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WINTER COLLEGE ON ATOMIC AND MOLECULAR PHYSICS

26 January - 18 March 1977

RESERVOIR THEORY - DENSITY OPERATOR METHOD

BACKGROUND

M. Scully et al. (Taken from
Laser Physics (1974)

XVI

RESERVOIR THEORY – DENSITY OPERATOR METHOD

16. Reservoir Theory — Density Operator Method

Our treatment of the quantum theory of radiation so far has predominantly involved entire systems, such as the electromagnetic field in a cavity. In most areas of quantum optics, however, we are interested in only part of the entire system. For example, in the laser problem, we want to know the field but are not particularly interested in what happens to the atoms. We find it convenient to separate the system of primary interest from that (or those) of secondary interest and to call the former simply the "system" and the latter the "reservoir." We can eliminate the reservoir by using the *reduced* density operator method in the Schrödinger (or interaction) picture or the noise operator method in the Heisenberg picture. In this chapter we discuss the density operator method, which provides computational convenience while stressing the statistical aspects of the problem. In Chap. 19 we use the noise operator approach, which often requires more calculation but offers a direct physical appeal in its resemblance to the classical Brownian motion problem.

In Sec. 16-1 we introduce the reservoir concept by considering a system consisting of a simple harmonic oscillator (e.g., radiation field) interacting with a reservoir of resonant, two-level atoms. The field is taken to vary little during the lifetime of an atom (as in the semiclassical theory of Chap. 8), allowing the use of a "coarse-grained" time rate of change for the field density matrix. The atom-field probability amplitudes derived in Chap. 14 are used in the derivation. With a good knowledge of this section, the reader is prepared for the quantum theory of the laser in Chap. 17.

In Sec. 16-2 we carry out a more general approach in which the density operator is used for the entire caculation. The formalism is illustrated by the atomic beam reservoir problem of Sec. 16-1 and allows an easy calculation

of the coherent-state representation. This latter equation of motion has the form of a two-dimensional Fokker-Planck equation. In Sec. 16-3 we develop the classical Fokker-Planck equation for the probability density $P(v, t)$ for a particle with velocity v in Brownian motion and for a random walk problem. The equations are discussed pictorially and are related to the quantum version of Sec. 16-2. In Sec. 16-4 a general density operator approach is used which places emphasis on the correlation developed between two interacting systems. The theory is applied to the damping of a field and of a two-level atom by a reservoir of simple harmonic oscillators. These cases are also discussed in the problems from the point of view of Sec. 16-2.

16-1. Light Field Damping by Atomic Beam Reservoir

To illustrate the reservoir concept in simple terms, we couple a simple harmonic oscillator representing a time-dependent electric field to a beam of two-level atoms, some of which are initially excited. The atomic energy distribution is characterized by a temperature T given by the Boltzmann distribution:

$$\frac{r_a}{r_b} = \exp(-\hbar\omega/k_B T), \tag{1}$$

where r_a and r_b are the numbers of atoms per second in the upper and lower levels which pass through the cavity (see Fig. 16-1). The effect of the beam on the simple harmonic oscillator is to bring it to the same temperature T, as we shall see through the derivation in this section.

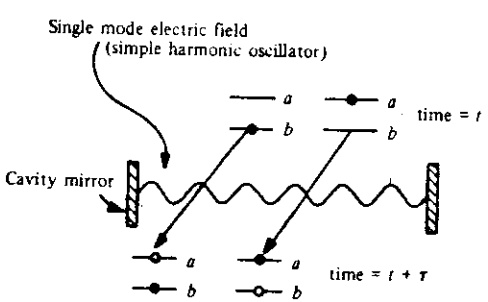

Figure 16-1. Diagram depicting passage of two-level atoms initially (time = t) in either the upper $|a\rangle$ or lower $|b\rangle$ state. Atoms initially in the upper state pass through the cavity at the rate r_a per second, and r_b atoms initially in the lower state pass through per second. The atoms interact with the field, bringing it to an equilibrium temperature T defined by (1), a fact ultimately demonstrated in Eq. (30).

In general, the field is described by a mixture of states conveniently represented by the field density operator:

$$\rho(t) = \sum_{\psi} P_\psi |\psi(t)\rangle\langle\psi(t)|. \tag{2}$$

Here P_ψ is the probability that the field has the state vector $|\psi\rangle$.

To derive the equation of motion for this density operator, we consider first the small change produced in the transit of a single atom, namely,

$$\delta\rho(t) = \rho(t + \tau) - \rho(t), \tag{3}$$

for which $\rho(t + \tau)$ is determined from an integration of a Schrödinger equation for the coupled atom-field system, as will be discussed shortly. A "coarse-grained" time rate of change for $\rho(t)$ is then given by the sum

$$\dot{\rho}(t) = r_a(\delta\rho)_a + r_b(\delta\rho)_b, \tag{4}$$

where $(\delta\rho)_a$ and $(\delta\rho)_b$ are the changes caused by the interaction with atoms injected in the upper and lower states, respectively. Equation (4) is an average time rate of change and, as such, does not describe fluctuations about this average which are responsible for the atomic contribution to shot noise in, for example, a laser. To treat these latter effects, we could include in (4) a random noise operator of the sort discussed in Chap. 19.

Two methods of integration for the coupled atom-field system are worth mentioning. For the first, the combined atom-field density operator $\rho_{a-f}(t)$ is found at time $t + \tau$ by means of an equation of motion (7.29) with the initial condition

$$\rho_{a-f}(t) = \rho(t) \otimes |\beta\rangle\langle\beta|. \tag{5}$$

Here an "outer product" (as distinguished from an inner product) of the initial field density operator $\rho(t)$ is made with the initial density operator $|\beta\rangle\langle\beta|$ for an atom injected into the energy eigenstate $|\beta\rangle$. The field operator $\rho(t + \tau)$ is then given by the trace over atomic coordinates:

$$\rho(t + \tau) = \mathrm{Tr}_{\mathrm{atom}}\{\rho_{a-f}(t + \tau)\}. \tag{6}$$

This field density operator can be used to compute the expectation value of an arbitrary operator $\mathcal{O}_f$ which depends on field coordinates alone, for

$$\mathcal{O}_f = \mathrm{Tr}_{a-f}\{\mathcal{O}_f \rho_{a-f}\} = \mathrm{Tr}_{\mathrm{field}}\{\mathcal{O}_f \rho\}.$$

The tracing operation in (6) automatically leaves $\rho(t + \tau)$ in a mixture of states even if $\rho(t)$ described a pure case (single state vector). Hence an equation of the form (2) for $\rho(t + \tau)$ would involve different $P\psi$'s from those in (2). This process of contraction (tracing) provides a fundamental, quantum-mechanical source of noise in the field. We will see more of this in Sec. 16-2 (and in Chap. 17) when field diffusion is considered.

A second integration method consists of integrating the *state vector* $|\psi(t)\rangle$ for the atom coupled to a *single* state of the field. This problem

was solved in Sec. 14-3 by the use of ordinary time-dependent perturbation theory. The resulting atom-field density operator is given by

$$\rho_{a-f}(t+\tau) = \sum_{\psi} P_{\psi} |\psi_{a-f}(t+\tau)\rangle\langle\psi_{a-f}(t+\tau)|, \tag{7}$$

which is then used in (5) for $\rho(t+\tau)$. The initial condition for the combined atom-field state vector is the outer product

$$|\psi_{a-f}(t)\rangle = |\psi(t)\rangle|\beta\rangle, \tag{8}$$

where the state vector $|\psi(t)\rangle$is any one of those appearing in(2). Inasmuch as this second method is considerably easier for the simple reservoir interaction at hand, we use it here and postpone further discussion of the straight density operator method to the general theory of Sec. 16-2. It is interesting to note that a very realistic model of the laser consists of simple extensions of the field-beam problem, and we will need in Chap 17 no more than our present method to give a fairly general quantum theory of the laser. In that development we use the strong-signal results of Sec. 14-3. Here we need consider only second-order perturbation theory solutions on resonance.

Our initial field state vector (at time t of atom injection) is given in the number representation and interaction picture as the superposition

$$|\psi(t)\rangle = \sum_{n} C_n(t)|n\rangle. \tag{9}$$

Corresponding to this is the initial field density operator:

$$\begin{aligned}\rho(t) &= \sum_{\psi} P_{\psi}|\psi\rangle\langle\psi| = \sum_{\psi} P_{\psi} \sum_{n}\sum_{m} C_n C_m^* |n\rangle\langle m| \\ &\equiv \sum_{n}\sum_{m} \rho_{nm}(t)|n\rangle\langle m|.\end{aligned} \tag{10}$$

The initial atom-field state vector is just the (outer) product:

$$\begin{aligned}|\psi_{a-f}(t)\rangle &= |\psi_{\text{atom}}(t)\rangle|\psi(t)\rangle \\ &= |\psi_{\text{atom}}(t)\rangle \sum_{n} C_n(t)|n\rangle.\end{aligned} \tag{11}$$

From (14.70) and (14.71), we know that the atom-field probability amplitudes $C_{a,n}(t)$ and $C_{b,n+1}(t)$ are coupled together in time according to the (on resonance) equations of motion

$$\dot{C}_{a,n}(t) = -ig(n+1)^{1/2} C_{b,n+1}(t), \tag{12}$$

$$\dot{C}_{b,n+1}(t) = -ig(n+1)^{1/2} C_{a,n}(t). \tag{13}$$

Hence we write the atom-field state vector at the later time $t+\tau$ as

$$|\psi_{a-f}(t+\tau)\rangle = \sum_{n} [C_{a,n}(t+\tau)|a\rangle|n\rangle + C_{b,n+1}(t+\tau)|b\rangle|n+1\rangle]. \tag{14}$$

Suppose first that the atoms are injected into the lower state, that is, $|\psi_{\text{atom}}(t)\rangle = |b\rangle$. Then the initial state vector for the combined atom-field system is

$$|\psi_{a-f}(t)\rangle = \sum_{n} C_{n+1}(t)|n+1\rangle|b\rangle, \tag{15}$$

where we have used $n+1$ instead of n to interface with the results in Sec. 14-3. Hence the probability amplitudes $C_{a,n}(t)$ and $C_{b,n+1}(t)$ in that section have the initial conditions $C_{an}(t) = 0$, $C_{b,n+1}(t) = C_{n+1}(t)$. Equation (14.74) then gives, to first-order,

$$C_{a,n}(t+\tau) \approx -ig\sqrt{n+1}\,\tau C_{b,n+1}(t) = -ig\tau\sqrt{n+1}\,C_{n+1}(t). \tag{16}$$

The corresponding amplitude for remaining in the lower state is, to second order in the interaction,

$$\begin{aligned}C_{b,n+1}(t+\tau) &\approx [1 - \tfrac{1}{2}g^2\tau^2(n+1)]C_{b,n+1}(t) \\ &= [1 - \tfrac{1}{2}g^2\tau^2(n+1)]C_{n+1}(t).\end{aligned} \tag{17}$$

These solutions could have been obtained, of course, by expansion of the sines and cosines in the exact resonant solution (14.82 and 14.83). The total density matrix elements $\rho_{a,n;a,m}(t+\tau)$ required in the trace of (6) are given by

$$\rho_{a,n;a,m}(t+\tau) = \sum_{\psi} P_{\psi} C_{a,n}(t+\tau) C_{a,m}^*(t+\tau), \tag{18}$$

in which the sum over state vectors refers to the field mixture. Hence, with (16) and (17), we have

$$\rho_{a,n;a,m}(t+\tau) \simeq g^2\tau^2\sqrt{(n+1)(m+1)}\,\rho_{n+1,m+1}(t) \tag{19}$$

$$\begin{aligned}\rho_{b,n;b,m}(t+\tau) &\simeq (1 - \tfrac{1}{2}g^2\tau^2 n)(1 - \tfrac{1}{2}g^2\tau^2 m)\,\rho_{nm}(t) \\ &\simeq [1 - \tfrac{1}{2}g^2\tau^2(n+m)]\rho_{nm}(t),\end{aligned} \tag{20}$$

that is,

$$\begin{aligned}\rho_{nm}(t+\tau) = g^2\tau^2\sqrt{(n+1)(m+1)}\,\rho_{n+1,m+1}(t) \\ + [1 - \tfrac{1}{2}g^2\tau^2(n+m)]\rho_{nm}(t).\end{aligned}$$

Hence the coarse-grained time rate of change of $\rho_{nm}(t)$ due to atoms initially in the lower state $|b\rangle$ is

$$\begin{aligned}\dot{\rho}_{nm}(t)|_{|b\rangle\text{ atoms}} &= r_b[\rho_{nm}(t+\tau) - \rho_{nm}(t)]_{|b\rangle\text{ atoms}} \\ &= -\tfrac{1}{2}\mathscr{R}_b(n+m)\rho_{nm} \\ &\quad + \mathscr{R}_b[(n+1)(m+1)]^{1/2}\,\rho_{n+1,m+1},\end{aligned} \tag{21}$$

where the rate coefficient

$$\mathscr{R}_b = r_b g^2\tau^2. \tag{22}$$

Similarly, for atoms injected into the $|a\rangle$ eigenstate, we have

$$\begin{aligned}C_{a,n}(t+\tau) &= [1 - \tfrac{1}{2}g^2\tau^2(n+1)]C_n(t), \\ C_{b,n+1}(t+\tau) &= ig\tau(n+1)^{1/2}\,C_n(t),\end{aligned} \tag{23}$$

which gives the contribution to the equation of motion for $\rho_{nm}(t)$:

$$\dot{\rho}_{nm}(t)\big|_{|a\rangle\text{ atoms}} = -\tfrac{1}{2}\mathscr{R}_a(n+1+m+1)\,\rho_{nm} + \mathscr{R}_a\sqrt{nm}\,\rho_{n-1,m-1}, \quad (24)$$

in which the rate coefficient

$$\mathscr{R}_a = r_a g^2 \tau^2. \quad (25)$$

Hence the total (coarse-grained) time rate of change of the density matrix element ρ_{nm} is given by

$$\begin{aligned}\dot{\rho}_{nm}(t) &= \dot{\rho}_{nm}\big|_{|a\rangle\text{ atoms}} + \dot{\rho}_{nm}\big|_{|b\rangle\text{ atoms}} \\ &= -\tfrac{1}{2}[\mathscr{R}_a(n+1+m+1) + \mathscr{R}_b(n+m)]\,\rho_{nm} + \mathscr{R}_a\sqrt{nm}\,\rho_{n-1,m-1} \\ &\quad + \mathscr{R}_b[(n+1)(m+1)]^{1/2}\,\rho_{n+1,m+1}. \end{aligned} \quad (26)$$

In particular, this gives the photon rate equation

$$\begin{aligned}\dot{\rho}_{nn} &= -[\mathscr{R}_a(n+1) + \mathscr{R}_b n]\rho_{nn} + \mathscr{R}_a n\rho_{n-1,n-1} \\ &\quad + \mathscr{R}_b(n+1)\rho_{n+1,n+1}. \end{aligned} \quad (27)$$

Here each term is simply understood in terms of the probabilities that atoms do (or do not) make transitions in the presence of a given number of photons. For example,

$$\begin{aligned}\mathscr{R}_a n\rho_{n-1,n-1} &= (\text{rate at which atoms are injected into state } |a\rangle) \\ &\quad \times (\text{probability of stimulated emission by an } n-1 \text{ photon field}) \\ &\quad \times (\text{probability of } n-1 \text{ photons}). \end{aligned} \quad (28)$$

We can further understand (27) in terms of Fig. 16-2, in which the flow of photon number probability is depicted. The $\mathscr{R}_a$ terms involve emissions and hence correspond to arrows pointing up in the diagram, while the $\mathscr{R}_b$ arrows

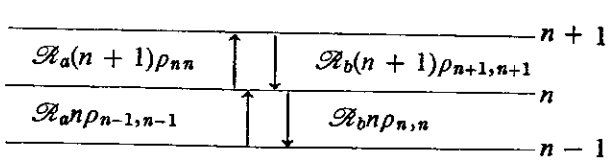

Figure 16-2. Diagram of photon number probability, ρ_{nn}. The $\mathscr{R}_a$ terms are due to emission, and the $\mathscr{R}_b$ to absorption. Each term is simply understood in terms of rates and transition probilities, as discussed in connection with Eq. (28).

point down. Equilibrium is obtained when the net flow between all pairs of levels vanishes, for example,

$$\mathscr{R}_a n\rho_{n-1,n-1} = \mathscr{R}_b n\rho_{nn}. \quad (29)$$

This condition for arbitrary n is referred to as detailed balancing. The solution of (29) with (1) is

$$\begin{aligned}\rho_{nn} &= \frac{\mathscr{R}_a}{\mathscr{R}_b}\rho_{n-1,n-1} \\ &= \frac{r_a}{r_b}\rho_{n-1,n-1} \\ &= \exp(-\hbar\omega/k_B T)\,\rho_{n-1,n-1}. \end{aligned}$$

Iteration of this result for successively lower n yields the Planck distribution for single mode:

$$\begin{aligned}\rho_{nn} &= \rho_{00}\exp(-n\hbar\omega/k_B T) \\ &= [1-\exp(-\hbar\omega/k_B T)]\exp(-n\hbar\omega/k_B T), \end{aligned} \quad (30)$$

where we have obtained ρ_{00} by imposing the normalization condition $\sum_n \rho_{nn} = 1$. Thus we see that in the steady state the field comes to the temperature of the atomic beam reservoir, as it should.

Another quantity of interest is the average photon number:

$$\langle n(t)\rangle = \sum_n n\rho_{nn}(t). \quad (31)$$

As shown in Prob. 7-1, the steady-state value of this number is given by the Bose-Einstein expression:

$$\langle n(\infty)\rangle \equiv \bar{n} = \sum_n n\rho_{nn} = \frac{1}{\exp(\hbar\omega/k_B T) - 1}, \quad (32)$$

in which ρ_{nn} is the steady-state distribution (30). Inserting the equation of motion (27) into (31), we find

$$\begin{aligned}\frac{d}{dt}\langle n(t)\rangle = \sum_n n\dot{\rho}_{nn}(t) &= \sum_n [-\mathscr{R}_a(n^2+n)\,\rho_{nn} - \mathscr{R}_b n^2\,\rho_{nn}(t) \\ &\quad + \mathscr{R}_b(n^2+n)\,\rho_{n+1,n+1}(t) + \mathscr{R}_a n^2\,\rho_{n-1,n-1}]. \end{aligned}$$

Letting $m = n+1$ in the summation over $\rho_{n+1,n+1}$ and $m = n-1$ in the summation over $\rho_{n-1,n-1}$, we find

$$\begin{aligned}\frac{d}{dt}\langle n(t)\rangle &= -\mathscr{R}_a[\langle n^2\rangle + \langle n\rangle] - \mathscr{R}_b\langle n^2\rangle \\ &\quad + \mathscr{R}_b\sum_{m=1}^{\infty}(m^2-m)\rho_{m,m} \\ &\quad + \mathscr{R}_a\sum_{m=0}^{\infty}(m^2+2m+1)\,\rho_{mm} \\ &= (\mathscr{R}_a - \mathscr{R}_b)\langle n\rangle + \mathscr{R}_a. \end{aligned} \quad (33)$$

Here the lone $\mathscr{R}_a$ term derives from a 1 in an emission factor $n+1$ and hence

can be regarded as the result of spontaneous emission. This implies a buildup of the radiation until an equilibrium is reached. In steady state, (33) yields

$$\langle n(\infty)\rangle = \frac{\mathscr{R}_a}{\mathscr{R}_b - \mathscr{R}_a} = \bar{n} \tag{34}$$

in agreement (of course) with (32).

Another quantity of particular interest is the electric field expectation value defined by [see (14.16)]

$$\begin{aligned}\langle E(t)\rangle &= \mathscr{E} \sin Kz \,\mathrm{Tr}\{\rho(t)(a + a^\dagger)\}\\ &= \mathscr{E} \sin Kz \,\mathrm{Tr}\{\rho(t)a\} + \text{c.c.}\\ &= \mathscr{E} \sin Kz \sum_{n=0}^{\infty} \langle n|\rho(t)a|n\rangle + \text{c.c.}\\ &= \mathscr{E} \sin Kz \sum_{n=0}^{\infty} \sqrt{n}\,\rho_{n,n-1}(t) + \text{c.c.}\end{aligned} \tag{35}$$

Inserting the equation of motion (26) for $\rho_{n,n-1}$ and using the techniques leading to (33), we can show (Prob. 16-4) that the positive frequency part of the field expectation value (35) has the equation of motion

$$\frac{d}{dt}\langle \mathscr{E}a\rangle = -\tfrac{1}{2}\mathscr{R}_b(\bar{n} + 1)^{-1}\langle \mathscr{E}a\rangle - i\nu\langle \mathscr{E}a\rangle. \tag{36}$$

apart from the sin Kz. Thus the average field decays to zero (dephases with zero ensemble average) in time with decreasing rate for larger temperatures. This does not imply that the field intensity vanishes in time; this quantity is proportional to the average photon number $\bar{n}$.

In the laser theory of Chap. 17, we use an atomic beam reservoir to simulate the finite cavity Q. For this, we suppose that the average electric field decays according to

$$\frac{d}{dt}\langle \mathscr{E}(t)\rangle = -\frac{1}{2}\frac{\nu}{Q}\langle \mathscr{E}(t)\rangle - i\nu\langle \mathscr{E}(t)\rangle \tag{37}$$

or, equivalently, that the average photon number has the decay term $-(\nu/Q)\langle n(t)\rangle$. Comparing (37) with (36), we see that the Q is given by

$$\frac{\nu}{Q} = \frac{\mathscr{R}_b}{\bar{n} + 1} = \frac{\mathscr{R}_a}{\bar{n}} = \mathscr{R}_b - \mathscr{R}_a, \tag{38}$$

that is,

$$\mathscr{R}_a = \frac{\nu}{Q}\bar{n}, \qquad \mathscr{R}_b = \frac{\nu}{Q}(\bar{n} + 1). \tag{39}$$

In these terms, the density matrix element ρ_{nm} has the equation of motion

$$\begin{aligned}\dot{\rho}_{nm} = &-\frac{1}{2}\frac{\nu}{Q}[2\bar{n}(n + m + 1) + n + m]\,\rho_{nm} + \frac{\nu}{Q}\bar{n}\sqrt{nm}\,\rho_{n-1,m-1}\\ &+ \frac{\nu}{Q}(\bar{n} + 1)[(n + 1)(m + 1)]^{1/2}\rho_{n+1,m+1}.\end{aligned} \tag{40}$$

16-2. Density Operator and Quantum Fokker-Planck Equations of Motion

We now consider the system-reservoir interaction from a totally operator point of view. This method is often more convenient in treating more complicated problems than the two-level atomic beam reservoir interacting with a simple harmonic oscillator. We illustrate the formalism with this simple problem and then derive a corresponding Fokker-Planck equation for the diagonal distribution $P(a)$ of Sec. 15-3. The reader will recall that $P(a)$ is ordinarily the probability density for the coherent state $|a\rangle$, although there exist pathological examples for which $P(a)$ has negative and highly nonsingular values. The Fokker-Planck equation is fairly easy to derive with the operator formalism of this section, although it is relatively difficult to obtain from the photon number representation (ρ_{nm}) found earlier by use of the state vector. The theory can also be used to treat the damping of a field or an atom by a reservoir of oscillators, as discussed in Prob. 16-14.

We consider in general a system denoted by A interacting with a reservoir denoted by B. The combined density operator is denoted by $\rho_{AB}(t)$, and the system reduced operator is given in extension of (6) by the trace over the reservoir coordinates:

$$\rho_A(t) = \mathrm{Tr}_B\{\rho_{AB}(t)\} = \sum_B \langle B|\rho_{AB}(t)|B\rangle. \tag{41}$$

At the initial time t, the system and reservoir are taken to be uncorrelated, and hence the initial value of $\rho_{AB}(t)$ is given by the outer product [like(5)]:

$$\rho_{AB}(t) = \rho_A(t) \otimes \rho_B(t). \tag{42}$$

For the atomic beam problem of Sec. 16-1, we derived a coarse time rate of change for the field-atom combination by writing an appropriate ρ_{AB} in terms of a superposition of state vectors whose time evolution was known from calculations in Sec. 14-3. Here we determine this time development for ρ_{AB} itself. Specially, in the interaction picture, the equation of motion for $\rho_{AB}(t)$ is

$$\dot{\rho}_{AB}(t) = -\frac{i}{\hbar}[\mathscr{V}_{AB}(t), \rho_{AB}(t)]. \tag{43}$$

Solving this through iteration as indicated in Sec. 7-4, we obtain $\rho_A(t + \tau)$ by tracing over the B subsystems:

$$\begin{aligned}\rho_A(t + \tau) = \mathrm{Tr}_B\{\rho_{AB}(t + \tau)\} &= \mathrm{Tr}_B\{\rho_A(t) \otimes \rho_B(t)\\ &+ \sum_s \left(-\frac{i}{\hbar}\right)^s \int_t^{t+\tau} dt_1 \int_t^{t_1} dt_2 \cdots \int_t^{t_{s-1}} dt_s\\ &\times [\mathscr{V}(t_1), [\mathscr{V}(t_2), \ldots, [\mathscr{V}(t_s), \rho_A(t) \otimes \rho_B(t)] \ldots]]\}.\end{aligned} \tag{44}$$

It is instructive to write the lowest-order terms explicitly:

$$\rho_A(t + \tau) = \mathrm{Tr}_B\{\rho_A(t) \otimes \rho_B(t) - \frac{i}{\hbar}\int_t^{t+\tau} dt'[\mathscr{V}(t'), \rho_A(t) \otimes \rho_B(t)] + \cdots\}$$

$$= \underbrace{\rho_A(t)}_{\substack{\text{(initial } A \text{ system}\\ \text{density matrix)}}} - \underbrace{\frac{i}{\hbar}\int_t^{t+\tau} dt' \,\mathrm{Tr}_B[\mathscr{V}(t'), \rho_A(t)\otimes\rho_B(t)] + \cdots}_{\substack{\text{(correlations produced by}\\ \text{interactions with } B \text{ system)}}}.$$

It is important to remember that $\rho_A(t+\tau)$ will be in general an impure case (statistical mixture)even though $\rho_A(t)$ may be a pure case (single state vector). The process of tracing over the unobserved states leads to an irreversible dynamics for the A system, that is, damping and "noise." We see this for the laser in Chap. 17.

Let us now illustrate this formalism by considering again the simple harmonic oscillator (e.g., electric field) damped by atomic beam reservoirs as depicted in Fig. 16-1. The initial atomic density matrix for a Boltzmann mixture of atoms in upper and lower levels is given by

$$\rho_{\text{atom}}(t) = Z^{-1}\begin{pmatrix}\exp(-\hbar\omega_a/k_BT) & 0\\ 0 & \exp(-\hbar\omega_b/k_BT)\end{pmatrix} = \begin{pmatrix}\rho_{aa} & 0\\ 0 & \rho_{bb}\end{pmatrix}, \tag{45}$$

where the normalization factor

$$Z = \exp(-\hbar\omega_a/k_BT) + \exp(-\hbar\omega_b/k_BT).$$

The frequency ω of Eq. (1) is given by the difference $\omega_a - \omega_b$. The on-resonance interaction energy required for (44) is given by (14.66) as

$$\mathscr{V} = \hbar g\sigma a^\dagger + \text{adjoint} = \hbar g\begin{pmatrix}0 & a\\ a^\dagger & 0\end{pmatrix}. \tag{46}$$

Keeping terms up to second order in $\mathscr{V}$ and capitalizing on the time independence of $\mathscr{V}$ (field on resonance), we have from (44) the fairly simple result

$$\rho(t+\tau) \equiv \rho_{\text{field}}(t+\tau) = \rho(t) - \frac{i}{\hbar}\tau\,\mathrm{Tr}_{\text{atom}}[\mathscr{V}, \rho_{a-f}(t)] - \frac{1}{2}\left(\frac{\tau}{\hbar}\right)^2\mathrm{Tr}_{\text{atom}}[\mathscr{V}, [\mathscr{V}, \rho_{a-f}(t)]]. \tag{47}$$

With (45), the initial condition for $\rho_{a-f}(t)$ is

$$\rho_{a-f}(t) = \rho(t)\otimes\begin{pmatrix}\rho_{aa} & 0\\ 0 & \rho_{bb}\end{pmatrix} = \begin{pmatrix}\rho_{aa}\rho(t) & 0\\ 0 & \rho_{bb}\rho(t)\end{pmatrix}.$$

The first-order contribution in (47) has only the off-diagonal elements

$$[\mathscr{V}, \rho_{a-f}(t)] = \hbar g\begin{pmatrix}0 & a\rho_{bb}\rho(t)\\ a^\dagger\rho_{aa}\rho(t) & 0\end{pmatrix} - \begin{pmatrix}0 & \rho_{aa}\rho(t)a\\ \rho_{bb}\rho(t)a^\dagger & 0\end{pmatrix}$$

$$= \begin{pmatrix}0 & a\rho_{bb}\rho(t) - \rho_{aa}\rho(t)a\\ a^\dagger\rho_{aa}\rho(t) - \rho_{bb}\rho(t)a^\dagger & 0\end{pmatrix}. \tag{48}$$

The second-order commutator is given with the use of (46) and (48) as

$$[\mathscr{V}, [\mathscr{V}, \rho_{a-f}(t)]] = \hbar^2g^2\begin{pmatrix}aa^\dagger\rho_{aa}\rho(t) - a\rho_{bb}\rho(t)a^\dagger & 0\\ 0 & a^\dagger a\rho_{bb}\rho(t) - a^\dagger\rho_{aa}\rho(t)a\end{pmatrix} + \text{adjoint}. \tag{49}$$

Substitution of (48) and (49) into (47) yields

$$\rho(t+\tau) \simeq \rho(t) - \tfrac{1}{2}g^2\tau^2[(aa^\dagger\rho - a^\dagger\rho a)\rho_{aa} + (a^\dagger a\rho - a\rho a^\dagger)\rho_{bb} + \text{adjoint}].$$

The coarse-grained time rate of change of $\rho(t)$ is then given by

$$\dot\rho(t) = r[\rho(t+\tau) - \rho(t)], \tag{50}$$

where r is the rate of injection of atoms per second. Noting here that $r_a = r\rho_{aa}$ and $r_b = r\rho_{bb}$, we obtain

$$\dot\rho(t) = -\tfrac{1}{2}\mathscr{R}_a[aa^\dagger\rho - a^\dagger\rho a] - \tfrac{1}{2}\mathscr{R}_b[a^\dagger a\rho - a\rho a^\dagger] + \text{adjoint}, \tag{51}$$

where $\mathscr{R}_a = r_ag^2\tau^2$ as in (25) and (22). The reader can verify quite easily (Prob. 16-7) that the equation of motion for the photon number matrix element ρ_{nm} is just that (26) obtained in Sec. 16-1 and hence that the equations of motion for the average photon number $\langle n(t)\rangle$ and the field $\langle E(t)\rangle$ are just (33) and (36), respectively. In terms of the cavity Q of (38), (51) reads

$$\dot\rho(t) = -\frac{1}{2}\frac{\nu}{Q}\left\{\bar n[aa^\dagger\rho - a^\dagger\rho a] + (\bar n + 1)[a^\dagger a\rho - a\rho a^\dagger]\right\} + \text{adjoint}. \tag{52}$$

A particularly interesting representation of the density operator equation of motion (51) is the coherent state. To find this, we substitute the coherent-state expansion (15.2)

$$\rho(t) = \int d^2a P(a,t)|a\rangle\langle a|$$

into (51), obtaining

$$\int d^2a\,\dot P(a,t)|a\rangle\langle a| = -\tfrac{1}{2}\int d^2a P(a,t)\{\mathscr{R}_a[aa^\dagger|a\rangle\langle a| - a^\dagger|a\rangle\langle a|a] + \mathscr{R}_b[a^\dagger a|a\rangle\langle a| - a|a\rangle\langle a|a^\dagger]\} + \text{adjoint}. \tag{53}$$

The expression in curly braces can be simplified through the use of the relation

$$a^\dagger|a\rangle\langle a| = (\partial_a + a^*)|a\rangle\langle a| \tag{54}$$

and its adjoint

$$|a\rangle\langle a|a = (\partial_a^* + a)|a\rangle\langle a|, \tag{55}$$

in which we have set

$$\partial_a = \frac{\partial}{\partial a} \tag{56}$$

for typographical simplicity. These relations follow (Prob. 16–8) from the formula

$$|a\rangle\langle a| = \exp(-aa^*)\exp(aa^\dagger)|0\rangle\langle 0|, \tag{57}$$

which, in turn, follows from Eq. (15.11) for the coherent state $|a\rangle$. In particular, we find in Eq. (53) that

$$\begin{aligned} a^\dagger a|a\rangle\langle a| - a|a\rangle\langle a|a^\dagger + \text{adjoint} \\ = a^\dagger a|a\rangle\langle a| - a|a\rangle\langle a|a^\dagger + \text{adjoint} \\ = [a(\partial_a + a^*) - aa^*]|a\rangle\langle a| + \text{adjoint} \\ = a\partial_a|a\rangle\langle a| + \text{adjoint} \end{aligned} \tag{58}$$

and similarly that

$$\begin{aligned} aa^\dagger|a\rangle\langle a| - a^\dagger|a\rangle\langle a|a + \text{adjoint} \\ = -[a\partial_a + \partial_a a^*]|a\rangle\langle a| + \text{adjoint.} \end{aligned} \tag{59}$$

We plan to substitute (58) and (59) into (53) and integrate the result by parts. In doing so, we encounter integrals like

$$\int d^2a\, Pa\partial_a|a\rangle\langle a| = aP|a\rangle\langle a|\Big|_{-\infty}^{\infty}\Big|_{-\infty}^{\infty} - \int d^2a\, \partial_a(aP)|a\rangle\langle a|. \tag{60}$$

The distribution $P(a, t)$ vanishes at the infinite limits, and therefore (60) becomes

$$\int d^2a\, Pa\partial_a|a\rangle\langle a| = -\int d^2a\, \partial_a\,(aP)\,|a\rangle\langle a|.$$

The $\partial_a\partial_{a^*}$ term requires two such integrations (yielding two minus signs). With these considerations, Eq. (53) becomes

$$\begin{aligned} \int d^2a\, \dot{P}\,|a\rangle\langle a| = -\int d^2a\, \{\tfrac{1}{2}(\mathscr{R}_a - \mathscr{R}_b)\,[\partial_a(aP) + \text{c.c.}] \\ - \mathscr{R}_a\partial_a\partial_{a^*}(P)|a\rangle\langle a|\}, \end{aligned}$$

where we obtain a complex conjugate from the adjoints since P is real. Identifying coefficients of $|a\rangle\langle a|$ in the integrands, we have the equation of motion for $P(a, t)$:

$$\begin{aligned} \dot{P}(a, t) = -\tfrac{1}{2}(\mathscr{R}_a - \mathscr{R}_b)\left\{\frac{\partial}{\partial a}\,[aP(a, t)] + \text{c.c.}\right\} \\ + \mathscr{R}_a\frac{\partial^2 P(a, t)}{\partial a\partial a^*}. \end{aligned} \tag{61}$$

In the next section we discuss a classical version of such an equation, which is called the Fokker-Planck equation. For the present we simply note that (61) can be used to calculate the time rates of change of various expectation values,

much as (26) for $\rho_{nm}(t)$ was used. In particular, the rate of change for the complex electric field value $\mathscr{E}a$ is given by

$$\begin{aligned} \frac{d}{dt}\langle\mathscr{E}a\rangle = \int d^2a\mathscr{E}a\dot{P}(a, t) = -\tfrac{1}{2}(\mathscr{R}_a - \mathscr{R}_b)\,\mathscr{E}\int d^2aa\left[\frac{\partial}{\partial a}(aP) + \text{c.c.}\right] \\ = -\tfrac{1}{2}(\mathscr{R}_b - \mathscr{R}_a)\,\mathscr{E}\int d^2a\, aP \\ = -\tfrac{1}{2}\,\mathscr{R}_b(\bar{n} + 1)^{-1}\langle\mathscr{E}a\rangle \end{aligned} \tag{62}$$

in agreement with the density matrix calculation of Eq. (36).

In terms of a cavity Q (38), the Fokker-Planck equation (61) reads

$$\dot{P}(a, t) = \frac{1}{2}\frac{\nu}{Q}\left\{\frac{\partial}{\partial a}\,[aP\,(a, t)] + \text{c.c.}\right\} + \frac{\nu}{Q}\bar{n}\,\frac{\partial^2 P(a, t)}{\partial a\partial a^*}. \tag{63}$$

16-3. The Fokker-Planck Equation

Having seen the Fokker-Planck equation in the preceding section on the damped oscillator, let us consider here how this equation arises naturally in stochastic problems and interpret its time evolution. The simplest problem is that of random walk along a line. The coordinate on the line can be, for example, a position or a velocity. We consider two possible methods, one involving discrete, small position changes, and the other involving (one-dimensional) velocity changes.

For the first method, we suppose that there are many positions of a line (Fig. 16-3) which are occupied with the probability $P(x_n, t)$. We desire the

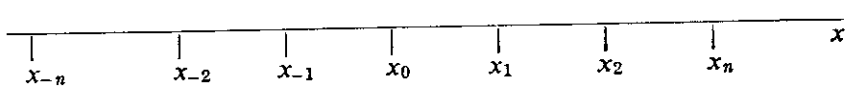

Figure 16-3. Discrete positions on a line which might be occupied by a particle.

equation of motion for this probability. We further suppose that the particle hops with a probability $p_\pm$ one position at a time, that is, $x_n \to x_{n\pm1}$. The change in probability of being at x_n in the time interval Δt is then given by

$$\begin{aligned} P(x_n, t + \Delta t) - P(x_n, t) = -(p_+ + p_-)\,P(x_n, t) \\ + p_-P(x_{n+1}, t) + p_+P(x_{n-1}, t). \end{aligned} \tag{64}$$

The reader will note that this equation is quite reminiscent of the photon rate equations in Sec. 16-1 and hence that a similar analysis can be applied to that

problem to obtain a Fokker-Planck equation. With the definition $x_{n\pm1} = x_n \pm \Delta x$ and appropriate first-order expansions of the P's, Eq. (64) yields

$$\frac{\partial}{\partial t} P(x_n, t)\,\Delta t = -(p_+ + p_-)P(x_n, t) + p_-\left\{P(x_n, t) + \Delta x\frac{\partial}{\partial x_n}P(x_n, t) + \frac{(\Delta x)^2}{2}\frac{\partial^2}{\partial x_n{}^2}P(x_n, t)\right\} + p_+\left\{P(x_n, t) - \Delta x\frac{\partial}{\partial x_n}P(x_n, t) + \frac{(\Delta x)^2}{2}\frac{\partial^2}{\partial x_n{}^2}P(x_n, t)\right\}. \quad (65)$$

This reduces to the Fokker-Planck equation:

$$\frac{\partial}{\partial t} P(x, t) = -M_1\frac{\partial}{\partial x}P(x, t) + \tfrac{1}{2}M_2\frac{\partial^2}{\partial x^2}P(x, t), \quad (66)$$

where we have taken Δx to be small and x_n to be x, and defined the moments

$$M_1 = (p_+ - p_-)\frac{\Delta x}{\Delta t}, \quad (67)$$

$$M_2 = (p_- + p_+)\frac{(\Delta x)^2}{\Delta t}. \quad (68)$$

We will interpret this equation after deriving it again in a slightly different context.

For this second method, the probability density of a particle scattering from a velocity v_0 at time t_0 to velocity v at time t is denoted by $P(v, t|v_0, t_0)$. We consider processes which are stationary in time, that is, which depend only on the time difference $s = t - t_0$. We then wish to find an equation of motion for the probability density $P(v, s|v_0)$. To this end, we note that the probability density of the particle's scattering into v in the time interval $s + \Delta s$ is

$$P(v, s + \Delta s|v_0) = \int dv_1\, P(v, s + \Delta s|v_1)\, P(v_1, s|v_0), \quad (69)$$

that is, for an initial v_0, the particle scatters in the interval s with probability $P(v, s|v_0)$ to the intermediate velocity v_1, and then at a time Δs later it scatters with probability $P(v, s + \Delta s|v_1)$ to v. The total probability density of scattering to v in the time $s + \Delta s$ is then given by summation over all intermediate velocities, v_1.

In the limit $\Delta s \to 0$, we may expand $P(v, s + \Delta s|v_0)$ to first order in Δs, obtaining

$$P(v, s + \Delta s|v_0) \approx P(v, s|v_0) + \frac{\partial}{\partial s}P(v, s|v_0)\,\Delta s. \quad (70)$$

With the definition $\Delta v = v - v_1$, the integrand of Eq. (69) is given by

$$P(v, s + \Delta s|v - \Delta v)\, P(v - \Delta v, s|v_0)$$

$$= \sum_{n=0}^{\infty}\frac{(-\Delta v)^n}{n!}\frac{\partial^n}{\partial v^n}P(v + \Delta v, s + \Delta s|v)\, P(v, s|v_0). \quad (71)$$

Inverting (70) and inserting (69) with (71), we find the equation of motion:

$$\frac{\partial}{\partial s}P(v, s|v_0) = \frac{1}{\Delta s}[P(v, s + \Delta s|v_0) - P(v, s|v_0)]$$

$$= \frac{1}{\Delta s}\sum_{n=1}^{\infty}\frac{\partial^n}{\partial v^n}\int d(\Delta v)\frac{(-\Delta v)^n}{n!}P(v + \Delta v, s + \Delta s|v)P(v, s|v_0)$$

$$= \sum_{n=1}^{\infty}(-1)^n\frac{\partial^n}{\partial v^n}[M_nP(v, s|v_0)], \quad (72)$$

where the moments (note that $\Delta t = \Delta s$)

$$M_n = \frac{1}{\Delta t}\int_{-\infty}^{\infty} d(\Delta v)\,\frac{(\Delta v)^n}{n!}\,P(v + \Delta v, s + \Delta s|v)$$

$$= \frac{\langle(\Delta v)^n\rangle}{\Delta t}. \quad (73)$$

For sufficiently small Δt and $n > 2$, $\langle(\Delta v)^n\rangle \to 0$ faster than Δt. Hence we can truncate (72) to two terms. We further integrate over all initial velocities to obtain the total probability density for a particle's having velocity v. We thus obtain the Fokker-Planck equation:

$$\frac{\partial}{\partial t}P(v, t) = -\frac{\partial}{\partial v}(M_1P) + \frac{1}{2}\frac{\partial^2}{\partial v^2}(M_2P). \quad (74)$$

In Fig. 16-4*a* we show how the first derivative term causes the center of the

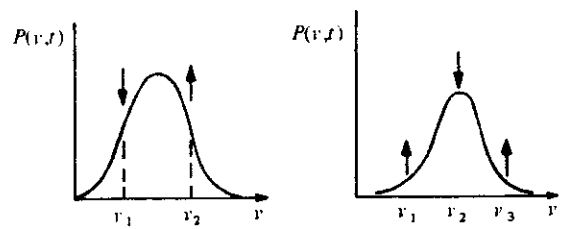

Figure 16-4. (a) For $M_1 > 0$, the first derivative $\partial(M_1P)/\partial v$ at $v = v_1$ is positive, causing $P(v_1, t)$ to decrease in time. For $v = v_2$, the derivative is negative, causing $P(v_2, t)$ to increase in time. Thus the center of gravity of $P(v, t)$ moves or "drifts" toward larger v. (b) Here the second derivative $\partial^2(M_2P)/\partial v^2$ is positive for $v = v_1$ and $v = v_3$, leading to increases in $P(v, t)$ in time, while for $v = v_2$ a decrease occurs, that is, the distribution spreads or diffuses.

distribution to shift or "drift" toward larger values of v if M_1 is positive and toward smaller values for M_1 negative. Mathematically, this is represented by the fact that the average v value has the equation of motion

$$\frac{d}{dt}\langle v(t)\rangle = M_1, \tag{75}$$

which is the content of Prob. 16-9. The first moment M_1 is consequently called the "drift term." In Fig. 16-4*b* we show how the distribution spreads or diffuses. In fact, the mean-square deviation

$$\langle v^2(t)\rangle - \langle v(t)\rangle^2 \tag{76}$$

has the equation of motion [$\Gamma = M_1/\langle v\rangle$]

$$\frac{1}{2}\frac{d}{dt}[\langle v^2(t)\rangle - \langle v(t)\rangle^2] = -\Gamma[\langle v^2(t)\rangle - \langle v(t)\rangle^2] + M_2, \tag{77}$$

as the reader can show by solving Prob. 16–10. For $M_1 = 0$ ($\Gamma = 0$), as may occur in (66), the distribution diffuses with the values

$$\langle x^2\rangle - \langle x\rangle^2 = 2M_2 t.$$

Hence M_2 is called the diffusion term. For $M_1 < 0$ ($\Gamma > 0$), (77) has the steady-state value

$$\langle v^2(t)\rangle - \langle v(t)\rangle^2 = \frac{M_2}{\Gamma},$$

in which the spreading speed is limited by the damping. We return to a discussion of this in Sec. 19-1 on Brownian motion.

The Fokker-Planck equation (63) for our atomic beam reservoir problem is a two-dimensional extension of (66). In it the complex variables a and a^* are independent. It is often convenient to use instead Cartesian or polar coordinates. We consider the latter here and leave the Cartesian representation to Prob. 16-11. With

$$a = r\exp(i\theta), \tag{78}$$

we transform to r and θ by writing

$$r^2 = aa^*, \qquad \theta = \tfrac{1}{2}i\ln(a^*/a). \tag{79}$$

These give the partial derivatives

$$\frac{\partial r}{\partial a} = \frac{1}{2}\frac{a^*}{r} = \tfrac{1}{2}\exp(-i\theta), \tag{80}$$

$$\frac{\partial\theta}{\partial a} = -\tfrac{1}{2}i\frac{\partial\ln(a)}{\partial a} = -\frac{1}{2}\frac{i}{a} = -\frac{1}{2}\frac{i}{r}\exp(-i\theta), \tag{81}$$

and hence by the chain rule

$$\frac{\partial}{\partial a} = \frac{\partial r}{\partial a}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial a}\frac{\partial}{\partial\theta}$$

$$= \tfrac{1}{2}\exp(-i\theta)\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right). \tag{82}$$

The partial $\partial/\partial a^*$ is given by the complex conjugate of (82):

$$\frac{\partial}{\partial a^*} = \tfrac{1}{2}\exp(i\theta)\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right), \tag{83}$$

which with (82), gives the second derivative (note that the purely imaginary terms have to cancel)

$$\frac{\partial^2}{\partial a\,\partial a^*} = \frac{1}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\theta^2}\right)$$

$$= \frac{1}{4r^2}\left(r\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\theta^2}\right). \tag{84}$$

Noting that

$$\frac{\partial}{\partial a}a = 1 + a\frac{\partial}{\partial a} = 1 + \tfrac{1}{2}r\frac{\partial}{\partial r} - \tfrac{1}{2}i\frac{\partial}{\partial\theta} = \frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}r^2 - \tfrac{1}{2}i\frac{\partial}{\partial\theta}, \tag{85}$$

we obtain the transformed Fokker-Planck equation:

$$\frac{d}{dt}P(r,\theta,t) = \frac{1}{2}\frac{v}{Q}\frac{\partial}{\partial r}\left[r^2P(r,\theta,t)\right] + \frac{1}{4}\frac{(v/Q)\bar{n}}{r^2}\left(r\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\theta^2}\right). \tag{86}$$

16-4. Generalized Reservoir Theory

We have dealt with the atomic beam reservoir problem at some length, since it illustrates the reservoir concept and the coarse-grained time derivative in a simple manner. We have considerable use for these concepts in Chap. 17 on the quantum theory of the laser. We now turn to the problem of two interacting systems in a more general context in which both systems are treated on an equal footing and emphasis is placed on the correlation developed between the systems. We apply the theory to the spontaneous decay of an excited two-level atom.

Consider two systems interacting with energy $\mathcal{V}(t)$. As in Sec. 16-2, the reduced density matrices ρ_A and ρ_B for the A and B systems, respectively, are given by traces over the complete set of states of the other system, that is,

$$\rho_A(t) = \mathrm{Tr}_B\{\rho_{AB}(t)\} = \sum_B\langle B|\rho_{AB}(t)|B\rangle, \tag{87}$$

$$\rho_B(t) = \mathrm{Tr}_A\{\rho_{AB}(t)\} = \sum_A\langle A|\rho_{AB}(t)|A\rangle. \tag{88}$$

Equations of motion for ρ_A and ρ_B are then, from (44) (in the interaction picture),

$$i\hbar\frac{d}{dt}\rho_A(t) = \mathrm{Tr}_B[\mathcal{V}(t), \rho_{AB}(t)], \tag{89}$$

$$i\hbar \frac{d}{dt} \rho_B(t) = \mathrm{Tr}_A[\mathscr{V}(t), \rho_{AB}(t)]. \tag{90}$$

Our procedure (following Lax) for solving these equations is to write the total

$$\rho_{AB}(t) = \rho_A(t) \otimes \rho_B(t) + \rho_c(t). \tag{91}$$

Here $\rho_c(t)$ represents the correlation which develops in time between the systems because of their interaction. Next we write the interaction term $[\mathscr{V}(t), \rho_A(t) \otimes \rho_B(t)]$ in the form

$$\begin{aligned}[\mathscr{V}(t), \rho_A(t) \otimes \rho_B(t)] &= \mathscr{V}(t)\rho_A(t) \otimes \rho_B(t) - \rho_A(t) \otimes \rho_B(t)\mathscr{V}(t) \\ &= \mathscr{V}(t)\rho_A(t) \otimes \rho_B(t) - \rho_B(t)\mathscr{V}(t)\rho_A(t) \\ &\quad + \rho_B(t)\mathscr{V}(t)\rho_A(t) - \rho_A(t) \otimes \rho_B(t)\mathscr{V}(t). \end{aligned} \tag{92}$$

Here $\mathscr{V}(t)$ includes both system and reservoir operators and thus implicitly contains an outer product. Using (92) and the fact that operators commute under a trace, we find

$$\begin{aligned}\mathrm{Tr}_B[\mathscr{V}(t), \rho_A(t) \otimes \rho_B(t)] &= \mathrm{Tr}_B[\mathscr{V}(t), \rho_B(t)]\rho_A(t) + [\mathrm{Tr}_B\{\rho_B(t)\mathscr{V}(t)\}, \rho_A(t)] \\ &= [\mathrm{Tr}_B\{\mathscr{V}(t)\rho_B(t)\}, \rho_A(t)] \\ &= [\mathscr{V}_A(t), \rho_A(t)], \end{aligned} \tag{93}$$

where

$$\mathscr{V}_A(t) = \mathrm{Tr}_B\{\mathscr{V}(t)\rho_B(t)\} \tag{94}$$

is a Hartree-type self-consistent energy for the A system. Similarly we find

$$\mathrm{Tr}_A[\mathscr{V}(t), \rho_A(t) \otimes \rho_B(t)] = [\mathscr{V}_B(t), \rho_B(t)], \tag{95}$$

where

$$\mathscr{V}_B(t) = \mathrm{Tr}_A\{\mathscr{V}(t), \rho_A(t)\} \tag{96}$$

is a Hartree energy for the B system. With (93)–(96) and (91), the equations of motion (89) and (90) become

$$i\hbar \frac{d}{dt} \rho_A(t) = [\mathscr{V}_A(t), \rho_A(t)] + \mathrm{Tr}_B[\mathscr{V}(t), \rho_c(t)], \tag{97}$$

$$i\hbar \frac{d}{dt} \rho_B(t) = [\mathscr{V}_B(t), \rho_B(t)] + \mathrm{Tr}_A[\mathscr{V}(t), \rho_c(t)]. \tag{98}$$

In order to integrate these expressions, we need an equation for the correlation operator $\rho_c(t)$. From (91), (97), and (98), this is

$$\begin{aligned} i\hbar\,\dot{\rho}_c(t) &= i\hbar\{\dot{\rho}_{AB}(t) - \dot{\rho}_A\rho_B - \rho_A\dot{\rho}_B\} \\ &= [\mathscr{V}(t), \rho_A \otimes \rho_B + \rho_c] - [\mathscr{V}_A(t), \rho_A[\rho_B - \mathrm{Tr}_B][\mathscr{V}(t)\rho_c]\rho_B \\ &\quad - \rho_A[\mathscr{V}_B(t), \rho_B] - \rho_A\mathrm{Tr}_A[\mathscr{V}(t), \rho_c]. \end{aligned} \tag{99}$$

We consider problems for which the Hartree energies $\mathscr{V}_A$ and $\mathscr{V}_B$ are zero (see Prob. 16-13), and we neglect the higher-order terms containing the commutator $[\mathscr{V}, \rho_c]$. In these approximations, we have the equation of motion

$$i\hbar \frac{d}{dt} \rho_c(t) = [\mathscr{V}(t), \rho_A(t) \otimes \rho_B(t)], \tag{100}$$

with the formal solution

$$\rho_c(t) = -\frac{i}{\hbar} \int_{t_0}^{t} dt'[\mathscr{V}(t'), \rho_A(t') \otimes \rho_B(t')], \tag{101}$$

where we have assumed that initially the two systems are uncorrelated, so that $\rho_c(t_0) = 0$. Equation (101) can be thought of as the beginning of an iterative solution for ρ_c in terms of ρ_A and ρ_B.

The operator equations of motion (97), (98), and (101) treat both systems on an equal footing, making it possible to study how each system reacts back on the other in the course of time. In general, the equations have to be solved in a self-consistent fashion. Their solution is simplified in the study of reservoir interactions for which one of the systems (the reservoir) is too large to respond significantly to energy inputs from the other. Good examples of this situation are provided by an ensemble of oscillators (e.g., photons or phonons) regarded collectively as a reservoir in interaction with a mode of the radiation field or an atomic system.

Suppose that $\rho_A(t)$ is the density matrix for the small system and $\rho_B(t)$ is the density matrix for the reservoir. The reservoir assumption allows us to take

$$\frac{d}{dt} \rho_B(t) \approx 0. \tag{102}$$

Hence we can set $\rho_B(t') = \rho_B(t_0)$ in Eq. (101) for $\rho_c(t)$. Substituting this modified value of $\rho_c(t)$ into (97) for $\rho_A(t)$ in which we drop the Hartree term, we find the equation of motion

$$\begin{aligned} \dot{\rho}_A(t) &= -\frac{1}{\hbar^2} \int_{t_0}^{t} dt'\, \mathrm{Tr}_B[\mathscr{V}(t), [\mathscr{V}(t'), \rho_A(t') \otimes \rho_B(t_0)]]. \\ &= -\hbar^{-2} \int_{t_0}^{t} dt'\, \mathrm{Tr}_B\{\mathscr{V}(t)\mathscr{V}(t')\rho_A(t') \otimes \rho_B(t_0) \\ &\quad -\mathscr{V}(t)\rho_A(t') \otimes \rho_B(t_0)\mathscr{V}(t')\} + \text{adjoint}. \end{aligned} \tag{103}$$

This is a valid equation for a system represented by ρ_A interacting with a reservoir represented by ρ_B. Note that, if the reservoir produces a shift in the frequency of the system (e.g., the Hartree term is not zero), this can be included in the unperturbed frequency of that system.

Because the t' integration is over correlation functions of the reservoir which are characterized by times short compared to those (decay times, etc.) of the system, we can often set

$$\rho_A(t') = \rho_A(t) \tag{104}$$

in the integrand of (103). This is called the "Markoff approximation." The models we treat immediately below allow us to work out the interaction-induced correlations explicitly and hence to note to what degree the Markoff approximation is valid. Note that (103) with (104), is obtained from the second-order approximation of (44) by letting the difference $\tau = t - t_0$ be small (Prob. 16-14).

16-5. Field and Atom Damping by Simple Harmonic Oscillator Reservoir

In this section we apply the generalized reservoir theory of Sec. 16-4 to the damping of a simple harmonic oscillator and of a two-level atom by a reservoir of simple harmonic oscillators. The field damping provides an alternative model for a laser cavity Q and is, in fact, the model used later in the Langevin treatment of laser oscillation (Sec. 20-2). The atom damping is just the Weisskopf-Wigner problem of Sec. 14-4, here treated with density operator techniques. In Prob. 16-14 this phenomenon is analysed with the perturbation method of Sec. 16-2.

We describe the single-mode field by the annihilation operator a, creation operator $a^\dagger$, and frequency Ω. We describe the simple harmonic oscillators of the reservoir by annihilation operators b_k, creation operators $b_k^\dagger$, and densely distributed frequencies ω_k. The interaction energy is then given by

$$\mathscr{V}(t) = \hbar \sum_k g_k a b_k^\dagger \exp[-i(\Omega - \omega_k)t] + \text{adjoint}, \tag{105}$$

where we have made the rotating-wave approximation as discussed in Sec. 14-3. We assume that the reservoir variables are distributed in uncorrelated thermal equilibrium mixtures of states. Hence the reservoir reduced density operator is the multimode extension of the thermal operator (15.38), namely,

$$\rho_B = \prod_k \exp(-\hbar\omega_k b_k^\dagger b_k / k_B T)\{1 - \exp(-\hbar\omega_k / k_B T)\}. \tag{106}$$

In Prob. 16-13, we show, using (106), that

$$\mathrm{Tr}_B(b_k^\dagger b_j \rho_B) = \bar{n}_k \delta_{k,j}, \tag{107}$$

where the thermal average boson number

$$\bar{n}_k = \frac{1}{\exp(\hbar\omega_k / k_B T) - 1}. \tag{108}$$

We further show that

$$\mathrm{Tr}_B\{b_k b_j^\dagger \rho_B\} = (\bar{n}_k + 1)\delta_{k,j}, \tag{109}$$

$$\mathrm{Tr}_B\{b_k b_j \rho_B\} = \mathrm{Tr}_B\{b_k^\dagger b_j^\dagger \rho_B\} = 0. \tag{110}$$

Hence insertion of the interaction energy (105) into the equation of motion (103) for $\rho_A(t)$ yields

$$\begin{aligned}\dot{\rho}_A(t) = -\int_{t_0}^{t} dt' \sum_k g_k^2 \{ & a^\dagger a \rho_A(t')(\bar{n}_k + 1)\exp[i(\Omega - \omega_k)(t - t') \\ & + a a^\dagger \rho_A(t')\bar{n}_k \exp[-i(\Omega - \omega_k)(t - t') \\ & - a^\dagger \rho_A(t')\, a\,(\bar{n}_k + 1)\exp[i(\Omega - \omega_k)(t - t')] \\ & + a\rho_A(t')a^\dagger \bar{n}_k \exp[-i(\Omega - \omega_k)\, t - t')]\} + \text{adjoint}.\end{aligned} \tag{111}$$

We now carry out the same procedure as was used in the Weisskopf-Wigner theory of spontaneous emission (Sec. 14-4). Specifically, we change the summation over k in (111) to an integration over frequency

$$\sum_k \to \int \mathfrak{D}(\omega) d\omega, \tag{112}$$

where $\mathfrak{D}(\omega)$ is the density of reservoir oscillator states. Furthermore, we replace the resulting $\mathfrak{D}(\omega)\ \bar{n}(\omega)\ g^2(\omega)$ by $\mathfrak{D}(\Omega)\ \bar{n}(\Omega)\ g^2(\Omega)$, inasmuch as this product varies little in the slowly varying region of the exponentials. Finally we set

$$\int_{t_0}^{t} dt' \exp[\pm i(\Omega - \omega)(t - t')] = \pi\delta(\Omega - \omega), \tag{113}$$

where we neglect the principal part, as discussed in Sec. 14-4. In these approximations, Eq. (111) becomes

$$\begin{aligned}\dot{\rho}_A(t) = & -\tfrac{1}{2}\gamma\{\bar{n}[a a^\dagger \rho_A(t) - a^\dagger \rho_A(t) a] + (\bar{n} + 1)[a^\dagger a \rho_A(t) - a\rho_A(t)a^\dagger]\} \\ & + \text{adjoint},\end{aligned} \tag{114}$$

where $\gamma = 2\pi\mathfrak{D}(\Omega)g^2(\Omega)$ is the Fermi golden rule rate constant derived in Eq. (14-105) and $\bar{n} = \bar{n}(\Omega)$. Equation (114) with γ replaced by ν/Q is the same as (52), obtained in Sec. 16-2 for a two-level atom reservoir. Thus the explicit model of a reservoir in thermal equilibrium does not affect our current results.

The equation of motion for a two-level atom damped by a reservoir of simple harmonic oscillators (spontaneous emission problem) can literally be obtained by the replacement of a and $a^\dagger$ by σ and $\sigma^\dagger$, respectively, in the interaction energy (105), in the intermediate result(111), and hence in Eq.(114). The explanations for the derivation are identical. We thus find for the two-level atom density operator $\rho_{\text{atom}}(t)$

$$\begin{aligned}\dot{\rho}_{\text{atom}} = & -\tfrac{1}{2}\gamma\{\bar{n}[\sigma\sigma^\dagger\rho_{\text{atom}} - \sigma^\dagger\rho_{\text{atom}}\sigma] \\ & + (\bar{n} + 1)[\sigma^\dagger\sigma\rho_{\text{atom}} - \sigma\rho_{\text{atom}}\sigma^\dagger]\} + \text{adjoint}.\end{aligned} \tag{115}$$

In particular, for zero temperature ($\bar{n} = 0$) and an initially excited atom [$\rho_{aa}(0) = 1$], Eq. (115) implies

$$\dot{\rho}_{aa}(t) = -\gamma\rho_{aa}, \tag{116}$$

which is just the Weisskopf-Wigner result derived in Sec. 14-4 with the use of the state vector. We return to this important problem in the Heisenberg picture in Chap. 19.

In conclusion, let us consider more carefully what reservoir characteristics are required for the damping to be truly Markoffian. The double integrals involved in (111) with (112) have the form

$$\int_0^{t-t_0} d\tau \int d\omega\, g^2(\omega)\mathcal{D}(\omega) \exp[i(\Omega - \omega)\tau]. \tag{117}$$

As the interval $t - t_0$ increases, the integration over τ acts as a frequency filter. Only reservoir modes with frequencies close to $\Omega(|\omega - \Omega| \lesssim 1/\tau)$ have significant effect on the motion of the system of interest. If the reservoir spectrum is densely and broadly distributed around Ω, the frequency integral becomes negligible as τ increases because of destructive interference. We call the time interval τ for which the frequency integral has appreciable value the correlation time τ_c. If little change occurs in the A system during this time(i.e., $\tau_c \ll 1/\gamma$), the Markoff approximation of Eq. (104) is valid. This approximation is also very useful in the noise operator approach of Chaps.19 and 20.

Problems

16-1. What is the average energy of an atom described by the density matrix of (19) and (20) to order g^2?

Ans. $\langle \mathscr{H}_{\text{atom}}(t+\tau)\rangle = \hbar\omega_b + \hbar(\omega_a - \omega_b)g^2r^2\langle n\rangle$.

16-2. To order g, calculate the expectation value of the dipole moment operator er, using the state vector (14).

Ans. $\langle er\rangle = \wp g\tau \sum_n (n+1)^{1/2}\, iC_{n+1}^*(t)\, C_n(t) + \text{c.c.}$

16-3. Derive Eq. (24) for $\dot{\rho}_{nm}|_{|a\rangle \text{ atoms}}$ by filling in the steps for Eqs. (23) and so forth.

16-4. Using Eq. (26) for ρ_{nm}, calculate the equation of motion for the ensemble average electric field $\langle E(t)\rangle$.

Ans. Equation (36).

16-5. Along the lines leading to (33) for $(d/dt)\langle n(t)\rangle$, calculate the corresponding equation for $\langle n^2(t)\rangle$ and its steady-state value.

16-6. What is the average photon number $\bar{n}$ of (32) for (*a*) $\sim$ zero temperature, (*b*) room temperature ($k_BT \approx \frac{1}{40}$ eV) and optical frequencies (1μ is 1.24 eV), (*c*) microwave frequencies (10,000 *MHz*)?

16-7. Derive the equation of motion (26) for ρ_{nm} by projecting the operator equation (51) into the number representation.

16-8. Show that Eqs. (54) and (55) for $a^\dagger|a\rangle\langle a|$ and $|a\rangle\langle a|a$ are true as outlined around Eq. (57).

16-9. Using the Fokker-Planck equation (74), show that the average velocity $\langle v(t)\rangle$ has the equation of motion (75).

16-10. Again using the Fokker-Planck equation (74), show that the mean-square deviation (76) has the equation of motion (77).

16-11. Show that, in the Cartesian representation defined by the transformation

$$a = x + iy, \qquad a^* = x - iy, \qquad x = \tfrac{1}{2}(a + a^*), \qquad y = \tfrac{1}{2}i(a - a^*), \tag{118}$$

the Fokker-Planck equation (63) is given by

$$\frac{\partial}{\partial t}P(x, y, t) = \frac{1}{2}\frac{\nu}{Q}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right)P(x, y, t) + \frac{1}{4}\frac{\nu}{Q}\bar{n}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)P(x, y, t). \tag{119}$$

16-12. Show that, for a two-level atom or a simple harmonic oscillator (SHO) interacting with a reservoir of SHO's, the Hartree energies vanish.

16-13. Show that the reservoir average

$$\langle b_j^\dagger b_k\rangle_B = \text{Tr}_B[b_j^\dagger b_k \rho_B] = \bar{n}_k\delta_{jk},$$

with $\bar{n}_k$ given by (108), in which the reduced density operator ρ_B is the many-mode thermal operator (106). Further show that

$$\langle b_j b_k^\dagger\rangle_B = (\bar{n}_k + 1)\delta_{jk}, \qquad \langle b_j b_k\rangle_B = \langle b_j^\dagger b_k^\dagger\rangle_B = 0.$$

16-14. Using the iterative expansion (44) to second order, show that both the two-level atom and the SHO systems damped by reservoirs of SHO's, are governed by

$$\dot{\rho}_A(t) \approx \frac{1}{\tau}[\rho_A(t+\tau) - \rho_A(t)] = -(\tau\hbar^2)^{-1}\int_t^{t+\tau} dt' \int_t^{t+\tau} dt'' \times \text{Tr}_B[\mathscr{V}(t')\mathscr{V}(t'')\,\rho_A(t)\otimes\rho_B(t) - \mathscr{V}(t'')\rho_A(t)\otimes\rho_B(t)\mathscr{V}(t'')] + \text{adjoint},$$

which also yields the equations of motion (114) and (115) in the Markoff approximation. The analysis here is highly analogous to that in Sec. 19-3, where the time development is determined by equations of motion for the annihilation and creation (or spin-flip) operators.

16–15. In Sec. 2–2, the energy density laws for the Planck and Wien lays are compared and shown to be fairy close in value. The difference between the two laws is that the Wien neglects stimulated emission. Calculate the photon distribution ρ_{nn} for this case from Eq. (27) by dropping the stimulated emission terms (those proportional to n). Then show that the second moment $\langle n^2\rangle$ is half the Planck value of Prob. 16-5.

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