$, three solutions $x_1 < x_2 < x_3$ for $y_m < y < y_M$ and one solution x_3 for $y > y_M$.

The linear stability analysis shows that points x_1 and x_3 are stable, whereas x_2 is unstable. For $y > C$ x_3 pratically coincides with the single atom solution x=y, so that we term x_3 "one-atom stationary state". On the other hand point x_1 arises from atomic cooperation, so that we call it "cooperative stationary state". With respect to saturation, we can also call x_3 "saturated stationary state" and x_1 "unsatured stationary state".

So for $C>4$ the system is bistable, and it is easy to see how such bistability originates the hystheresis cycle. One must simply exchange the axes x and y in Fig. 1, so that one has a plot of the transmitted light $E_T \propto x$ versus the incident light $E_I \propto y$. We stress that this plot closely rensembles the numerical plot given in ref.(26). For low values of E_I (i.e. of y) the system is in the coperative stationary state x_1. Increasing E_I the transmitted light slowly increases following x_1, until in correspondence of the value $y = y_M$ the system is forced to make a discontinuous transition to the one-atom stationary state x_3. If we conversely start from a high value of E_I, the system is in the stationary state x_3. Decreasing E_I the transmitted light decreases continuously following x_3, until in correspondence of the value $y = y_m$ the system jumps to the stationary state x_1. The hysteresis cycle of E_T versus E_I is shown in Fig. 2a. Notice that the jump is of a factor $C \gg 1$ at both transition points.

On the other hand taking into account that the total fluorescent intensity I_F is proportional to the total population of the upper level N_+, and that from eq. (6.I0b)

$$(6.I3) \qquad N_{+,st} = \frac{N}{2} - \Delta_{st} = \frac{N}{2}\,\frac{x^2}{1+x^2}$$

we have that also I_F exhibits an hysteresis cycle . The upper branch of the plot shown in Fig.2(b) corresponds to the satured stationary solution x_3 , in which $N_{r,st}$ is simply proportional to the number of the atoms N. The lower branch corresponds to the unsatured stationary solution x_1 ; from the explicit expression of x_1 one can show ([29]) that in this state the fluorescence intensity is inversely proportional to the number of atoms. This cooperative effect is indeed amazing!

The description of the spectrum of the fluorescent light requires the analysis of the fluctuations of the field in the stationary state. In ref.(23) these fluctuations are put in simple relation with the fluctuations of the polarization in the stationary state. Of course the study of these fluctuations requires a fully quantum machanical analysis, which remains still to be done. However, on the basis of the regression theorem we can assume that the time dependence of the polarization fluctuations in the stationary state is well reproduced by the transient behaviour of the polarization in its approach to the stationary state. Hence we are led to study this approach by means of the time evolution equations(6.6). This analysis is fairly simple when the photon lifetime K^{-1} is the shortest characteristic time in play, i.e. $K \gg \gamma_\perp, \gamma_\parallel, \gamma_R, \varepsilon_I$ ... etc. In fact in such conditions we can adiabatically eliminate the reaction field from eqs.(6.6) putting $\dot{E}=0$. Thus one obtains the following coupled nonlinear equations for S, Δ :

$$(6.14a) \qquad \dot{S} = -\frac{2\gamma_R}{N} S\Delta + \varepsilon_I \Delta - \gamma_\perp S$$

$$(6.14b) \qquad \dot{\Delta} = \frac{2\gamma_R}{N} S^2 - \varepsilon_I S - \gamma_\parallel \left(\Delta - \frac{N}{2}\right).$$

The linear analysis of eqs.(6.14) around the stationary states shows that x_1 is a stable nodus, x_2 is a saddle point and x_3 is a stable focus. More precisely, the monotonic approach to x_1 is ruled by an exponential $\exp(\lambda_1 t)$ with

$$(6.25) \qquad \lambda_1 = -2\gamma_\parallel \left(1 - \frac{c}{c+\sqrt{c^2-y^2}}\right), \quad 1 \ll y \lesssim c.$$

On the other hand the oscillatory approach to x_3 is ruled by the complex exponentials $\exp(\lambda_{3\pm} t)$ where

$$(6.26) \qquad \lambda_{3\pm} = -\frac{\gamma_\perp+\gamma_\parallel}{2} \pm i\varepsilon_I, \quad y \geq c \gg 1.$$

We recall that the presence of oscillations in the polarization fluctuations is a necessary condition for the existence of a Dynamical Stark Shift (DSS), which in fact appears when the shift ε_I becomes larger than the linewidth. Hence in the spirit of the regression theorem we shall consider a necessary condition for the DSS the presence of oscillations in the approach to the stationary state, identifying the imaginary part of the damping constant with the shift. Simultaneously, the real part will be considered as an estimate of the linewidth. Now from eq. (6.25) we see that we should not expect any DSS when the system is in the cooperative stationary state. Not only, but increasing E_I (i.e. y) one should observe a line narrowing. This is a critical slowing down which strongly rensembles what occurs in tunnel diodes ([30]). Only when the system jumps to the single atom stationary state it should suddenly appear a DSS, with two largely separated sidebands because the condition $y > c \gg 1$ implies that $\varepsilon_I \gg (\gamma_\perp+\gamma_\parallel)/2$. The meaning of the condition $y > c$ (i.e. $E_I > E_I^{(c)}$) for the appearance of the DSS in a dense system ($c \gg 1$) becomes particularly clear in the case $\gamma_\perp \simeq \gamma_\parallel = \gamma$. In this case one has from eqs. (6.9) and(6.12) that $y/c \simeq \varepsilon_I/\gamma_R$. Then $y > c$ means $\varepsilon_I > \gamma_R$. So the condition for the appearance of the DSS $\varepsilon_I > \gamma$, given by the one-atom theory, has to be replaced by $\varepsilon_I > \gamma_R (\gg \gamma)$. This substitution of a one-atom decay

rate with a much larger cooperative decay rate is rather expected; what is amazing is the discontinuous appearance of the DSS which occurs when E_I crosses the value $E_I^{(+)}$.

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FIGURE CAPTIONS

Figure 1

Plot of the function $y = \frac{2cx}{1+x^2} + x$ for $C=1$ and $C=20$

For $C \gg 4$ one has $x_M = 1$, $y_M = c$, $x_m = \sqrt{2c}$, $y_m = \sqrt{8c}$.

Points x_1 and x_3 are stable, whereas points x_2 are unstable.

Figure 2

Hysteresis cycles of (a) the transmitted field and (b) the total fluorescent intensity. Full (dotted) line arrows indicate the variation obtained increasing (decreasing) the input field E_I. The upper (lower) part of the plots corresponds to the stationary solution x_3 (x_1).

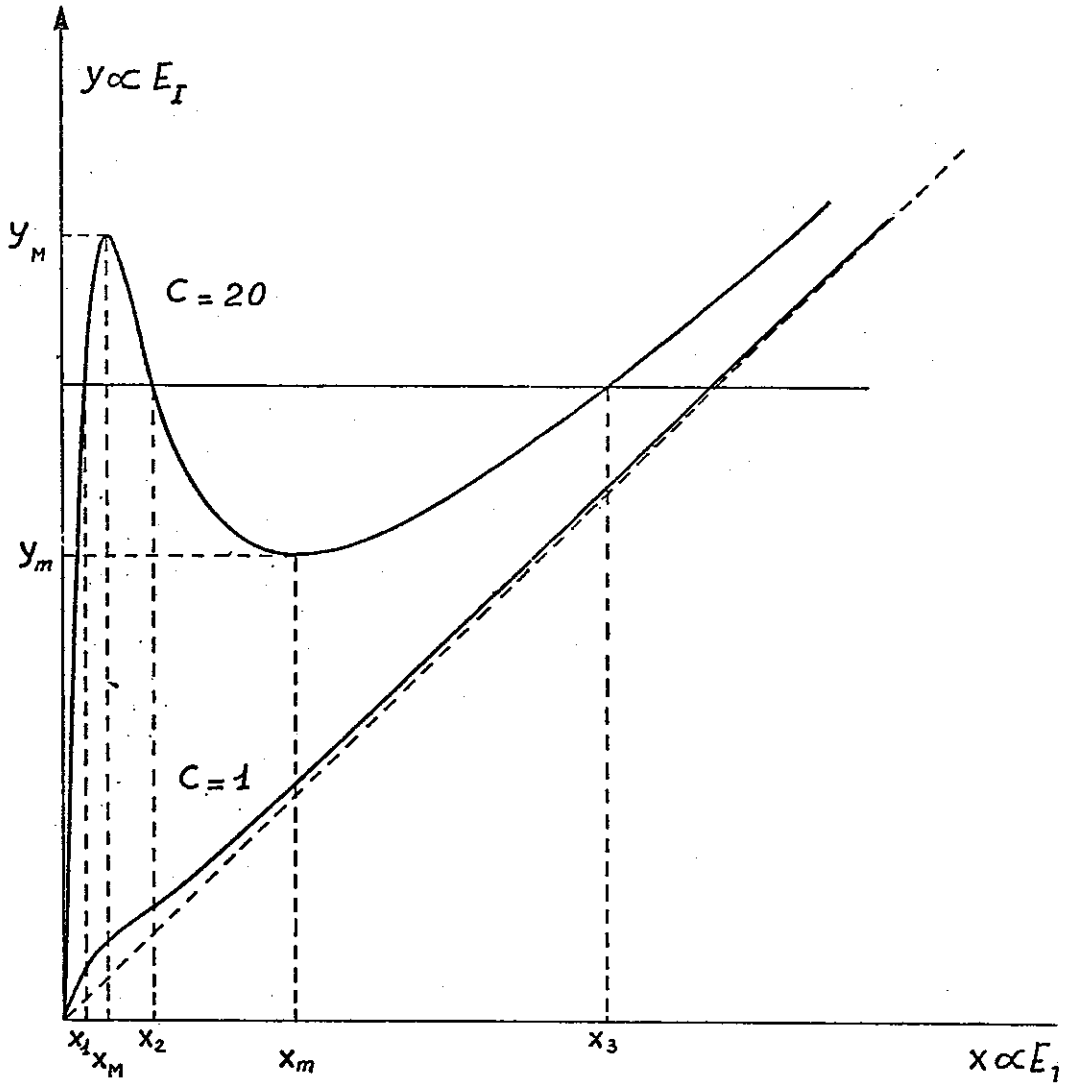
y ∝ E_I
y_M
y_m
C = 20
C = 1
x_1
x_M
x_2
x_m
x_3
x ∝ E_1

Fig. 1

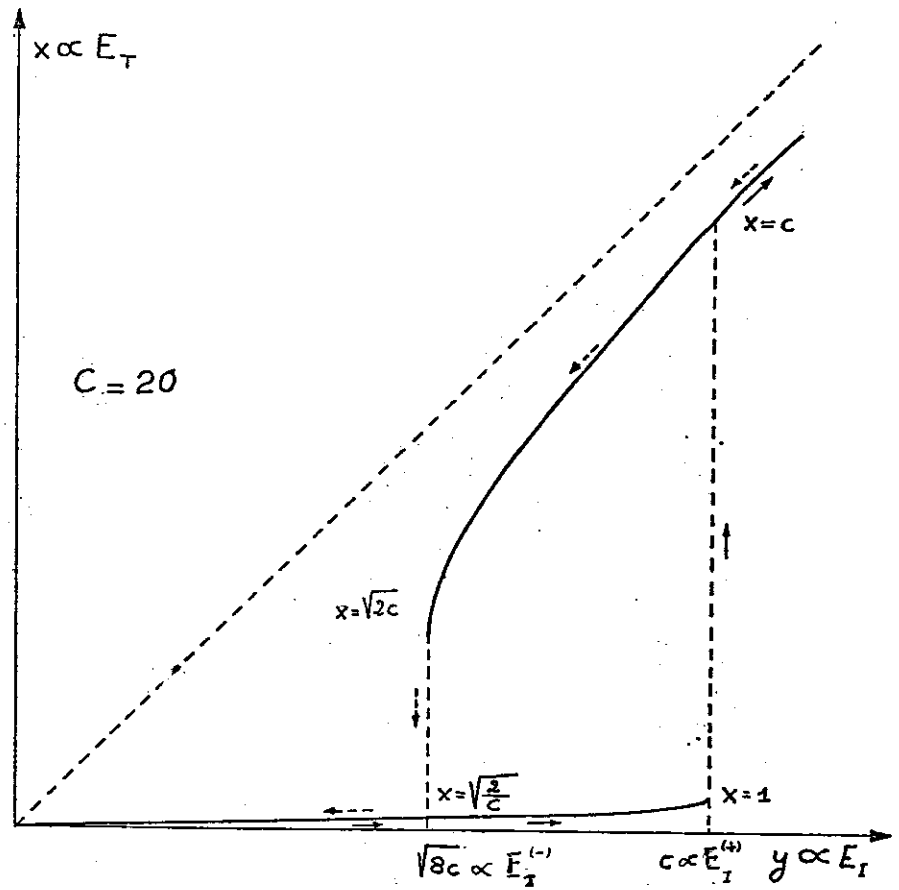
x ∝ E_T
C = 20
x = c
x = √2c
x = √(2/C)
x = 1
√8c ∝ E_I^(-)
c ∝ E_I^(+)
y ∝ E_I

Fig. 2 a

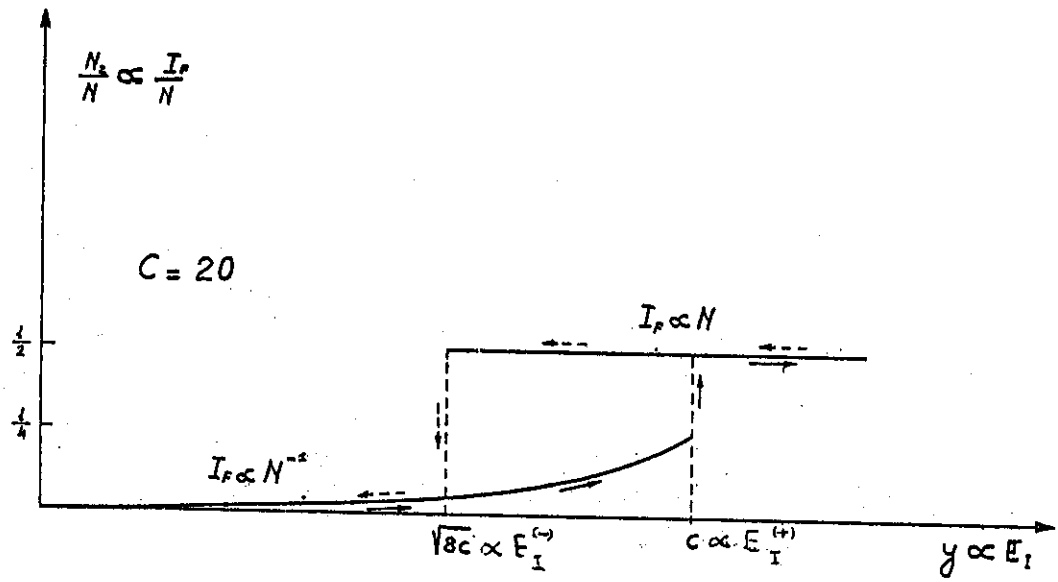
$\frac{N_2}{N} \propto \frac{I_F}{N}$
C = 20
$I_F \propto N$
$\frac{1}{2}$
$\frac{1}{4}$
$I_F \propto N^{-\alpha}$
$\sqrt{8c} \propto E_I^{(-)}$
$c \propto E_I^{(+)}$
$y \propto E_I$

Fig. 2 b