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# "On the Equivalence and Properties of Noisy Neural and Probabilistic Ram Nets"

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# ON THE EQUIVALENCE AND PROPERTIES OF NOISY NEURAL AND PROBABILISTIC RAM NETS

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A formal identity is demonstrated between the equations describing the dynamical evolution of a net of noisy RAMs and those for a net of noisy neurons, under certain assumptions on the latter. The dynamical evolution of such noisy nets is investigated both analytically and by computer, and found, in the generic case, to evolve to a unique stable fixed point for the RAM/neuron output probabilities.

It is obvious that living neural nets represent a solution to the problem of the construction of intelligent machines. However, the exact nature of that solution has not yet been discovered. This is so despite several decades of intensive experimental and theoretical effort, going back to the work of McCullough and Pitts [1], whose description of the neuron as a binary decision element (BDN) has informed much later work in the field. The activity of nets of BDNs was also expressed in mathematical form [2], which led to further attempts to explain brain activity [3], though without any fundamental breakthrough. The McCullough–Pitts model was subsequently extended by many authors, most notbly Little [4], to include an account of the stochastic nature of synaptic transmission, the so-called "noisy neuron". Little's model was not concerned with the details of synaptic physiology, all sources of noise (such as random variations in the number and size of the packets of neurotransmitter released by a presynaptic neuron, or spontaneous leakage of transmitter substance into the synaptic cleft) being absorbed into a single, neuron-dependent "spontaneity parameter"; authors such as Shaw and Vasudevan [5] were later able to inject a greater degree of biological realism by modelling more closely the details of synapse–synapse interaction. However there are alternatives to models of the Little type, for example the model due to Taylor [6,7]. This is roughly equivalent to that of Shaw and Vasudevan in the degree of biological precision involved, but represents a somewhat different approach to stochastic neurodynamics, one which emphasises the primary role of individual neuronal firing probabilities.

The Taylor model for a noisy neural net will be shown in this paper to be formally equivalent in its dynamical development to a network of "noisy" random access memories (RAMs); this equivalence holds out the prospect of a direct hardware realisation of noisy neural networks, and the possibility of designing nets which are capable of displaying a modicum of "intelligent" behaviour and are able to perform useful work. Conventional, deterministic RAMs are used throughout the electronics industry, and nets of RAMs with a limited level of noise have been suggested recently [8] to enhance the capabilities of the WISARD pattern recognition machine

[9], which in its original form used purely deterministic memory units. The realisation of the Taylor model as a noisy RAM-net – with each electronic unit playing the processing role of a single neuron – would be very much in the spirit of current connectionist approaches to AI [10]. To date the vast part of the research into parallel distributed processing (PDP) models of cognition has been done using simulations on conventional serial computers; such simulations are slow and costly. Any direct implementation of PDP ideas in hardware, using some form of neuron-analogue, would surely be of great interest since it would allow the construction and study of much larger and more complex networks than is practicable with serial (or "modestly parallel") simulations.

Current research in VLSI technology, aimed at building progressively faster machines, is tending toward a situation in which only nearest-neighbour interactions between components will be practicable; such a restriction on connectivity would represent a severe limitation for a machine of the type we envisage. However, it should be noted that for many applications (for example in the field of computer vision) this search for ever-increasing speed is related to the need to simulate on a serial machine the operations of a vastly parallel one. Since we would be working with a machine of the latter type there does not appear to be any difficulty in hardware implementation associated with transmission times between different RAMs (if commercially available RAMs with response times of microseconds were used, they could be joined by wires of metres in length). We do not feel that a restriction to microsecond response times would be unduly damaging, given that such a "neurocomputer", although slow by some standards, would nevertheless be able to operate at speeds several orders of magnitude greater than the biological hardware which inspired it.

In addition to demonstrating the formal equivalence of the Taylor model and the noisy RAM net we present an analysis of the time-development of such noisy nets. In particular both analytic and computational techniques are used to determine this behaviour, and our results compared with those of other models.

We start by summarising the assumptions and fundamental results of the Taylor model [6] of noisy neural nets. The assumptions are as follows:

(1) There are $N$ neurons, with arbitrary connectivity, and total activity at time $t$ described by the probabilities $p_j(t)$ $(1 \leqslant j \leqslant N)$ that the neurons will fire.

(2) Time is assumed discrete, $t=0, \tau, 2\tau, \ldots$, in terms of a smallest unit of time, $\tau$, related to the refractory periods, synaptic and other delays, etc. for the various neurons.

(3) Neural firing is determined by the total amount of transmitter substance arriving on a given neuron from the other neurons (or itself), with no spatial delay of post-synaptic potential over the dendrites or cell body up to the axon hillock.

(4) Transmitter substance is released into the synaptic cleft in a Poison process with frequency $t_1^{-1}$ and lifetime $t_{dec}$. The probability density function of transmitter substance is then

$$\rho_s(q) = e^{-\lambda} \sum_{n \geqslant 0} (n!)^{-1} \lambda^n \delta(n q_0 - q) , \qquad (1)$$

where $\lambda = t_{dec}/t_1$ and $q_0$ is the quantum of transmitter contained in a synaptic vesicle. The suffix s denotes that the activity arising from (1) is spontaneous. The parameters $\lambda$ and $q_0$ will, in general, depend on the indices $j$ and $i$, where $j$ denotes the presynaptic neuron, $i$ the postsynaptic one (and $q_0$ may be negative for an inhibitory effect).

(5) The arrival of a nerve impulse from the $j$th to the $i$th cell causes the release of $n_{ji}$ number of vesicles into the synapse; the density function will then be

$$\rho_d^{(ij)}(q) = \delta(q - n_{ji} q_0^{(ij)}) . \qquad (2)$$

(6) The $i$th neuron fires at a given time if the total amount of transmitter substance (spontaneous, due to the leakage through the function $\rho_s$ of (1), or excited through $\rho_d$ of (2)) is larger than some critical value $q_c$, which again may depend on $i$.

The above assumptions determine the equations governing the time development of the set $\{p_j(t)\}$ as [8]

$$p_i(t+1) = \int_{q_c^{(i)}}^{\infty} dq \prod_{j=1}^{N} \int dq_{ji}\, \delta\Big(q - \sum_j q_{ji}\Big)$$

$$\times \prod_j \{p_j(t)\, \rho_d^{(ij)}(q_{ji}) + [1 - p_j(t)]\, \rho_s^{(ij)}(q_{ji})\} . \qquad (3)$$

This is a dynamical equation in which the r.h.s. is a polynomial of degree $N$ in the variables $\{p_j(t)\}$, $1\leq j\leq N$. The simplest case is for $N=2$, when (3) takes the form

$$p_1(t+1)=(\alpha_0\bar{p}_1\bar{p}_2+\alpha_1\bar{p}_1p_2+\alpha_2p_1\bar{p}_2+\alpha_3p_1p_2)(t)\,, \quad (4a)$$

$$p_2(t+1)=(\beta_0\bar{p}_1\bar{p}_2+\beta_1\bar{p}_1p_2+\beta_2p_1\bar{p}_2+\beta_3p_1p_2)(t)\,, \quad (4b)$$

with $\bar{p}=1-p$ and with the constants

$$\alpha_0=\int_{q_c^{(1)}}^{\infty}\mathrm{d}q\int\mathrm{d}q_1\,\mathrm{d}q_2\,\delta(q-q_1-q_2)\times\rho_s^{(11)}(q_1)\rho_s^{(12)}(q_2)\,, \quad (5a)$$

$$\alpha_1=\int_{q_c^{(1)}}^{\infty}\mathrm{d}q\,\rho_s^{(11)}(q-q_0^{(12)})\,, \quad (5b)$$

$$\alpha_2=\int_{q_c^{(1)}}^{\infty}\mathrm{d}q\,\rho_s^{(12)}(q-q_0^{(11)})\,, \quad (5c)$$

$$\alpha_3=\theta(q_0^{(11)}+q_0^{(12)}-q_c^{(1)})\,. \quad (5d)$$

The $N$-neuron case can be given as a generalisation of (4), (5). Thus if $\boldsymbol{a}$ denotes a binary $N$-vector $\boldsymbol{a}=(a_1,...,a_N)$, $a_i=0$ or 1, then the probabilities $P_{\boldsymbol{a}}$ for net activity $\boldsymbol{a}$ related to the probability vector $\boldsymbol{p}=(p_1,...,p_N)$ may be defined as [9]

$$P_{\boldsymbol{a}}(\boldsymbol{p})=\prod_{r=1}^{N}p_r^{a_r}\bar{p}_r^{\bar{a}_r}\,. \quad (6)$$

Then (3) has the general form

$$P_i(t+1)=\sum_{\boldsymbol{a}}\alpha_{\boldsymbol{a}}^{(i)}P_{\boldsymbol{a}}(\boldsymbol{p}(t))\,, \quad (7)$$

with the $\alpha_{\boldsymbol{a}}^{(i)}$ constants defined by a set of equations analogous to (5). It might be thought that the simplicity of (7) makes the $2^N$ state variables $\{P_{\boldsymbol{a}}\}$ more appropriate to use than the $N$ individual firing probabilities $\boldsymbol{p}$; however it was shown in ref. [9] that identities between the $P_{\boldsymbol{a}}$'s make the linearity of (7) (and corresponding possibility of a markovian analysis of the system) only an illusion, and that the $p_j$ are in general the more suitable variables (the recent report of Clark [3] also contains a careful discussion of this issue, in the context of his own markovian dynamical model). It is relevant to remark that the polynomality on the r.h.s. of (7) has been noted as useful in ref. [10], where elements with such inputs were called "sigma pi" units [2, ch. 2].

It is possible to relax some of the above assumptions and still preserve the polynomial form of (3). Thus the distributions (1) and (2) may be modified, for example to include the variability in size of $q_0^{(ij)}$ (as Shaw and Vasudevan did), or the assumption of a sharp cut-off at $q_c$ for the response of the total neuron $i$. It is also possible to discuss the dynamical development in continuous time [6,7] though that will be considered in more detail elsewhere.

We now turn to the noisy RAM net equations which represent the other side of the identity we wish to demonstrate. A single $N$-RAM is assumed to have $N$ binary inputs, so has $2^N$ possible input vectors each of which goes to one of the $2^N$ possible addresses. We let the set $\{\boldsymbol{a}\}$ denote these vectors, using the same notation as above. It can be seen that an $N$-RAM can perform any one of $2^{2^N}$ binary functions on its $N$ inputs. The resulting outputs are therefore given by the same functions $P_{\boldsymbol{a}}(\boldsymbol{p})$ of the inputs $\boldsymbol{p}$ as defined in (6), and a general $N$-RAM has output

$$F(\boldsymbol{p})=\sum_{\boldsymbol{a}}u_{\boldsymbol{a}}P_{\boldsymbol{a}}(\boldsymbol{p})\,, \quad (8)$$

where $\boldsymbol{a}$ may be regarded as labels for the different addresses, and where, in the deterministic case, the constants $u_{\boldsymbol{a}}$ are equal to zero except for one, which has value unity. The behaviour of a net of such RAMs can be investigated and these deterministic networks have been analysed in detail by a number of authors [11].

Noise has been added in ref. [8] to RAM outputs by assuming that for each binary input vector $\boldsymbol{i}$ (or equivalently address location $\boldsymbol{a}$) the output is one with a probability 0, $\frac{1}{2}$ or 1. In order to draw a parallel between the noisy RAM and noisy neuron we extend that idea further, having output one for each of the addresses of the RAM with some fixed probability between 0 and 1 (thus we may regard the constant $u_{\boldsymbol{a}}$ in (8) as the probability for giving output 1 at the associated address $\boldsymbol{a}$). This is a natural extension of RAM activity, one which would be expected to allow a much more flexible encoding of the features of the external world than the case where the output probabilities are restricted to {0,1} or {0, $\frac{1}{2}$, 1}.

We may describe the connectivity of a net of such noisy RAMs by introducing a connectivity matrix $C^{(i)}$, which will be an $N\times n$ matrix if there are $n$ $N$-RAMs in the net, so that $C^{(i)}\boldsymbol{p}$ is the binary input entering the $i$th RAM, where the net is in state $\boldsymbol{p}=(p_1, ..., p_n)$. Assuming a discrete time-development and synchronous operation of the network we obtain from (8) that

$$p_i(t+1)=\sum_a u_a^{(i)} P_a(C^{(i)}\boldsymbol{p}(t)) , \tag{9a}$$

$$\text{with } 0\leqslant u_a^{(i)} \leqslant 1 . \tag{9b}$$

For $n=N$ and $C^{(i)}=1$, the identity matrix, the set of equations (9) has identical form to (7), with identification of the constants $u_a^{(i)}$ and $\alpha_a^{(i)}$. Thus we have arrived at the following result:

*Every neural net composed of N noisy neurons, and satisfying assumptions (1)–(6) above, has identical dynamical development to that of a net composed of N noisy N-RAMs with suitable connectivity.*

It appears that the converse is also true (although then the constants (5a)–(5d) have to be defined in terms of more general probability distributions, since otherwise $\alpha_3$ will only have the value 0 or 1, when $N=2$, for example). It is also clear that a net of $n$ noisy $N$-RAMs, for $n>N$, will behave identically to a net of noisy neurons in which only $N$ contribute output to any one of the neurons. These ideas may be further extended to the case of a set of $N$-RAMs with different values of $N$, so giving the more general result:

*The dynamical behaviour of any net of noisy neurons satisfying assumptions (1)–(6) (with arbitrary $p_a$, $p_d$ replacing (1) and (2)) can be mirrored by that of some net of noisy RAMs, and conversely.*

The above result indicates that further investigation of the behaviour of (9) is of interest both from the viewpoint of modelling brain activity and that of developing intelligent machines. The remainder of this paper, then, is a study of the equations (9). These form an infinite family of families of polynomial maps. Thus for each $N$, (9) is a polynomial map of degree $N$ from $[0,1]^N$ into itself. Furthermore, for each $N$, the map depends on $2^N\times N$ constants. The existence of such an infinite set of maps was already realised in 1971 [6], but their properties were not understood then, in spite of attempts at analysis [12]. The development of electronic computers, and the incereased understanding of the properties of maps of finite (or infinite) dimensional spaces into themselves associated with the development of chaos [13] have now allowed a better understanding of the behaviour of the maps (9), and in particular of their dependence on the parameters $u_a^{(i)}$.

To begin with let us consider the $N=2$ case

$$p_1(t+1)=(\alpha_0\bar{p}_1\bar{p}_2+\alpha_1\bar{p}_1 p_2 +\alpha_2 p_1\bar{p}_2+\alpha_3 p_1 p_2)(t) ,$$
$$p_2(t+1)=(\beta_0\bar{p}_2\bar{p}_1+\beta_1\bar{p}_2 p_1 +\beta_2 p_2\bar{p}_1+\beta_3 p_2 p_1)(t) , \tag{10}$$

which can be realised as the net of two 2-RAMs of fig. 1a. A detailed analysis has been given of two-dimensional maps [14], where numerous maps with chaotic behaviour have been described; it would seem likely that this simple network could also display such behaviour. It is possible to analyse (10) analytically, since the two fixed points $(p_{1\pm}, p_{2\pm})$ can be obtained as the solution of a quadratic equation in one or other of the variables $p_1$, $p_2$ by elimination of the other from the fixed point equations for (10). If we denote the r.h.s. of (10) as $F(\boldsymbol{p})$ $(\boldsymbol{F}=(F_1, F_2))$, then a stability analysis can be given in terms of the eigenvalues of the $2\times 2$ matrix $\mathrm{d}F(\boldsymbol{p}_\pm)$. Using $10^6$ randomly-generated values of the parameters $\alpha_0$, ..., $\beta_3$

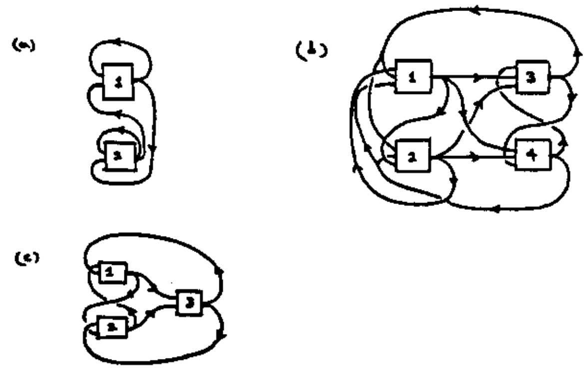

Fig. 1. (a) The detailed connections between two 2-RAMs described analytically by eq. (10). (b) The connections between four 3-RAMs used in computer simulation to analyse the number and nature of convergence to fixed points. (c) The connections between the three 3-RAMs used in the same computer simulation as (b).

in (10) we found, somewhat surprisingly, that in all cases $p_-$ was the only fixed point in $[0,1]^2$, and either had eigenvalues $\lambda$ of $dF(p)$ with $|\lambda|<1$ (so no Hopf bifurcation), or one real eigenvalue $\lambda_-$ with $\lambda_- \leq 1$. In the latter case the associated one-dimensional map was investigated and always found stable. In a further simulation, using parameters from the set $\{0, \frac{1}{2}, 1\}$, the only case when convergence to a fixed point did not occur was for parameters such that $p_+$ lay on the boundary of $[0, 1]^2$. In that case there was convergence to $p_-$ under iteration except for these special values of the $\alpha$'s and $\beta$'s for which the 2-cycles $(u_1, c) \leftrightarrow (c, u_2)$, $u_1 = u_2 \in \{0, 1\}$ or $(u_1, c) \leftrightarrow (1-c, u_2)$, $u_1 \neq u_2 \in \{0, \frac{1}{2}, 1\}$, occurred. One could also have cases where (10) were linear, when up to a 4-cycle could occur. Though it was unfortunately not possible to obtain a purely analytic result, we feel that the combined analytic/computational evidence is sufficiently strong to conjecture that *for generic values of the parameters, the dynamical evolution of an $N=2$ noisy net is always to a unique stable fixed point; for special values of the parameters on the boundary of $[0, 1]^8$ the evolution may be to a 2- or 4-cycle, in which case the detailed behaviour may depend on the initial state of the network.*

One cannot extend this approach to the case where $N>2$: even for $N=3$, with equations of the form

$$p_1(t+1) = (a + bp_2 + cp_3 + dp_2p_3)(t) ,$$
$$p_2(t+1) = (e + fp_1 + gp_3 + hp_1p_3)(t) ,$$
$$p_3(t+1) = (l + mp_1 + np_2 + qp_1p_2)(t) , \qquad (11)$$

elimination of, say, $p_2$ and $p_3$ to determine the fixed points of (10) gives a quintic equation for $p_1$. Thus there is no analytic solution. Of course the Brouwer fixed point theorem shows that there is always a fixed point of (9) in $[0,1]^N$, for each $N$ and any set of parameters $u_a^{(i)}$. Moreover this can be proven unique by the contraction mapping principle, for values of the parameter $i$ suitably close to $u_0$ (where $0_0 = (0, ..., 0)$). This can be seen immediately for $N=2$ in (10), and by a generalisation to higher $N$, since for any solutions $p(t)$, $p'(t)$ of (10),

$$|p_1(t+1) - p'_1(t+1)| \leq |\alpha_1 - \alpha_0|\ |p_1(t) - p'_1(t)| + |\alpha_2 - \alpha_0|\ |p_2(t) - p'_2(t)| + |\alpha_3 + \alpha_0 - \alpha_1 - \alpha_2| \times [|p_1(t) - p'_1(t)| + |p_2(t) - p'_2(t)|], \qquad (12)$$

with a similar equation for $|p_2(t+1) - p'_2(t+1)|$. Thus (10) is a contraction map if, for example

$$\max\{(|\alpha_1 - \alpha_0| + |\alpha_2 - \alpha_0| + 2|\alpha_3 + \alpha_0 - \alpha_1 - \alpha_2|),\ (|\beta_1 - \beta_0| + |\beta_2 - \beta_0| + 2|\beta_3 + \beta_0 - \beta_1 - \beta_2|)\} < 1 . \qquad (13)$$

However the region given by (13) is considerably smaller than the total regional $[0,1]^8$ of $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3)$, and becomes an even smaller proportion of the allowed region for higher $N$.

We have investigated the behaviour of eq. (9) for higher $N$ by wholly computational techniques. The program chose at random values of the memory and initial-state parameters in $[0,1]^M$, where $M = 2^N \times N$ or $2^N \times (N+1)$ is the total number of parameters. For each such choice, interations were made of (9), and the iterations deemed to have converged if the difference $|p_1(t+1) - p_1(t)| < 0.001$. $5 \times 10^4$ such runs were done for a net of four 3-RAMs connected maximally, as in fig. 1b, or a net of three 3-RAMs as in fig. 1c. In both cases we again discovered no behaviour other than convergence to a fixed point. It was found difficult to proceed to much higher $N$ for such maximally connected nets, so $n$ $N$-RAMs ($n \gg N$) were connected together randomly, and the above iteration procedure (with convergence criterion $|u_i(t+1) - u_i(t)| < 0.001$, $1 \leq i \leq n$) repeated, using randomised starting values, but the same memory content parameter and connectivity for each 100 runs. The results of these computations are presented in fig. 2 where the ordinate is the average run length to convergence, the average being taken over 100 runs for each $N$. The graph shows increasingly fast convergence as N grows, with a limit of around 4 iterations.

Our computational results thus lead us to propose that:

*(i) For generic values of the network parameters, the dynamical evolution of a net composed of $n$ $N$-RAMs has only 1 stable fixed point.*

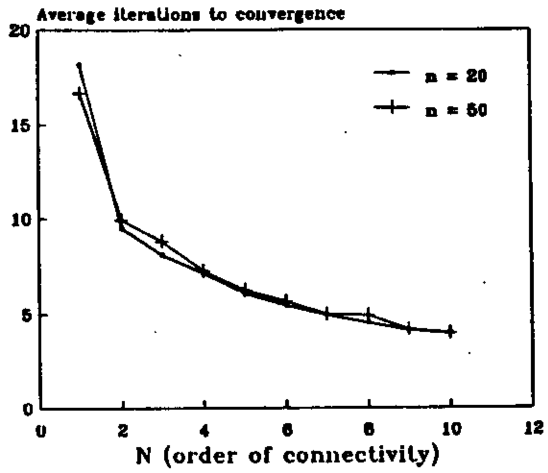


Fig. 2. The average run length (averaged over 100 runs) to convergence, for a randomly connected net of $n$ $N$-RAMs, for various values of $n$ and $N$. Convergence was tested by $|u_i(t+1)-u_i(t)|<0.001$ for $1\leqslant i\leqslant n$.

*(ii) If $n\gg N\geqslant 10$ the convergence is extremely fast, within 4 iterations.*

*(iii) Cyclic behaviour can only occur for boundary values of the parameters, with maximum cycle length 2N.*

Again we stress that we do not have an analytic proof of the above results, but that there is strong support from our computer analyses. The results are, to us, somewhat surprising, since the class of mappings (9) certainly contains chaotic maps for some ranges of the parameters; it contains, for example, the Hénon map [15]

$$p_1(t+1)=1-ap_1^2(t)+p_2(t)\,,$$
$$p_2(t+1)=bp_1(t)\,, \qquad (14)$$

for which the dynamical behaviour becomes chaotic when $a\sim 1.4$, $b\sim 0.3$. However the restricted parameter range (9b) seems to avoid the chaotic regime completely, having for $N=2$ only period 2 or 4 maps on the boundary of the parameter space. Thus eqs. (9) lead to rather different dynamical behaviour than other models of neural nets, many of which do exhibit chaos such as that of ref. [16].

How do we reconcile our results with those of other neural modellers? It is first worthwhile to point out that whilst many authors have indeed reported complex dynamical behaviour leading to chaos [16–18] this has not been observed in equations of the type (9); indeed there has been no previous study of such systems of equations, to our knowledge. It should also be noted that Clark [3] has applied non-equilibrium statistical mechanics to a Little-type model (which is assumed to define an aperiodic, irreducible, homogenous finite Markov chain) and has discovered that the generic behaviour of such a model is evolution toward a single, stable fixed point. Furthermore we note that at the moment there is insufficient neurobiological data available to make any choice between the various "noisy neuron" models, some of which exhibit chaos or metastability, whilst others, like ours, apparently do not. We do not feel our results contradict those of other workers, since the model we have analysed differs in its structure from those which have been found to give rise to complex attractor sets and chaos; it represents an alternative path of enquiry, although one which has the singular advantage of a possible hardware implementation. With this last point in mind , we note that it is not yet clear to what extent chaotic activity occurs in living neural nets (see also the numerous references in ref. [20]) and especially whether such activity is of benefit to an organism; if chaos turns out to be something nature prefers to avoid it would clearly be an undesirable feature to build into our machines. Thus the result that the generic time development of our noisy nets is that of rapid convergence to a single fixed point may indicate that the Taylor model extracts a useful and hardware implementable feature of the activity of living neurons whilst discarding other, unhelpful, features such as chaos.

It may be interesting to note that the response of one of the simplest living neural nets, that of the retina of Limulus can be described well by a system of linear equations for neural activity which also have a single fixed point [21]. The possible description of this living system, and others, within the framework of the present paper is under investigation.

There are three outstanding problems to be addressed before we can understand the way in which the properties of the noisy nets described above could be used in intelligent behaviour. These are (a) extension to a continuous time description, (b) the nature of the input→output transformations which such nets could perform, and (c) the manner in which learning might be achieved in these nets: what is the algorithm for modifications of the parameters $u_\alpha^{(i)}$?

Problem (b) may be resolved in general by taking certain of the $p_i$'s in (9) as given, input, variables and others as the given output variables. The total system of equations (9) is thus reduced to a polynomial mapping of the remaining variables, which have parameters $u_a^{(i)}$ which are functions of the input and output variables. If the latter vary more slowly than the number of iterations to convergence (of the order of 4 iteration times, say 5 ms) then the output will be a faithful representation of the fixed point of the net, which would itself be a function of the input. We propose to discuss the problems (a)–(c), as well as how such fixed point nets may be useful, for example as pattern recognisers, in more detail elsewhere [22].

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