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COLLEGE ON NEUROPHYSICS:
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"Computation of Optical Flow"

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Computation of Optical Flow

- Preliminaries $\begin{cases} \text{Definitions} \\ \text{Limitations} \end{cases}$
- Aperture Problem Revisited
- A simple solution to the aperture problem
- Implementation and error analysis
- Need of a "stable" description of optical flow
- Discontinuities of optical flow

3 D Motion $\quad \vec{V} = \vec{V}(\vec{X})$

$\vec{X} = (X_1, X_2, X_3)$

$\vec{V} = (V_1, V_2, V_3)$

imaging device

$\vec{x} = f\,\dfrac{\vec{X}}{X_3}$

$\vec{v} = f\,\dfrac{\vec{e}_3 \times (\vec{V} \times \vec{X})}{X_3^{\,2}}$

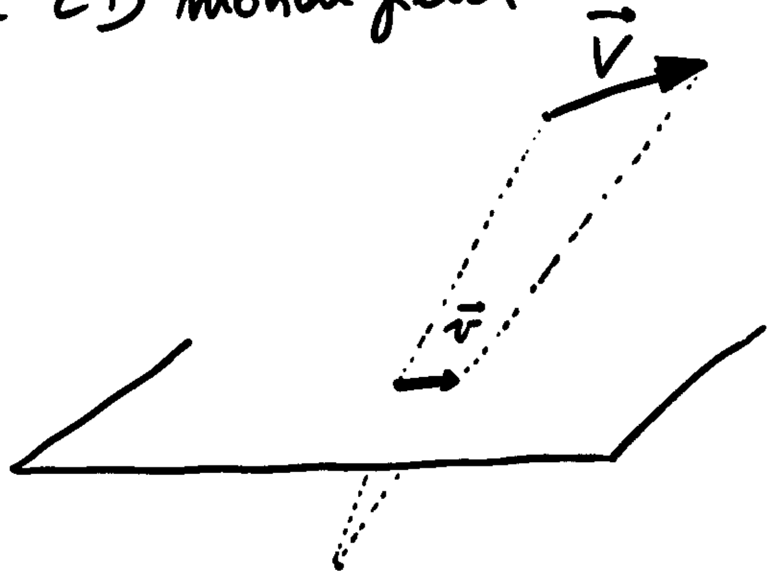
2D Motion Field
Optical Flow

topologically and metrically close

qualitative and quantitative features

$E(x, y, t)$ image brightness at location x on the image plane at tim

The 2D motion field

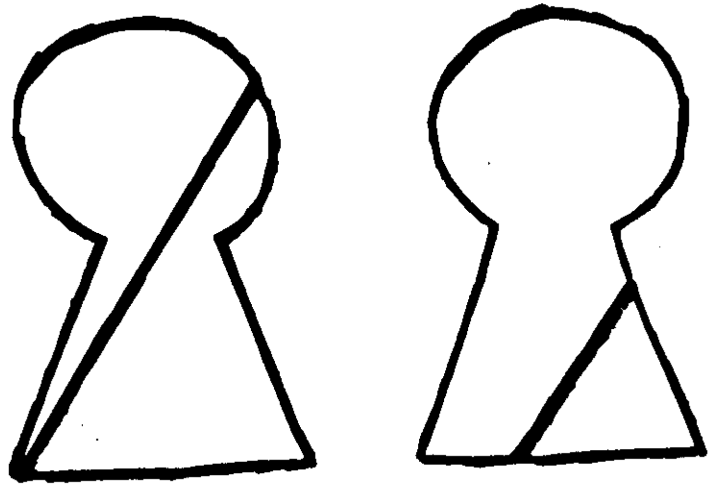


It can be recovered by means of matching techniques

It can be estimated by means of differential techniques

The Optical Flow is the estimate of the 2D motion field according to a given differential techniques

The "aperture" problem



$$\frac{dE}{dt} = 0 \qquad \text{1 equation for 2 unknowns}$$

$$\Downarrow$$

underconstrained problem

Regularization Theory:

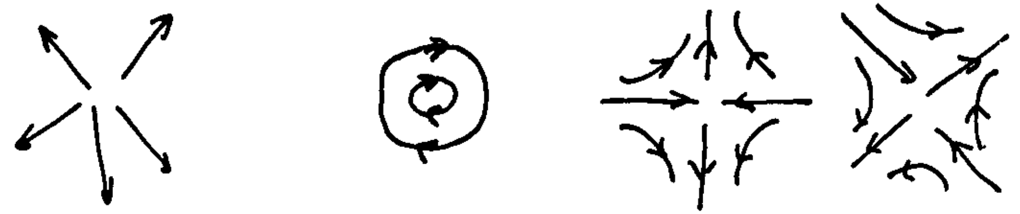
add constraints to the problem so that its solution becomes unique (and depends continuously upon changes on data)

The theory of deformable objects says that an infinitesimal deformation can always be decomposed in:

- a rotation
- an expansion.
- shear 1.
- shear 2.

A linear 2D vector field can be always decomposed in

$$\alpha\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \beta\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} + \gamma\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \delta\begin{pmatrix}-1 & \\ 1 & 1\end{pmatrix}$$



exp.   rot   shear 1   shear 2

$$\frac{d}{dt}\frac{E_x}{E_y}=0 \quad\Rightarrow F^* = \text{const}$$ direction of gradient

$$\frac{d}{dt}\left(E_x^2+E_y^2\right)=0 \quad\Rightarrow G^* = \text{const}$$ amplitude of gradient

$$\frac{d}{dt}\left(E_x E_y\right)=0 \quad\Rightarrow H^* = \text{const}$$ shear I

$$\frac{d}{dt}\left(E_x^2-E_y^2\right)=0 \quad\Rightarrow I^* = \text{const}$$ shear II

to solve the aperture problem E^* and F^* must intersect transversally

$$\Downarrow$$

$$\det H = \det \operatorname{Hess} E(x,y,t) = E_{xx}E_{yy}-E_{xy}^2 \neq 0$$

if any two of F^*, G^*, H^*, I^* are stationary

$$\frac{d}{dt}\operatorname{grad} E = 0 \qquad \begin{aligned}\tilde{u}\,E_{xx}+\tilde{v}\,E_{xy}+E_{xt}&=0\\ \tilde{u}\,E_{xy}+\tilde{v}\,E_{yy}+E_{yt}&=0\end{aligned}$$

$$\vec{\tilde{v}} = \vec{v} + H^{-1}\left(M^T \operatorname{grad} E + \operatorname{grad}\frac{dE}{dt}\right)$$

$$H = \begin{pmatrix} E_{xx} & E_{xy} \\ E_{xy} & E_{yy}\end{pmatrix} \qquad M^T = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{pmatrix}$$

if $\frac{dE}{dt}=0$

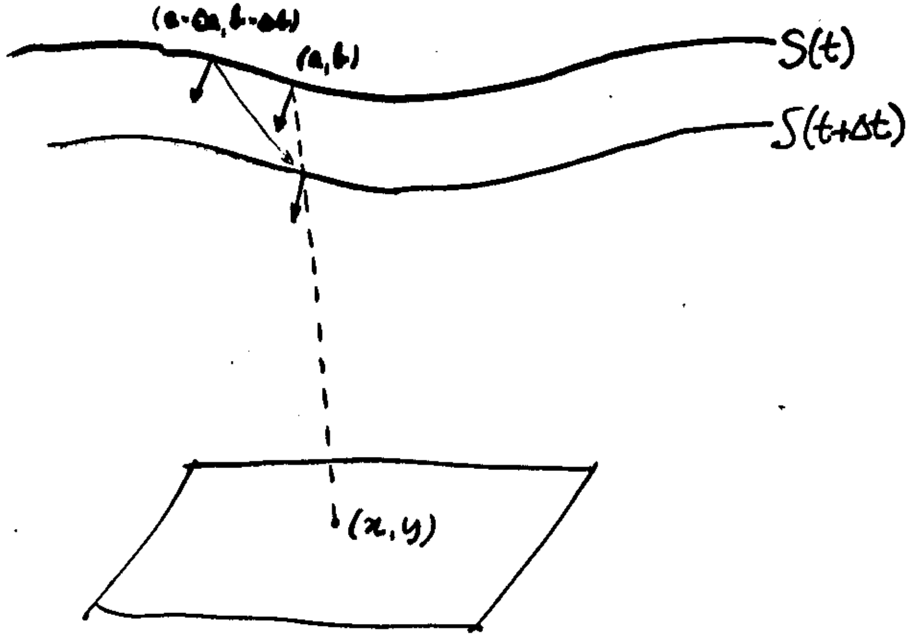
$$\vec{\tilde{v}}_1 = \vec{\tilde{v}} - H^{-1}\left(\left(\text{Jacobian }\vec{\tilde{v}}_0\right)^T \operatorname{grad} E\right)$$

$$\vec{\tilde{v}}_2 = \ldots\ldots$$

Numerical implementation

- Smooth the image brightness over space and time
- Compute derivatives
- Solve $\frac{d}{dt}\vec{\nabla} E = 0$ wherever

  $\left| \det \text{Hess } E \right| > \varepsilon > 0$
- "Regularize" the obtained vector field



SCENE RADIANCE $\Longleftrightarrow$ IMAGE IRRADIANCE

$$L = \rho \frac{\vec{I} \cdot \vec{N}}{I} + \sigma \left(\frac{\vec{R} \cdot \vec{D}}{RD} \right)^{n}$$

Lambertian term specular term

Phong model (1975)

Variables which depend on intrinsic parameters (texture)

Variables which depend on position in space (illumination, point of view, direction of reflection)

Lambertian Translation

$$v_\perp = O_F$$

Lambertian Rotation

$$v_\perp - O_F = \rho \frac{\vec{N} \cdot \vec{I} \times \vec{\omega}}{\| \nabla E \|}$$

Specular Translation

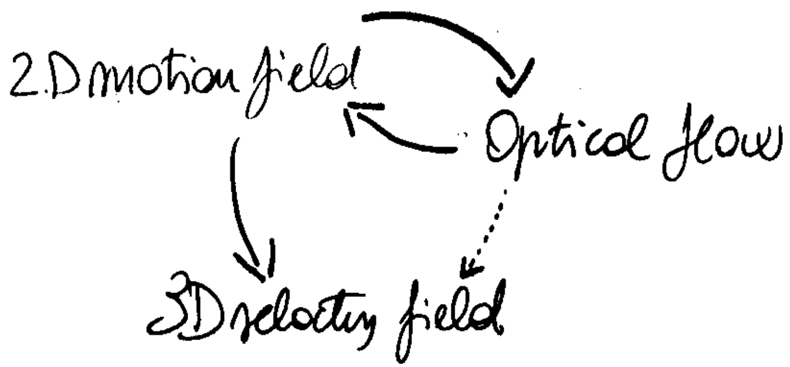
$$v_\perp - O_F = \frac{\sigma n}{D^3} \left(\frac{\vec{D} \cdot \vec{R}}{DR} \right)^{n-1} \frac{(\vec{F} \times \vec{D}) \cdot (\vec{R} \times \vec{D})}{\| \nabla E \|}$$

Specular Rotation

$$v_\perp - O_F = g(\sigma, n, \vec{D}, \vec{R}, \vec{\omega}, \vec{x_0}, \vec{N}, \vec{n}, \nabla E)$$

$$\lim_{\|\vec{\nabla}E\| \to \infty} |v_\perp - o_\perp| = 0$$

It might be that OPTICAL FLOW and MOTION FIELD allow the same qualitative description (Theory of structural stability)



3D motion estimation

- Mathematical properties of the motion field
  - Structural Stability
  - Limit Sets
- Characteristic curve and qualitative description
- 3D motion from singular points: pure translation, pure rotation, and general motion.

The motion field is a planar vector field usually continuous and with continuous first spatial derivatives

It is the "phase portrait" of a dynamical system

What about its limit sets?

What about its singular points?

The 2D motion field is a planar vector field (flow or dynamical system)

$\Downarrow$

1 - Peixoto Theorem
structurally stable dynamical systems are dense (generic, typical, ...). Therefore they do not show chaotic behaviour and have nice properties.

2 - Structurally stable planar vector fields have:
- a finite number of singular points which are elementary (isolated) and non degenerate.
- have a finite number of limit cycles
- do not have saddle-saddle points separatrix.

3 - Bendixon - Poincaré Theorem.

4 - Index of points.

5 - Well established theory of bifurcations.

Structural Stability

a vector field X is structurally stable if all vector fields Y close to X are topologically equivalent to X.

- we need a topology

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

$$\| \ \| = | f_1^2 - f_2^2 |$$

. . . . .

- we need a definition of equivalence

if there is an homeomorphism from trajectories of X to trajectories of Y which preserves singularities.

$$\dot{x} = f(x,y,\lambda)$$
$$\dot{y} = g(x,y,\lambda)$$

$(\tilde{x},\tilde{y})$ is a singular point if

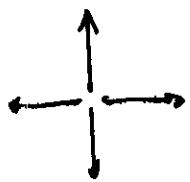
$$f(\tilde{x},\tilde{y},\lambda)=0$$
$$g(\tilde{x},\tilde{y},\lambda)=0$$

$$M = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(\tilde{x},\tilde{y})}$$

λ_1, λ_2 eigenvalues of M

FOCUS

$\lambda_1 = \lambda_2$ real



NODE

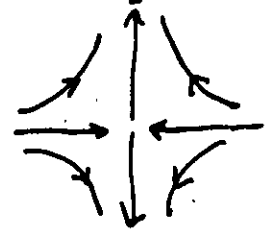
λ_1, λ_2 real

$\lambda_1 \cdot \lambda_2 > 0$

SADDLE POINT

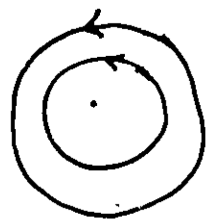
λ_1, λ_2 real

$\lambda_1 \cdot \lambda_2 < 0$



CENTER

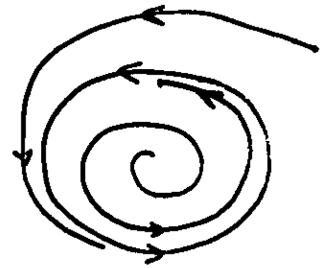
$\lambda_{1,2} = \pm i\alpha$



SPIRAL

$\lambda_{1,2} = \beta \pm i\alpha$

LIMIT CYCLE



General Motion

$$\vec{V} = \vec{\omega} \wedge \vec{R} + \vec{T}$$

at a given instant we can always reduce this motion to an equivalent such that $\vec{\omega} \wedge \vec{T} = 0$ and $\vec{\omega} \cdot \vec{A} = 0$

so that

$$\vec{V} = \vec{\omega} \wedge \vec{X} - \vec{\omega} \wedge \vec{A} + \vec{T}$$

because $\vec{V} = \vec{T}_\omega + \vec{T}_\perp + \cancel{x}\vec{\omega} \wedge \vec{X} - \vec{\omega} \wedge \vec{A}$

and $\vec{T}_\perp$ $\vec{\omega} \wedge \vec{A}$ are both orthogonal $\vec{\omega}$.

the characteristic curve Γ_{gm} is the intersection of

$$X_A^2 - A X_A + X_{\omega A}^2 = 0 \qquad \text{cylinder}$$

$$\omega X_\omega X_{\omega A} + T X_A = 0 \qquad \text{hyperbolic paraboloid}$$

Translation $\quad \vec{V} = \vec{T}$

the locus of points Γ_T such that

$$\vec{e}_z \wedge (\vec{T} \wedge \vec{X}) = 0$$

is a straight line parallel to $\vec{T}$ through the optical focus.

Rotation $\quad \vec{V} = \vec{\omega} \wedge \vec{R} - \vec{\omega} \wedge \vec{A}$

let consider

$$(\vec{\omega} \wedge \vec{X} - \vec{\omega} \wedge \vec{A}) \wedge \vec{X} = 0$$

by projecting along $\vec{\omega}$, $\vec{A}$ and $\vec{\omega} \wedge \vec{A}$

$$X_\omega (X_A - A) = 0$$

$$X_A^2 - A X_A + X_{\omega A}^2 = 0$$

$$X_{\omega A} X_\omega = 0$$

solutions are

$(X_\omega, A, 0)$ axis of rotation

$(0, X_A, X_{\omega A})$ such that $X_A^2 - A X_A + X_{\omega A}^2 = 0$

[illegible] $(0, 0, 0)$

Characteristic Curve

The characteristic curve of a motion Γ is the locus of points in the space $\mathbb{R}^3$ such that their perspective projection is a singular point, that is a point $\vec{x}$ such that

$$\vec{v}(\vec{x}) = 0$$

It is possible to define for each motion a characteristic curve Γ_T, Γ_R e Γ_{gm} independently of the shape of the object.

The singular points for a given motion and a given object are the projections of the intersection of Γ with the visible surface of the moving object.

Properties of singular points

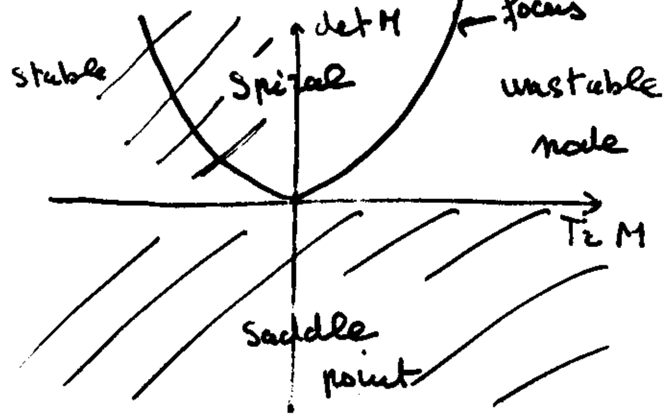
1 - $P(\Gamma_T)$ is a point

2 - $P(\Gamma_R)$ is a pair of two lines

3 - $P(\Gamma_{gm})$ is rather arbitrary

The nature of a singular point is described by the behaviour of the linear approximation of the field around it. The qualitative nature is described by the eigenvalues of the matrix $M_{ij} = \left.\frac{\partial v_i}{\partial x_j}\right|_{\text{singular point}}$

$$\lambda^2 - \mathrm{Tr}\, M \cdot \lambda + \mathrm{Det}\, M = 0$$



we have

$$M_{ij} = \frac{1}{f\vec{\alpha}\cdot\vec{x}}\Big((\vec{\alpha}\cdot\vec{e}_i)(\vec{x}\cdot\vec{x})\vec{e}_3\wedge\vec{\omega}\cdot\vec{e}_j - (\vec{\alpha}\cdot\vec{e}_i)(\vec{x}\cdot\vec{\omega})$$

$$\vec{e}_3\wedge\vec{x}\cdot\vec{e}_j - (\vec{\alpha}\cdot\vec{x})(\vec{x}\cdot\vec{e}_i)\vec{e}_3\wedge\vec{\omega}\cdot\vec{e}_j +$$

$$+ (\vec{\alpha}\cdot\vec{x})(\vec{x}\cdot\vec{\omega})\vec{e}_3\wedge\vec{e}_i\cdot\vec{e}_j\Big) - \frac{V_3}{X_3}\delta_{ij}$$

and also

$$\mathrm{Tr}\, M = -2\frac{V_3}{X_3} + \frac{\vec{\alpha}\wedge\vec{x}\cdot\vec{\omega}}{\vec{\alpha}\cdot\vec{x}}$$

$$\mathrm{Det}\, M = \frac{V_3^2}{X_3^2} - \frac{V_3}{X_3}\frac{\vec{\alpha}\wedge\vec{x}\cdot\vec{\omega}}{\vec{\alpha}\cdot\vec{x}} + \frac{(\vec{\alpha}\cdot\vec{\omega})(\vec{x}\cdot\vec{\omega})}{\vec{\alpha}\cdot\vec{x}}$$

where $\vec{\alpha}$ is the unit normal vector to the tangent plane to the moving surface at the point $\vec{X}$ which projects onto the image plane in $\vec{x}$.

$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ are the three unit vectors parallel to X, Y Z axis

Recovering Translation

1 - The point $P(\Gamma_T) = f\left(\frac{T_1}{T_3}, \frac{T_2}{T_3}\right)$

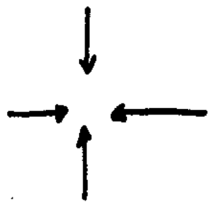
2 - The eigenvalues of $P(\Gamma_T)$ are equal and real

$$\lambda = -\frac{T_3}{X_3}$$

$\frac{1}{\lambda}$ time to crash

Computation of time to crush

for pure translation the singular point (the focus of expansion) is a pure node



locally the 2-D vector field is

$$v_x = \lambda x + \dots$$
$$v_y = \lambda y + \dots$$

the analysis of singularities of the 2-D Motion field says

time to crush $\tau = \frac{1}{\lambda}$ $\qquad \lambda = \frac{v_x}{x} = \frac{v_y}{y}$

accuracy of more than 90%

Recovery of rotation

A - immobile point $\vec{V} = 0$

does not move on the image plane

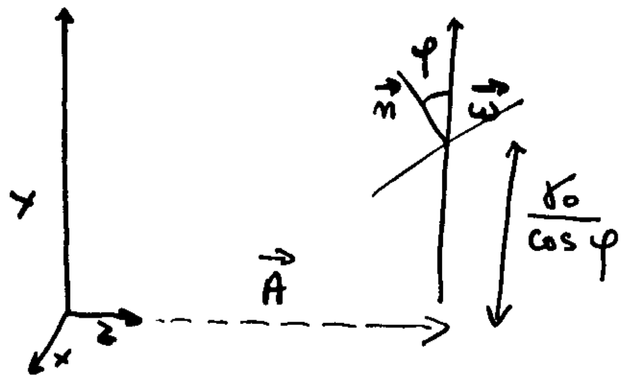
$$Tr\, M = \frac{\vec{\alpha} \wedge \vec{x} \cdot \vec{\omega}}{\vec{\alpha} \cdot \vec{x}}$$

$$Det\, M = \frac{(\vec{\alpha} \cdot \vec{\omega})(\vec{x} \cdot \vec{\omega})}{\vec{\alpha} \cdot \vec{\omega}}$$

we obtain in the TrM, $Det\, M$ plane

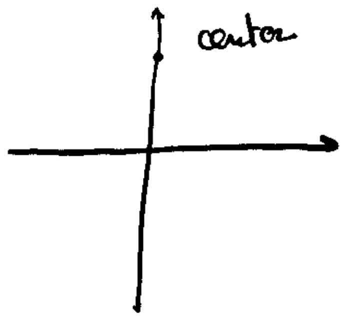
$$\left(\det M - \frac{g^2\omega^2}{g^2-1}\right)^2 + \frac{g^2\omega^2}{g^2-1}\, tr^2 M = \frac{g^2\omega^4}{(g^2-1)^2}$$

where $g = \frac{\gamma_0}{A \sin\varphi}$

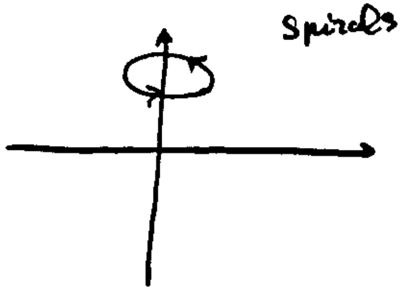


φ is the angle between the normal to the surface and the rotation axis.

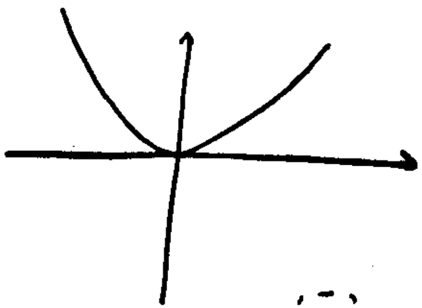
$g = \infty$
$\varphi = 0$



$g > 1$

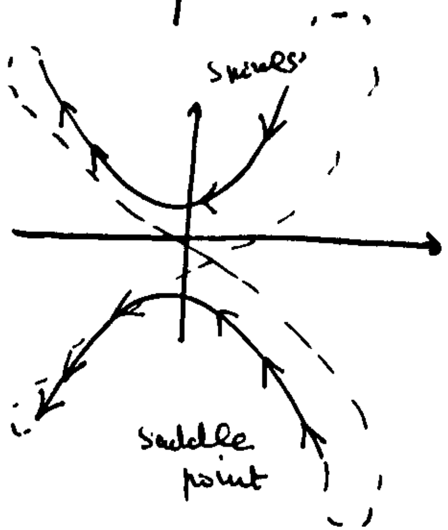


$g = 1$



bifurcation point

$g < 1$



Computation of angular velocity

for pure rotation when the axis of rotation is perpendicular to the surface, the immobile point is a pure center

locally the 2-D vector field is

$$v_x = ax + by + \dots$$
$$v_y = cx - ay + \dots$$

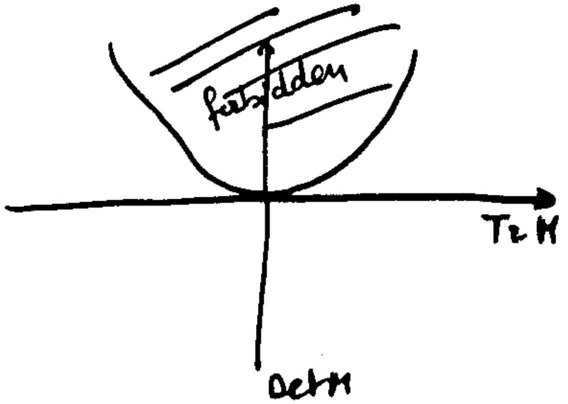
The analysis of singularities of the 2-D motion field says

$$\omega = \det \begin{vmatrix} a & b \\ c & -a \end{vmatrix}$$

accuracy $\sim 90\,\%$

B- Singular points $\vec{V} = \vec{\omega} \wedge \vec{X} - \vec{\omega} \wedge \vec{A}$

$$\mathrm{Det}\, M = \frac{\mathrm{Tr}^2 M}{4} - \frac{1}{4} \left(\frac{\vec{\alpha} \wedge \vec{\omega} \cdot \vec{x}}{\vec{\alpha} \cdot \vec{V}} \right)$$



- cannot be spirals

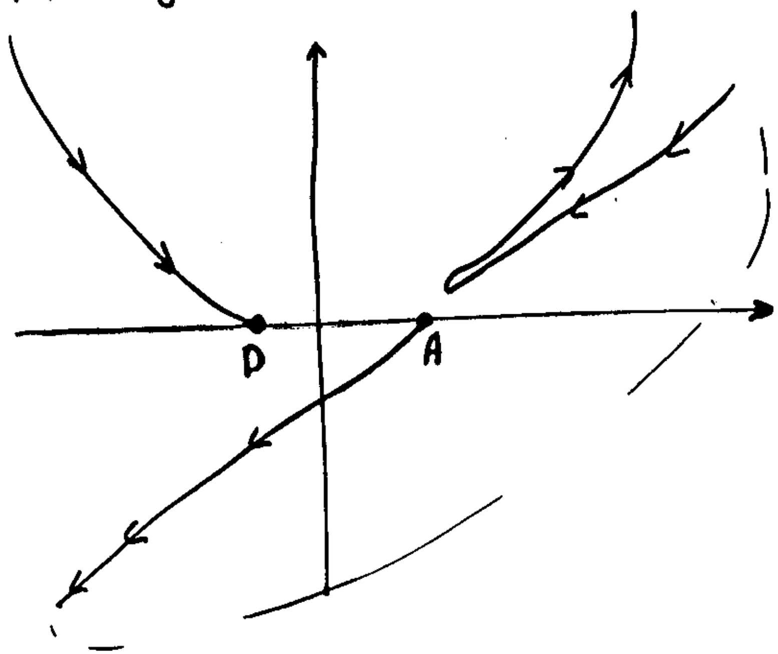
- They move always on a straight line in the image plane.

When the object is a plane we have

$$\mathrm{Tr}\, M = \frac{\omega}{2} \frac{\cos \omega t \mp 3\sqrt{1 - g^2(t)}}{g - \sin \omega t}$$

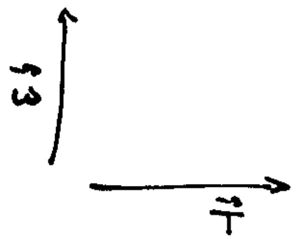
$$\mathrm{Det}\, M = \frac{\omega^2}{2} \frac{1 - g^2(t) \mp \cos \omega t \sqrt{1 - g^2(t)}}{(g - \sin \omega t)^2}$$

$g(t) = 2g - \sin \omega t$ $\qquad \pm$ for the 2 s.p.



General Motion

A – $\vec{\omega} \cdot \vec{T} = 0$ orthogonal rotation



we can rewrite as

$$\vec{V} = \vec{\omega} \wedge \vec{X} - \vec{\omega} \wedge \vec{A}'$$

where $\vec{A}'$ depends on time

$$\vec{A}' = \vec{A}_0 + \vec{T} t$$

$$|\vec{A}'| = |\vec{A}_0| \cdot \sqrt{1 + \frac{2 \vec{A}_0 \cdot \vec{T}}{A_0^2} t + \vec{T}^2 t^2}$$

formally similar to rotation but with variable $\vec{A}$.

The "immobile" point moves on the image plane along a straight line.

B – axial rotation

$$\vec{T} \parallel \vec{\omega}$$

$$\vec{V} = \vec{\omega} \wedge \vec{X} - \vec{\omega} \wedge \vec{A} + \vec{T}$$

at every time

the characteristic curve T_{gm} does not move with time.

we have

$$\det M = \frac{1}{4} T z^2 M - \left(\frac{\vec{\alpha} \wedge \vec{V} \cdot \vec{\omega}}{\vec{\alpha} \cdot \vec{V}} \right) + \frac{(\vec{\alpha} \cdot \vec{\omega})(\vec{X} \cdot \vec{\omega})}{\vec{\alpha} \cdot \vec{X}}$$

when $T \to \infty$ $\qquad \downarrow$ $\qquad 0$

and we have only spirals.