INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

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H4.SMR/303 - 28

WORKSHOP
GLOBAL GEOPHYSICAL INFORMATICS WITH APPLICATIONS TO RESEARCH IN EARTHQUAKE PREDICTIONS AND REDUCTION OF SEISMIC RISK

(15 November - 16 December 1988)

DYNAMICAL SYSTEMS
THE STRUCTURE OF CHAOS

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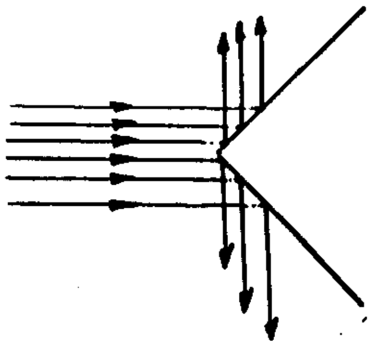
Los Angeles
U.S.A.

Seismic Dynamical Systems are Nonlinear

- Instability
  ex: Buckling
- Sensitivity to initial conditions



In our case, non-linear effects such as bifurcations, etc. can be induced by

- Time delays
- Non-linearity

separately!

Example of bifurcation due to time delay

$$\dot{x} = -\alpha x$$

solution: $x = Ae^{-\alpha t}$

Consider

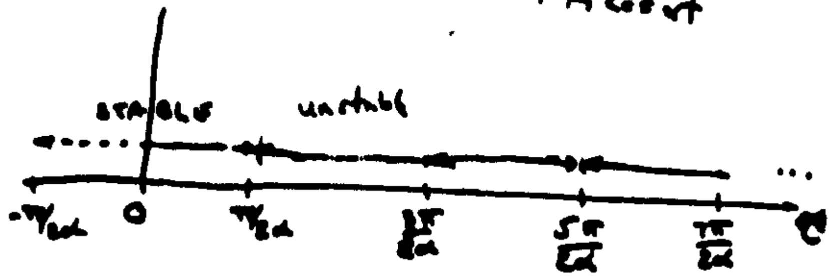
$$\dot{x}(t) = -\alpha x(t-\tau)$$

i.e. $\dot{x}(t+\tau) = -\alpha x(t)$

if $\tau = \pi/2\alpha$

solution: $x = +A \sin \alpha t$

$= +A \cos \alpha t$



$e^{-\gamma t} \sin \gamma t$, $\gamma > 0$

$e^{\gamma t} \sin \gamma t$, $\gamma > 0$

V-7

Giornale dell' Istituto Italiano degli Attuari
7 (1936) 74-80

SULLA TEORIA DI VOLTERRA DELLA LOTTA PER L'ESISTENZA

A. KOLMOGOROFF.

SUNTO. — L'A. studia le equazioni differenziali che si riferiscono alla lotta per l'esistenza, analoghe a quelle già considerate dal Volterra, facendo delle ipotesi di carattere puramente qualitativo sulla forma delle equazioni stesse.

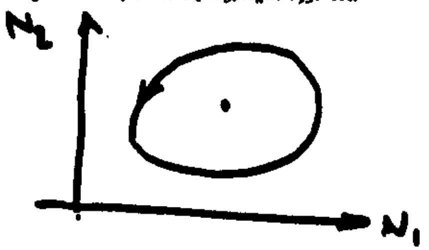
1. La questione delle azioni reciproche di una specie mangiante e di una specie mangiata si riduce nelle ricerche di Vito Volterra [1] alla considerazione delle equazioni differenziali

$$\frac{dN_1}{dt} = (\varepsilon_1 - \gamma_1 N_2) N_1 ,$$

$$[1_0] \qquad \frac{dN_2}{dt} = (-\varepsilon_2 + \gamma_2 N_1) N_2 ,$$

dove N_1 e N_2 sono delle quantità di individui rispettivamente della specie mangiata e della specie mangiante che dipendono dal tempo t, ε_1 e $-\varepsilon_2$ i loro coefficienti di accrescimento e γ_1, γ_2 delle costanti. È naturale che le espressioni analitiche scelte dal Volterra per i secondi membri delle equazioni $[1_0]$ possono essere considerate soltanto come prima approssimazione dello stato reale delle cose. Diversi autori hanno proposto altre relazioni per esprimere la dipendenza delle derivate $\frac{dN_1}{dt}$ e $\frac{dN_2}{dt}$ dalle quantità N_1 ed N_2. Rinunciando a queste ipotesi speciali, la cui scelta è del tutto arbitraria, scriviamo le equazioni delle azioni reciproche sotto la forma seguente

[1] Cfr. per es. V. VOLTERRA, *Ricerche matematiche sulle associazioni biologiche*, « Giornale dell'Istituto Italiano degli Attuari », Anno II, n. 3, luglio 1931-IX.



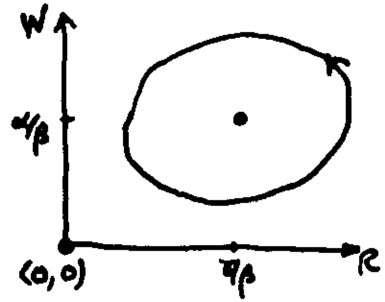
THE FABLE OF THE WOLVES AND THE RABBITS

$$\dot{W} = -\gamma W + \beta R W$$

$$\dot{R} = \alpha R - \beta R W$$

(Rabbits eat grass
Wolves eat rabbits)

FIXED POINT: $\dot{W} = 0 \rightarrow R = \gamma/\beta$ $\quad \dot{R} = 0 \rightarrow W = \alpha/\beta$ $\quad$ OR $W = 0,\ R = 0$



POINCARE'S THEOREM
(2-D)

A FURTHER EXAMPLE OF TIME DELAY EFFECTS:

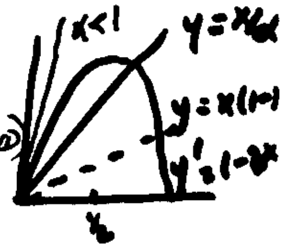
Logistic Equation

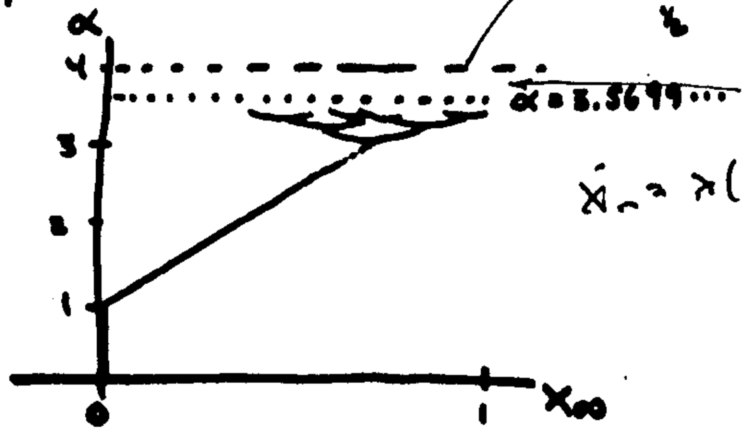
(THE FABLE OF THE FISH POND)

$0 < X_1 < 1$

$0 < \alpha < 4$

$$X_{n+1} = \alpha X_n (1 - X_n)$$

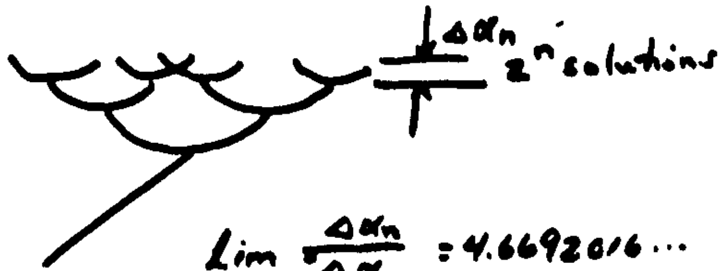




$\dot{x}_n = \lambda(1-x)$

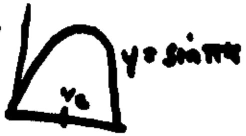
M.J. Feigenbaum
Phys. Letts 1975

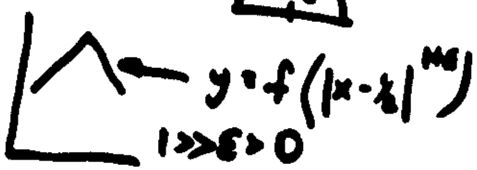
predictability?



$$\lim_{n\to\infty} \frac{\Delta\alpha_n}{\Delta\alpha_{n+1}} = 4.6692016\ldots$$

$$X_{n+1} = \lambda \sin \pi X_n$$

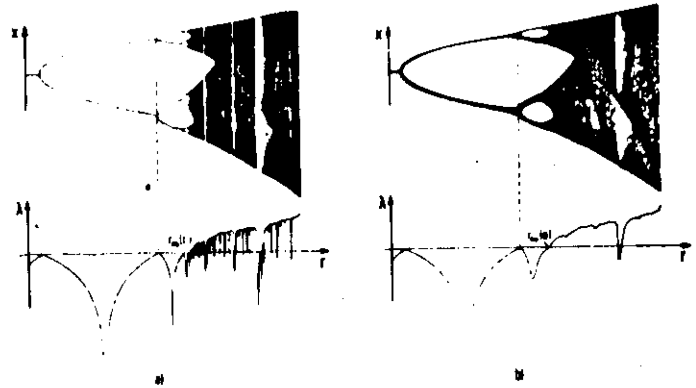




Here, ζ_n are Gaussian-distributed variables with averages

$$\langle \zeta_n \zeta_{n'} \rangle = \sigma^2 \delta_{n,n'} \tag{3.69}$$

(similarly their Fourier components ζ_k are Gaussian-distributed), and σ measures the intensity of the white noise. We recall that the new Fourier components $|a_k^{n+1}|$ of a 2^{n+1}-cycle are a factor of μ^{-1} smaller than the old components $|a_k^n|$. This means that any finite external noise eventually suppresses all subharmonics above a certain n, as shown in Fig. 36c.



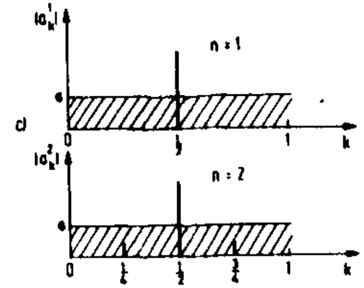


Fig. 36: a) Iterates of the logistic map and its Liapunov exponent λ, compared to b) the corresponding quantities in the presence of external noise with amplitude $\sigma = 10^{-3}$ (after Crutchfield, Farmer and Huberman, 1982). Although the noise washes out the fine structure in the iterates and in λ, there is still a sharp transition to chaos which is indicated by the change of sign of λ in b). c) Suppression of subharmonics in the presence of white noise σ. Note that the subharmonic amplitudes are reduced by a factor μ^{-1} for $n \to n+1$ (schematically).

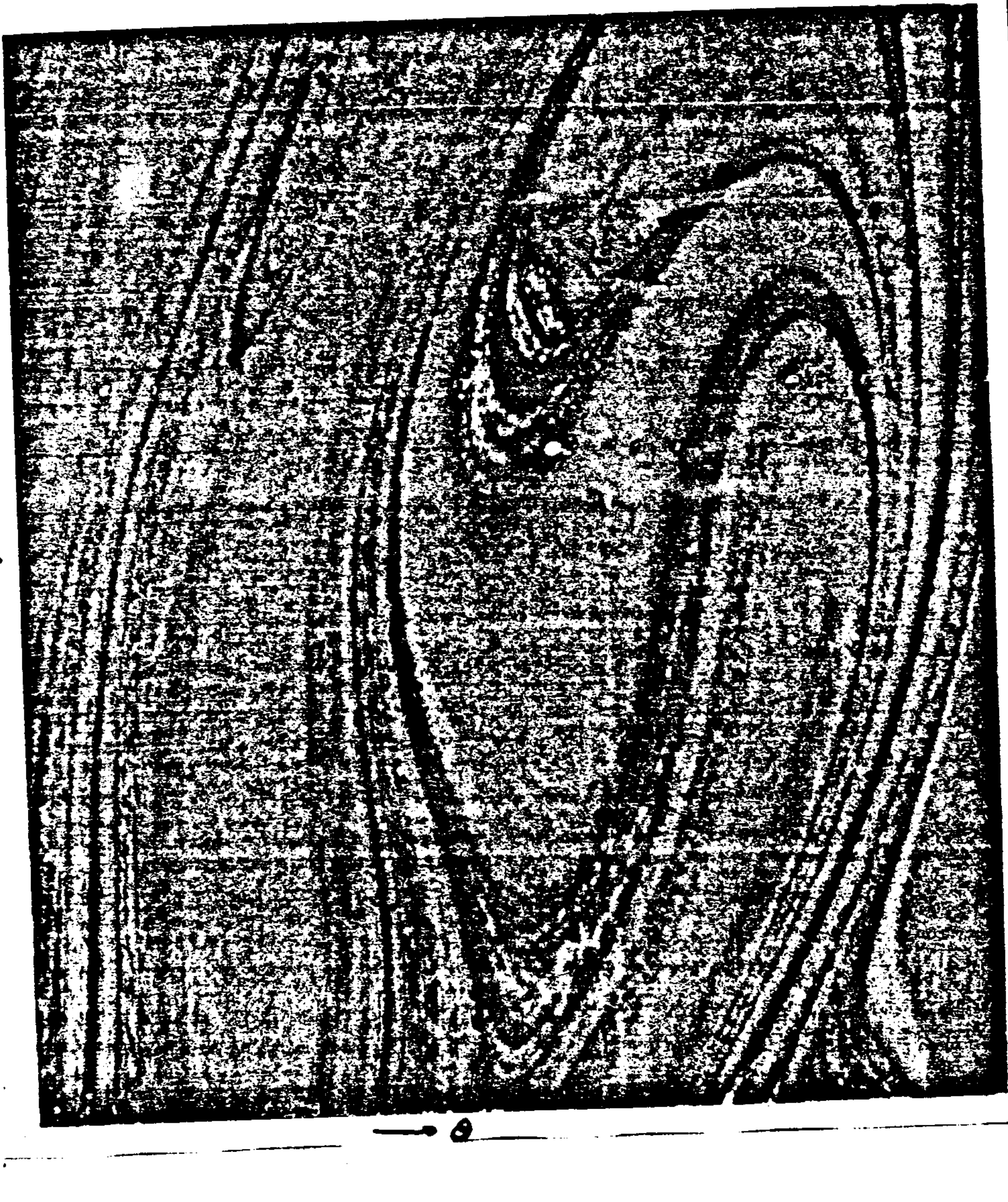
REF.
E. OTT, SCIENCE, 238, 1987 585

Fixed Pts:

$\theta = -.477$ | $\theta = -.471$
$\dot{\theta} = -.609$ | $\dot{\theta} = .287$

$$\ddot{\theta} + \nu\dot{\theta} + \omega_0^2 \sin\theta = f \cos \omega_0 t$$

$\nu = 0.2$
$\omega_0 = 1.0$
$f = 2.0$



the "continuous" part of the Fourier power spectrum increases by orders ude, especially near zero-frequency, i.e. for very long periods. The same experimental Turbulence spectra. Hence, we speak of the '*Onset of* t'. Yet, sharp peaks at the basic frequency remain in Fig. 21H. When the velops more than just one circular band, at other parameter values, those disappear as well, cf. Fig. 24. These pictures of the Rössler attractor ined by the Santa-Cruz (/Eugene, OR) group on an *analogue*-computer [232, facilitates parameter searches for interesting attractors. The is not high enough therefore to resolve the behavior inside the strange the Hénon attractor (3.8) this will be done, with the aid of a digital [260], in the next section.

nservative system we have many Feigenbaums, of different basic periods m_0, ously present at a given value of μ. There are many interweaving branches ent trees coming very close together in infinite hierarchies, cf. sections For the attracting (repelling) branches of these trees in a dissipative th infinite hierarchies of different trees would be difficult to construct ose enough to an attractor, *all* orbits move towards it forever. In a ve system we find Feigenbaums of different basic periods m_0 (in $m_0 2^n$) fferent μ-regions, one after the other [215,240,235:234], sometimes with tween them. This makes it even more difficult to predict where the "last" [240,215] than in a conservative system, cf. section 2.6, and where le chaos might set in.

n sequences have been established *rigorously* for mappings of the interval to itself which are of the form

$$y_{t+1} = f(|y_t|^{1+\varepsilon}, \mu), \qquad 0 < \varepsilon \ll 1, \qquad (3.33)$$

y_{t+1} is smooth, except at f's maximum at $y = 0$ [237], under some nical conditions on f [237]. These results involve perturbation expansions are restricted to small ε values, whereas the interesting case is $\varepsilon = 1$ (3.9). lytical results have been extrapolated and extended to N-dimensional [227].

doubling bifurcations are observed in real turbulence experiments as we in section 3.4. Below we discuss chaotic, non-periodic, attractors.

ge Attractors

descriptive, but less attractive, name for these objects might be ic Attractors', since the *motion along the attractor should be 'ergodic'*, er 2, and '*mixing*' [250-256,272,270,92,90-98], i.e. t-dependent correlation should vanish as $t \to \infty$. The latter precludes periodic attractors. A non-attractor has infinitely many intersection points with a transverse icular") surface. Yet, there cannot be any continuous 'curve' in this of section' with the attractor passing through *every* point of an 'interval' urve; otherwise it would not be an attractor *along* that 'interval' [270]. bility remaining is for the Strange Attractor to pass through a 'Cantor nfinite number of points which are not dense on any 'interval' [168]. of such attractors have been constructed [250-272,169]. None of the above s of a Strange Attractor are easy to test, for a given system of equations rent opinions can be heard on the strangeness of particular attractors [270]. It is apparent however that there is a bewildering variety of types of s, in between Strange Attractors and simple Limit Cycles, cf. [250-270]. you that there also is a great variety of conservative systems - in between and integrable systems - as we saw in chapter 2. Some attractors [250-260] ger than others. An attractor is often called 'Strange' already if it onsist of a single *periodic* orbit [272]. In that case no chaotic behavior ed along the attractor and 'Aperiodic Attractor' might be a more te name [236,250,257,259,262].

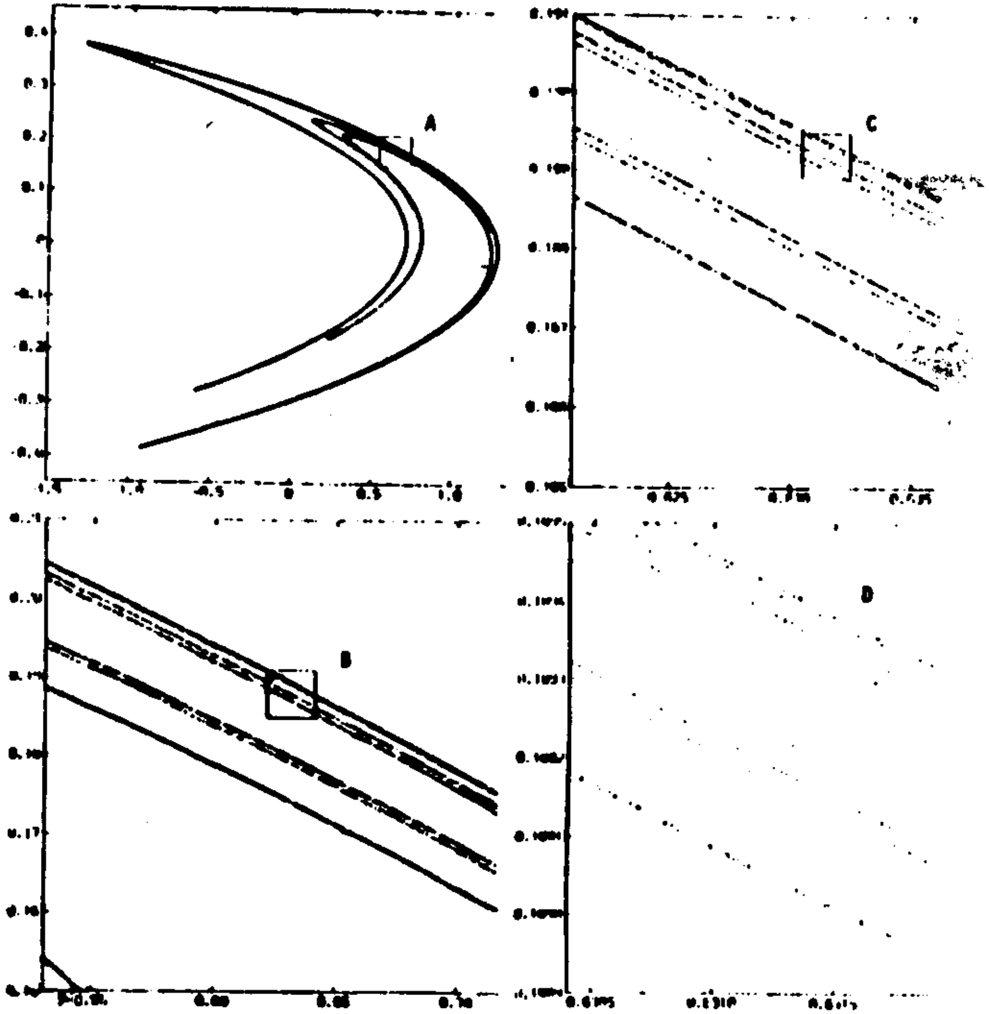


Figure 22

The Hénon-Attractor, at a = 1.4 and b = 0.3 in (3.8). Horizontally plotted is x_t, vertically: bx_{t-1} (taken from [260]). The complete attractor (A) seems simple, yet a 15 fold magnification of the little "box", in A, shows more "curves" (B). A further magnification (x10), of the small box in B, shows several more (C) "curves". A final magnification (x10), of the box in C, again shows new ones (D). This suggests a 'Cantor Set' cross-section. Along the curves points are repelled. Points are "transversally" attracted to the curves. Note the conservative curves in Figs. 6b,c. Text: see next page.

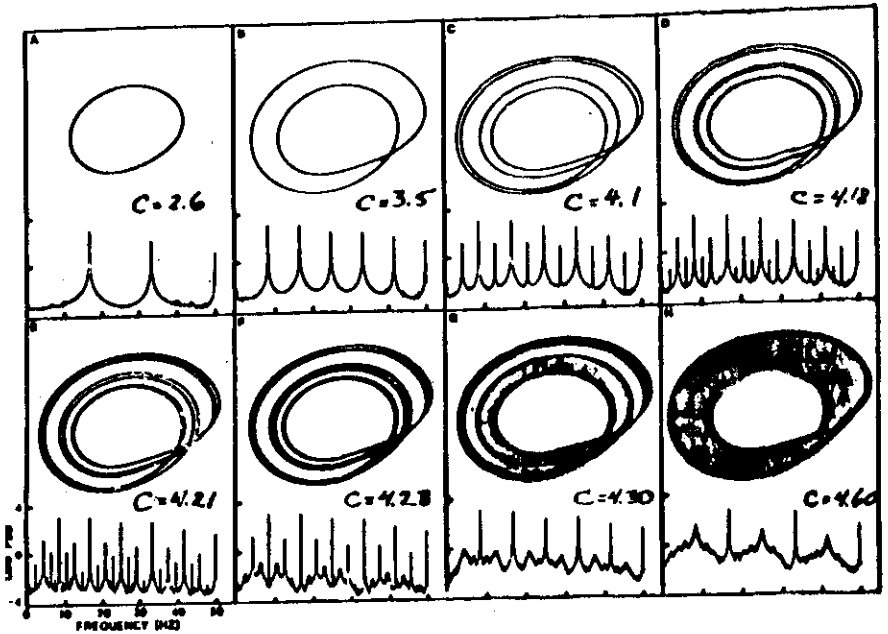
Question for example Hénon's mapping

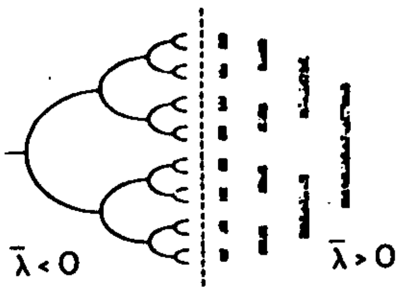
$$x_{t+1} = bx_{t-1} + 1 - ax_t^2, \qquad \text{cf. (3.8)},$$

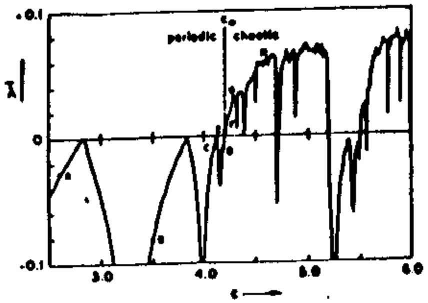
Feigenbaum (periodic) attractor

$$\dot{x} = -(y+z)$$
$$\dot{y} = x + 0.2y$$
$$\dot{z} = 0.2 + xz - Cz$$

Rössler (1976)



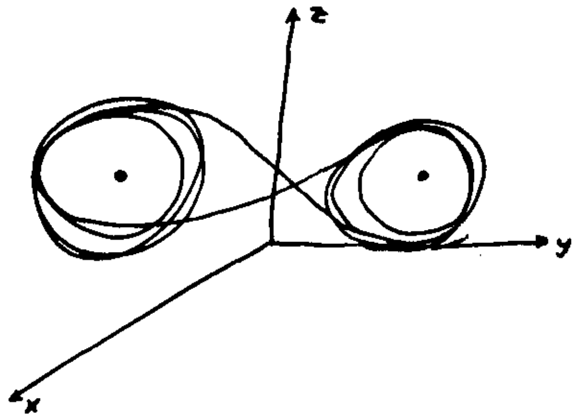




Crutchfield et al. (1980)

Lorenz (Benard convection)
J. Atmos. Sci. 20, 1963, 130

$$\dot{x} = a(y-x)$$
$$\dot{y} = bx - y - xz$$
$$\dot{z} = -cz + xy$$



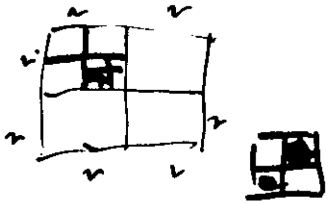
$a = 10$, $c = 8/3$

$b = 166.$	periodic
$b = 166.1$	intermittent periodicity interrupted by chaotic bursts.
$b = 166.2$	chaotic

Can we predict the future for a strange attractor?
Lyapunov exponents.

Renormalization

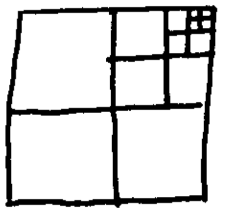
Crack fusion



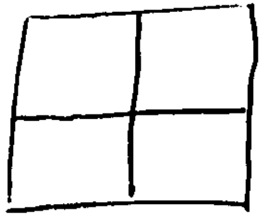
We divide the scale of crack sizes up by lengths scaled as powers of two.

let X_n be the number of cracks of size $a_0 2^n$
$n = 0, 1, 2, \dots$

(for example if $a_0 = 1$, we have 1cm
2cm
4cm
8cm
etc. cracks)



consider successive quarterings of squares.
In the n^{th} order (with 4 $(n-1)$ boxes) in 2-D



Rule of fusion: two n cracks fuse to form an $(n+1)$ crack
the fusion of an n with an m crack forms an n crack
$n > m$.

an (n) box is fractured if
4 $(n-1)$ boxes are fractured
or 3 $(n-1)$ " " "
or 2 $(n-1)$ " " " half of the cases

Healing is important?

$$\dot{X}_n = r f(X_{n-1}) - X_n$$

chosen $f(X_{n-1})$ to have a critical point
$f(x) = x$

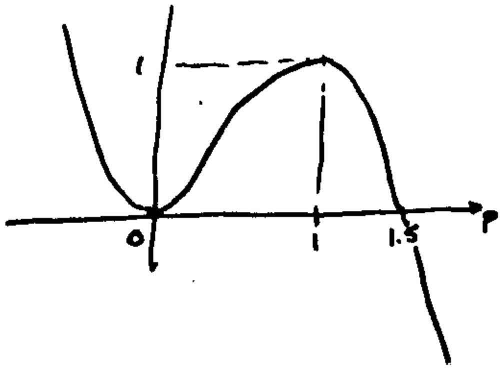
if p_{n-1} is prob. that an $n-1$ box be fractured
then

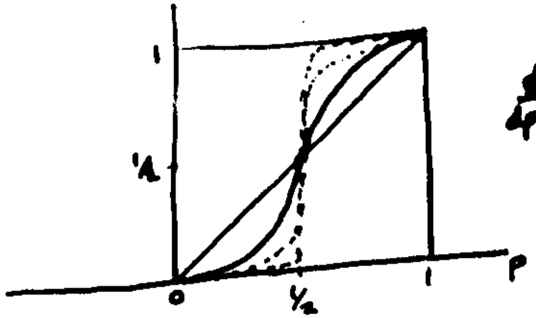
$$p_n = 3p_{n-1}^2 - 2p_{n-1}^3$$

Now give the smallest size objects an initial chance!

$$f(x) = 3x^2 - 2x^3$$

$P_{n+1} = P_n^2(3-2P_n)$





$\frac{d^2}{dp^2}(3p^2-2p^3)=0$ at $p=\frac{1}{2}$

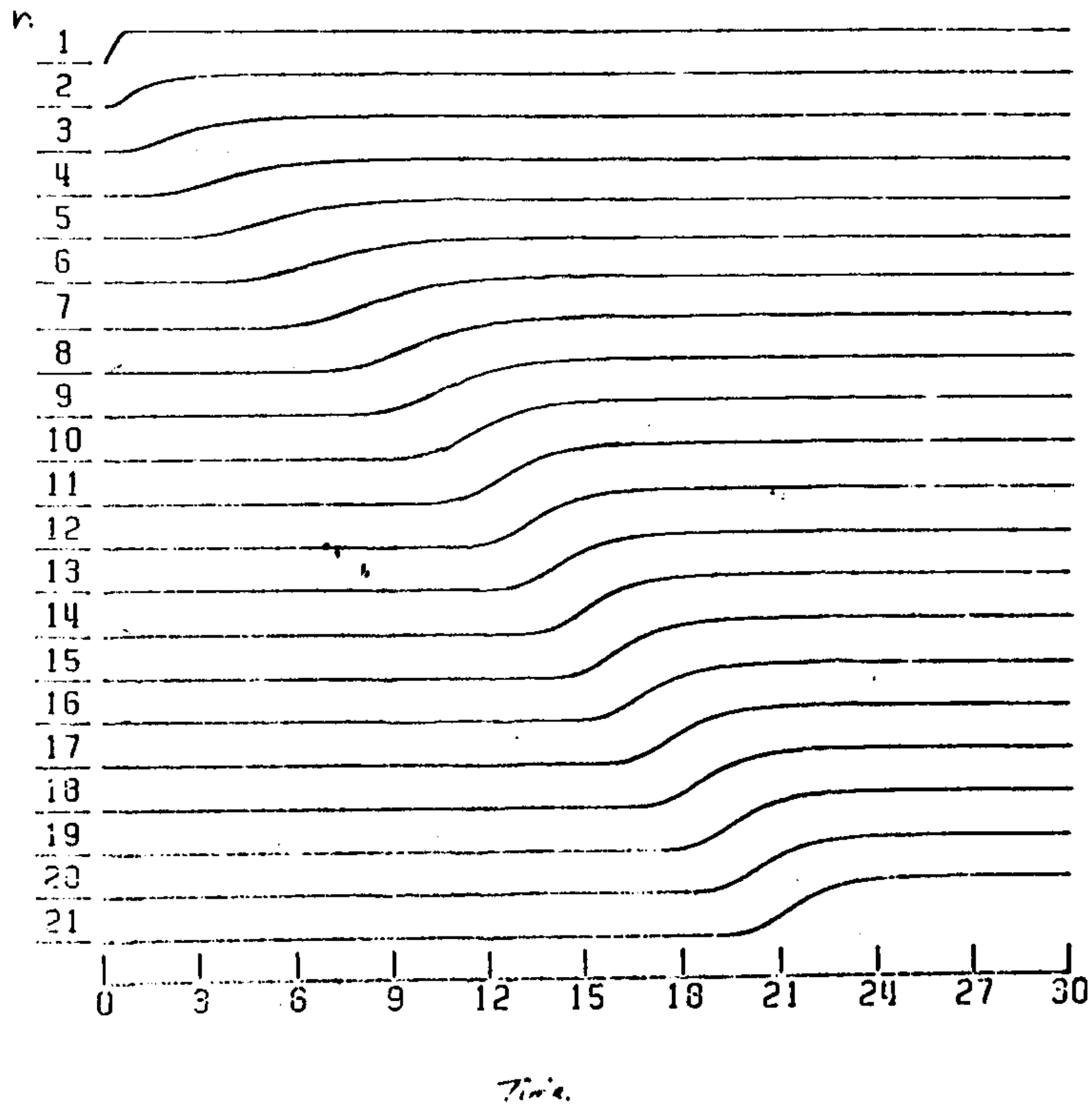
if $f(p)=3p^2-2p^3$

$f(\frac{1}{2})=\frac{1}{2}$

If $P_{n+1} = 3P_n^2 - 2P_n^3$

then $\lim_{n\to\infty} P_n$ is $=0$ $\quad 0<p_1<\frac{1}{2}$

$=1$ $\quad \frac{1}{2}<p_1<1$



Consider

$\dot{X}_n = f(X_{n-1}) - X_n$ ← healing

$f(x) = 3x^2 - 2x^3$ (crack fusion)

Let $f(x) = H(x - \tfrac{1}{2})$ i.e. $f(x) = 0$ $0 \le x < \tfrac{1}{2}$
$= 1$ $1 \ge x > \tfrac{1}{2}$

with initial conditions

$f(X_n) = 0$ at $t = 0$, $n \neq 0$

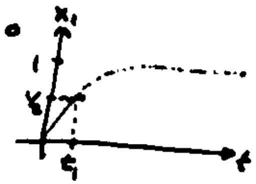
$f(X_0) = H(t)$

$\dot{X}_1 = H(t) - X_1$ $X_1 = 0$ $t < 0$

$\dot{X}_1 = 1 - X_1$ $t > 0$

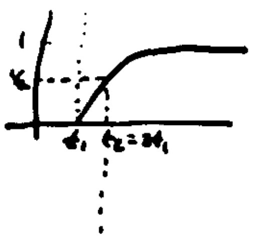
$\Downarrow$

$X_1 = 1 - e^{-t}$

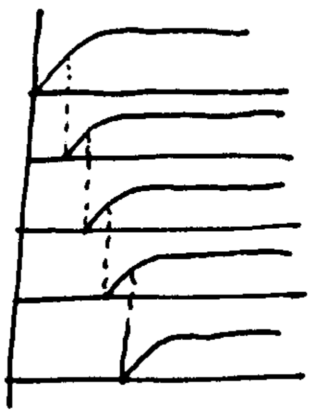


$\dot{X}_2 = -X_2 \Rightarrow X_2 = 0$ $t < t_1$

$\dot{X}_2 = 1 - X_2 \Rightarrow X_2 = 1 - e^{-(t - t_1)}$ $t > t_1$



etc.



Length:
Let crack sizes be 2^n $-\infty < n \le N$

Number:
Let population in category n be x_n

Fusion:
If 2 cracks of size n fuse, they form a crack of size $n+1$

If 2 cracks of dissimilar sizes $n > m$ fuse, they form a crack of size n.

Aftershocks:
Large shocks have a 'fallout' of small cracks

Creep:
The time at which a large crack is formed is later than the onset of the process of fusion

Goals: Instabilities

Is it possible to produce clustering, especially of large earthquakes ~~[illegible]~~ the steady input of shocks originating in external (i.e. plate tectonics) causes?

Clustering to be repetitive (more or less)

For two crack size categories let L & B be the number of cracks at time t. Then we write

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② ③

$$\frac{dB}{dt} = \gamma \tilde{L}^2 - \alpha B$$

① ④ ⑤ ⑤ ②

$$\frac{dL}{dt} = \mu(L,B,t) + \eta \tilde{L}^2 - \nu \tilde{L}^2 L - \kappa L B - \gamma L^2$$

$\mu = 5(1-3\tilde{L}^2)$

—— source terms, positive feedback

—— negative feedback

1. Plate Tectonics
2. Fusion of two L-cracks
3. Healing (Dietrich, 1978) $\hat{\mu} - \hat{\mu}_0 \sim \log t$
4. Aftershocks
5. Fusion of an L & a B crack

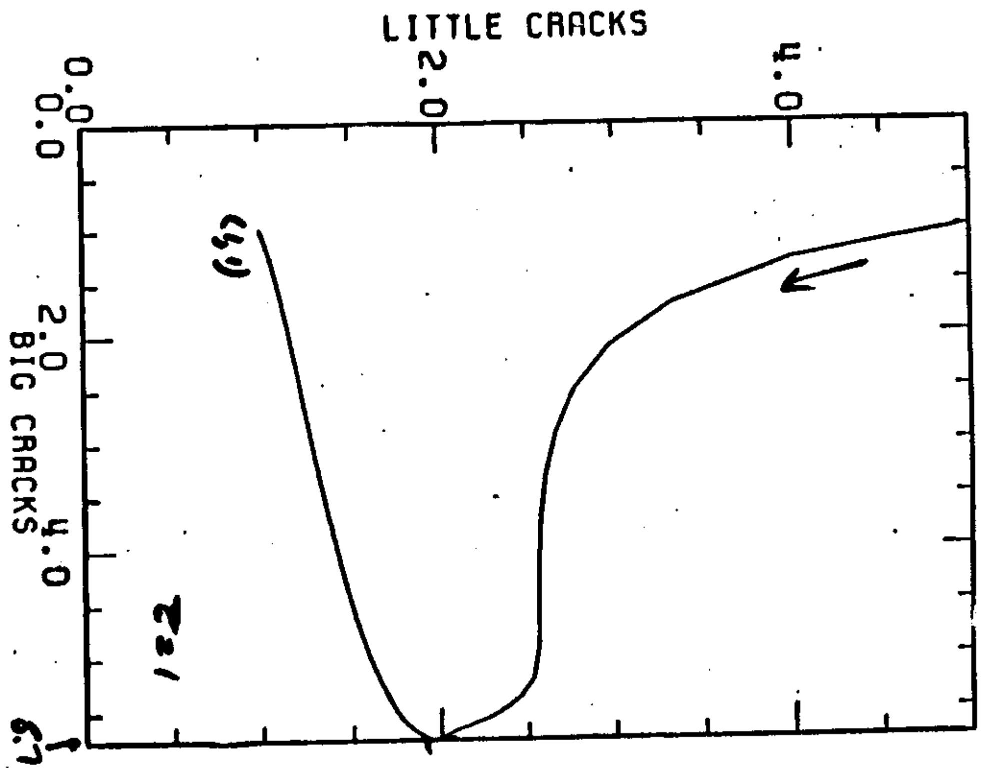
$$\tilde{L}(t) = L\left(t - \tau e^{-\varepsilon\sqrt{L(t')}}\right) \approx L\left(t - \tau e^{-\varepsilon\sqrt{L(t)}}\right)$$

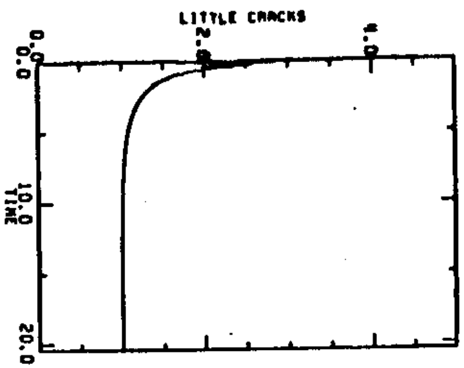
$$t - \tau < t' < t$$

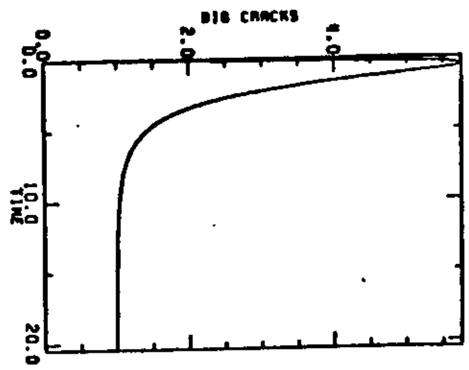
Numerical examples
($\mu=1$), $\eta=3$, $\nu=1$, $\kappa=2$, $\gamma=1$, $\alpha=1$, $S=6$
(equilibrium pt. = $L=1$, $B=1$)

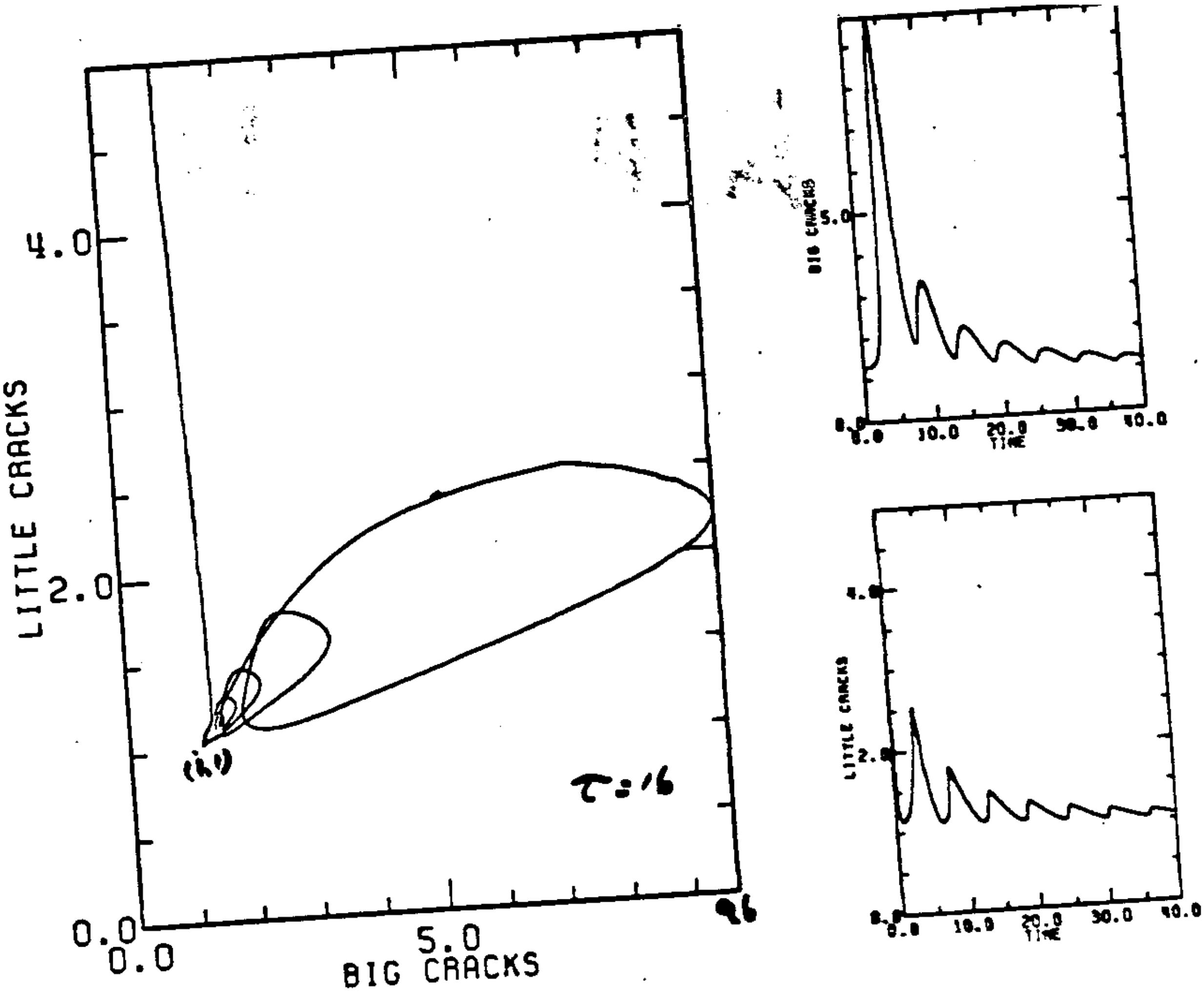
$\varepsilon=1$

Initialize: $L = 1 + 4e^{\alpha t}$, $B=1$ for $t<0$








LITTLE CRACKS
BIG CRACKS
0.0
2.0
4.0
5.0
(1,1)
τ=16
TIME
10.0
20.0
30.0
40.0

5b

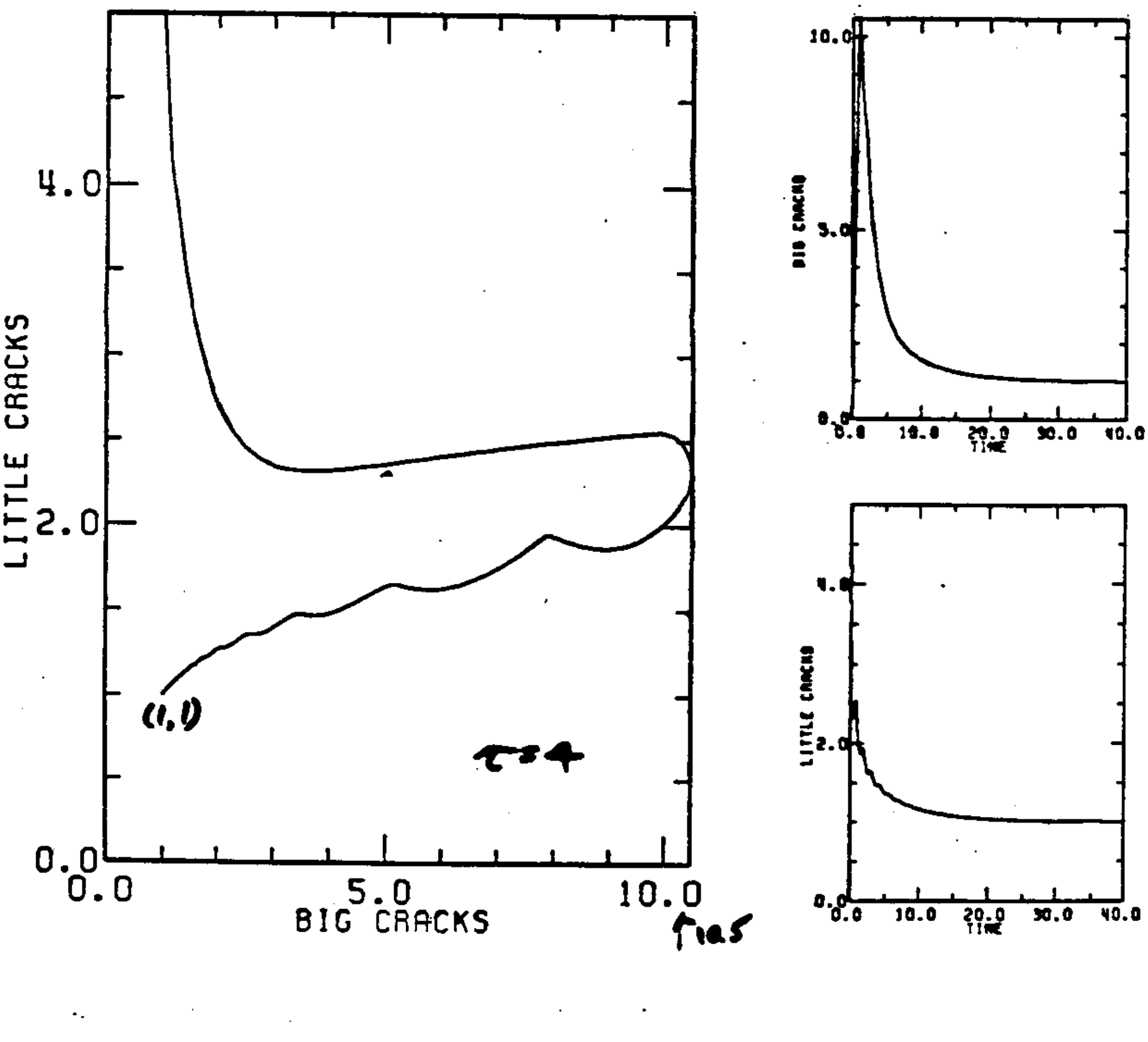

LITTLE CRACKS
BIG CRACKS
0.0
2.0
4.0
5.0
10.0
(1,1)
τ=4
TIME
20.0
30.0
40.0

Fig 5

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$$\dot{B} = \gamma \tilde{L}^2 - \alpha B$$

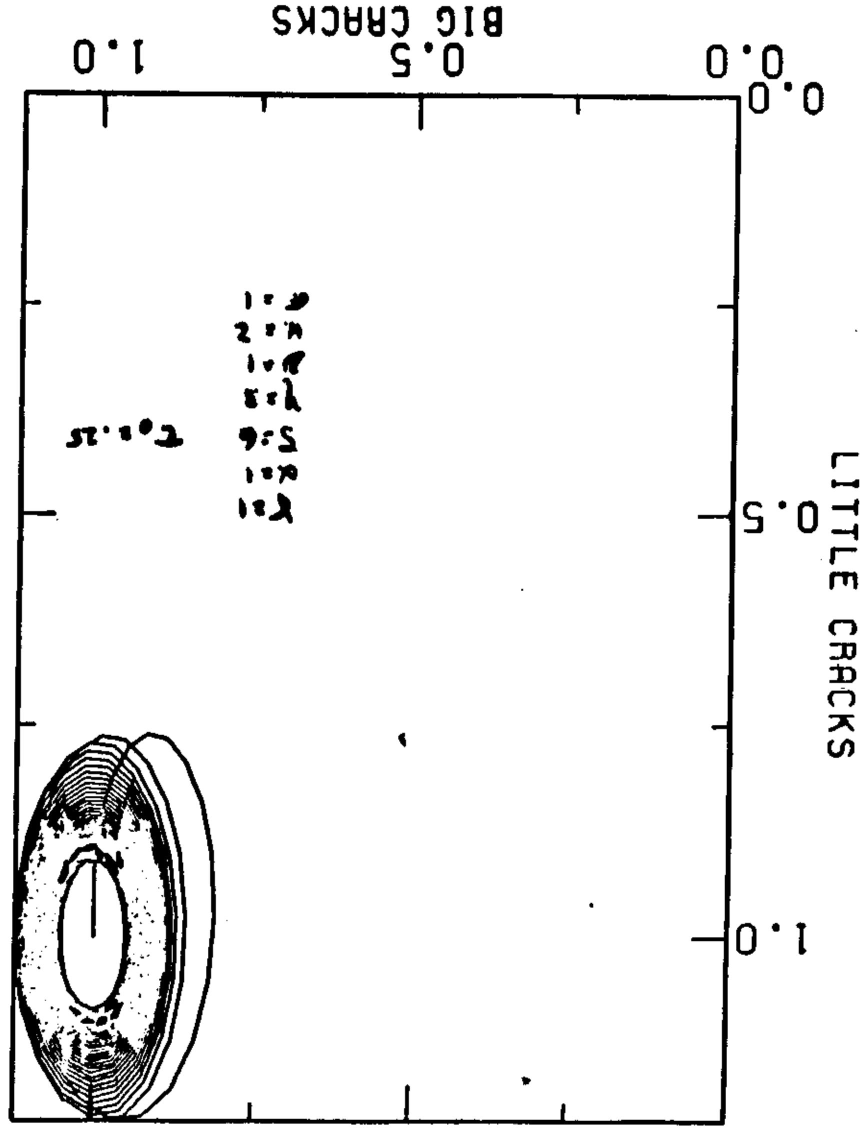
$$\dot{L} = \mu(t) + \eta \tilde{L}^2 - \nu \tilde{L}^2 L - \kappa L B - \gamma L^2$$

$$\dot{\mu} = \zeta(1 - \xi \tilde{L}^2)$$

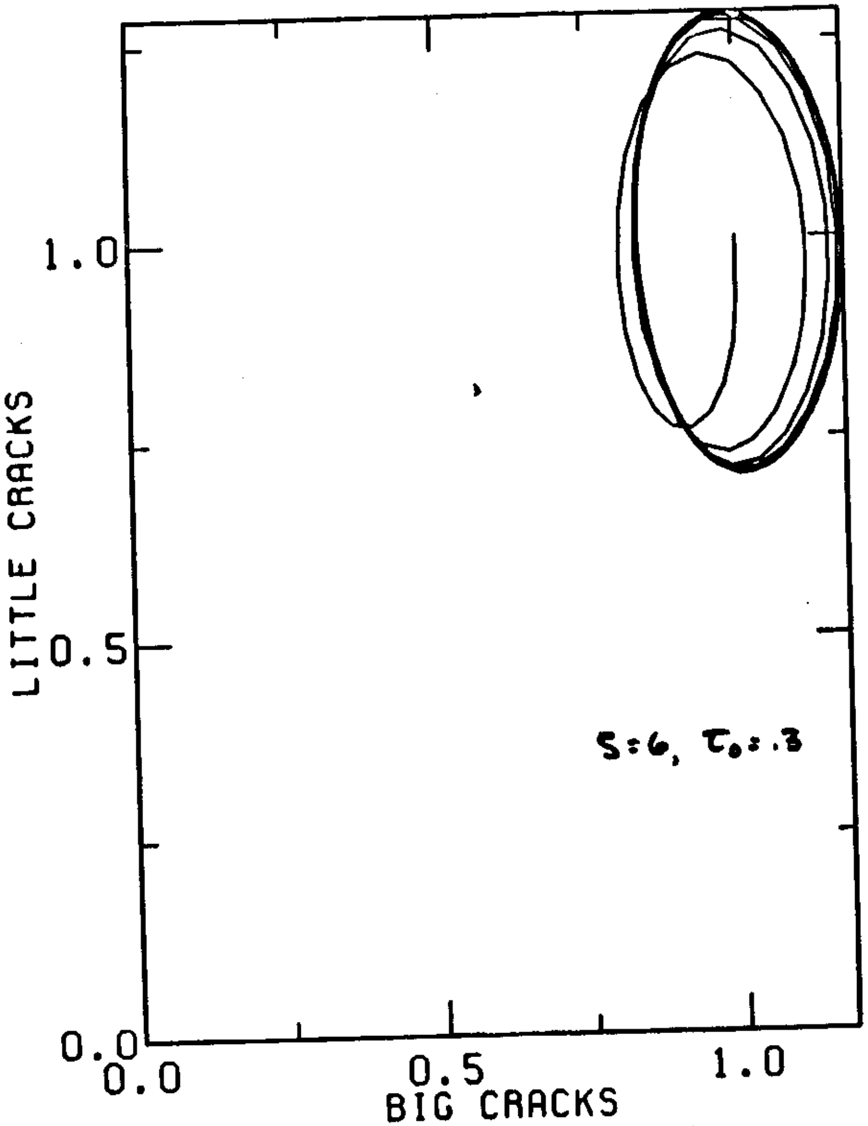
$$\xi = \frac{(1+\delta)\mu}{\delta \gamma \mu}$$

for $\gamma = 1$
$\alpha = 1$
$\zeta = 6$
$\eta = 3$
$\nu = 1$
$\kappa = 2$
$\gamma = 1$

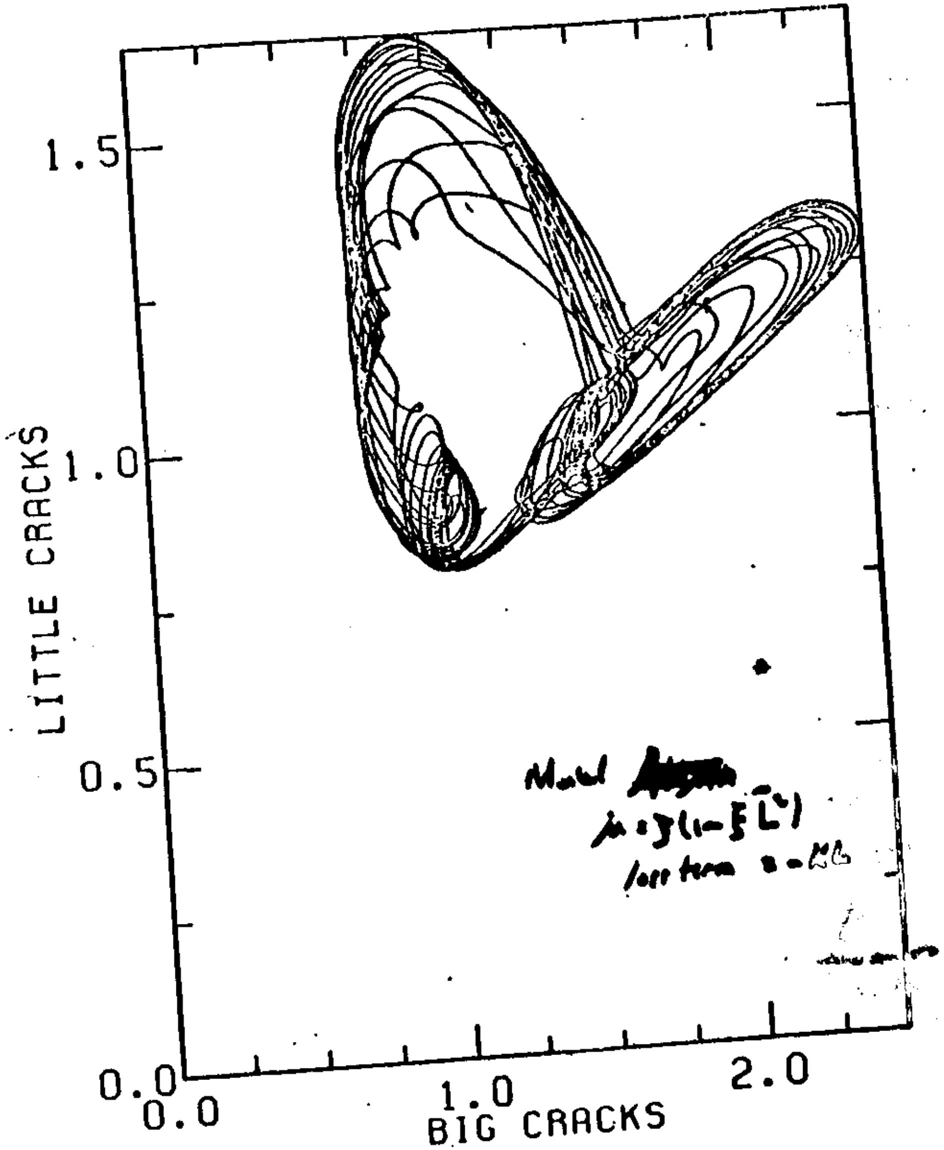
, Hopf bifurcation at $\tau_0 = .27$



V-19 30

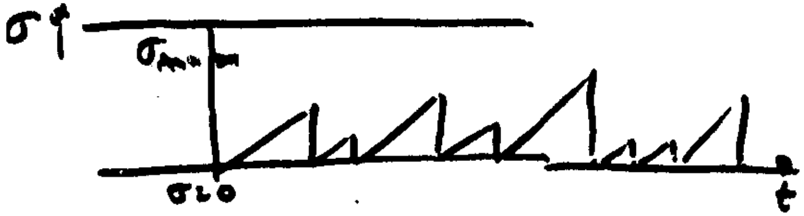


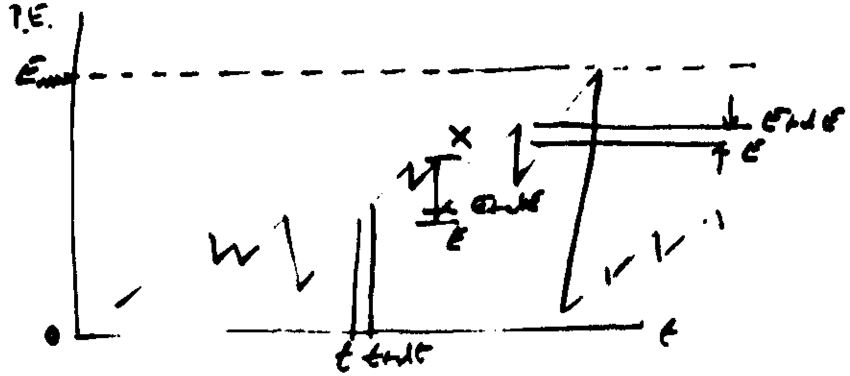
LITTLE CRACKS
BIG CRACKS
0.0
0.5
1.0
0.0
0.5
1.0
$S=6,\ \tau_0=.3$



LITTLE CRACKS
BIG CRACKS
0.0
0.5
1.0
1.5
0.0
1.0
2.0
Model
$\mu=\gamma(1-\xi\bar{L}^2)$
loss term = $-LB$

(slip)
Stress-prediction





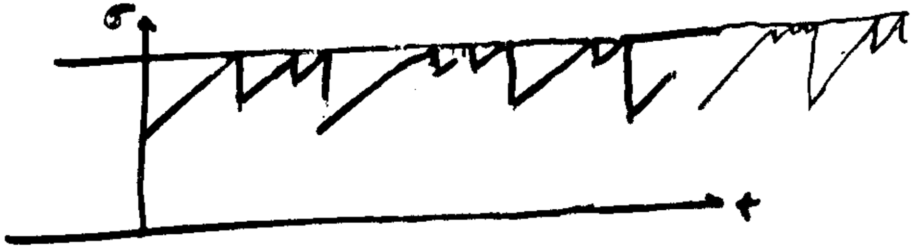
$P(E,t)\,dE$ is prob. that syst. is in state bet. E & E+dE at time t

$\lambda(E)\,dt$ is prob. that an eq. will occur at time between t & t+dt when syst. is in state E

$T(X,E)\,dE$ is cond. prob. that if eq. occurs when syst. is at state X it will have a final state bet. E & E+dE

α is rate of input of tectonic energy (assumed constant).

Time Prediction



if $\log_{10} \dot{N} = 4.77 - .85 M$

Pred. return time Poisson Process M>8 = 110 y.

For Poisson process mean = std. dev.

If $M_{max} = \infty$
$M_0 = 4$
$M_{cutoff} = 8$ $\sqrt{\text{variance}} = 75$ yrs.

Then

$$\lambda(E)P(E,t) + \alpha \frac{\partial P(E)}{\partial E} + \frac{\partial P(E,t)}{\partial t} = \int_E^{E_{max}} \lambda(X) P(X)\, T(X,E)\, dX$$

Normalization

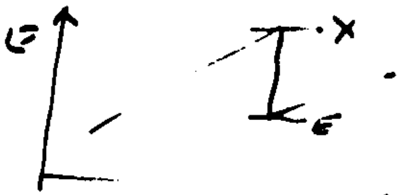
$$\int_0^X T(X,E)\,dE = 1$$

$$\int_{E_{min}=0}^{E_{max}=1} P(E)\,dE = 1$$

L. Knopoff
Reviews of Geophysics
1974

Theorem of Knopoff & Markushevich (Comput. Seismol 1984)
also Lomnitz-Adler: (BSSA 1986)
The system is always stable and stationary if Markovian
Define Seismicity

$$S(t) = \int P(x,t)(dx)\lambda(x)(dt)\, T(x,E)\,(x-E)\,dE$$

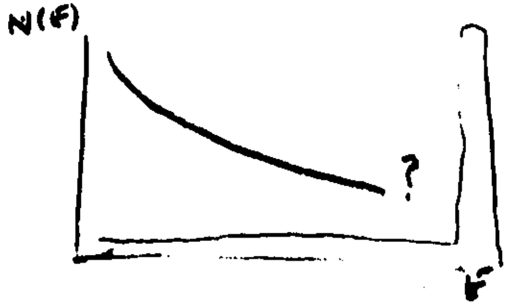


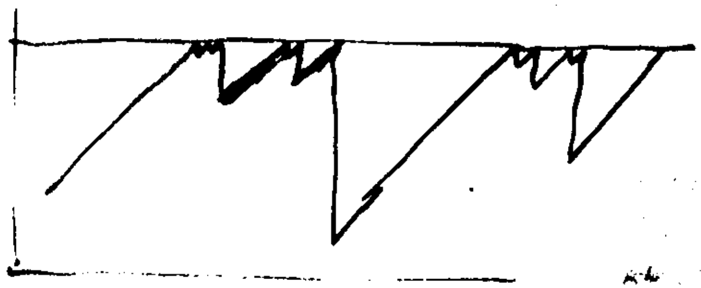
Now introduce time delays

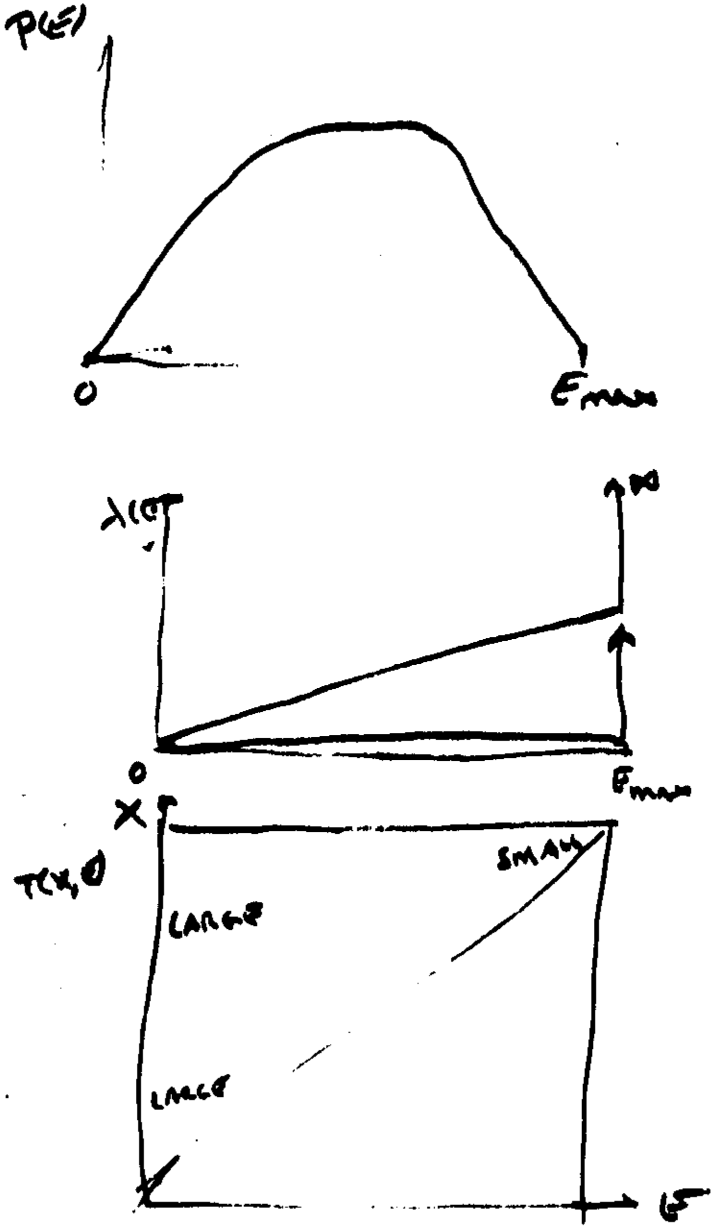
$$\lambda(E) = \lambda_0(E) + b\,S(t-\tau)$$

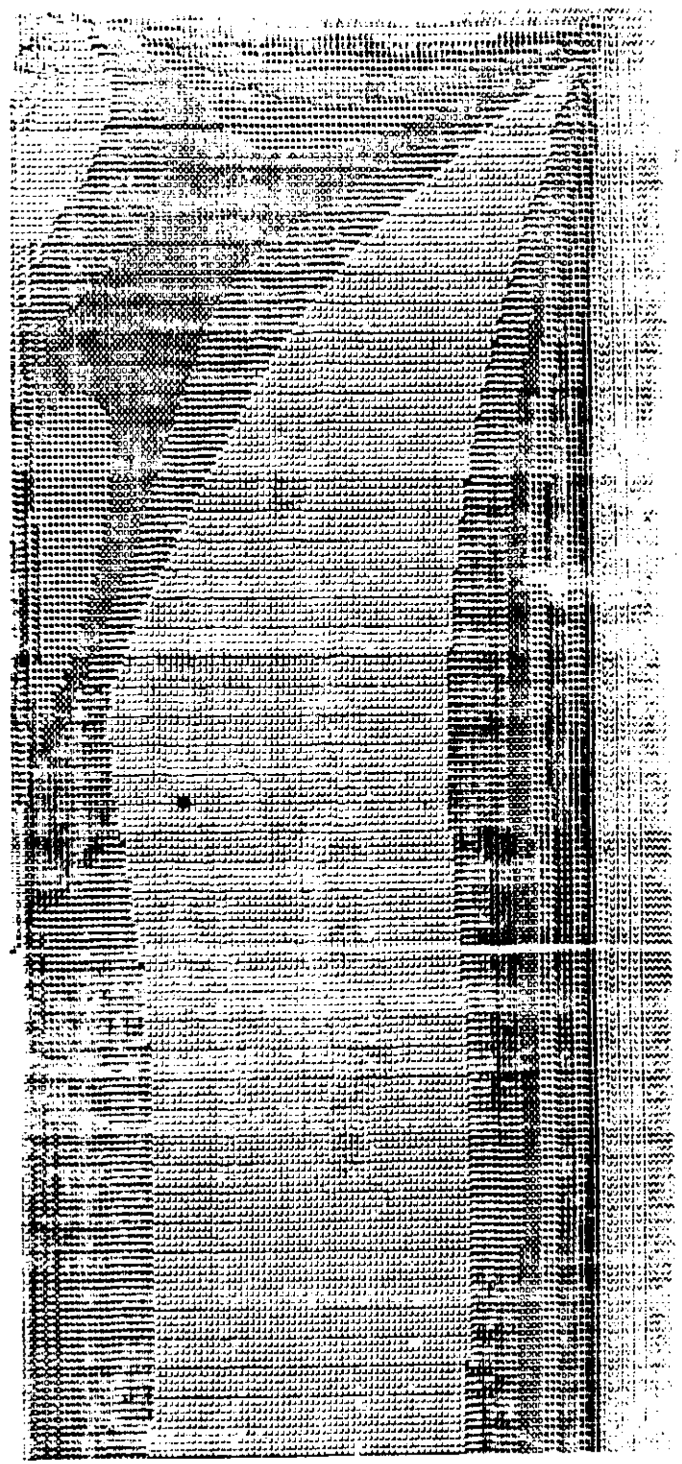
Is the system still stable & stationary?

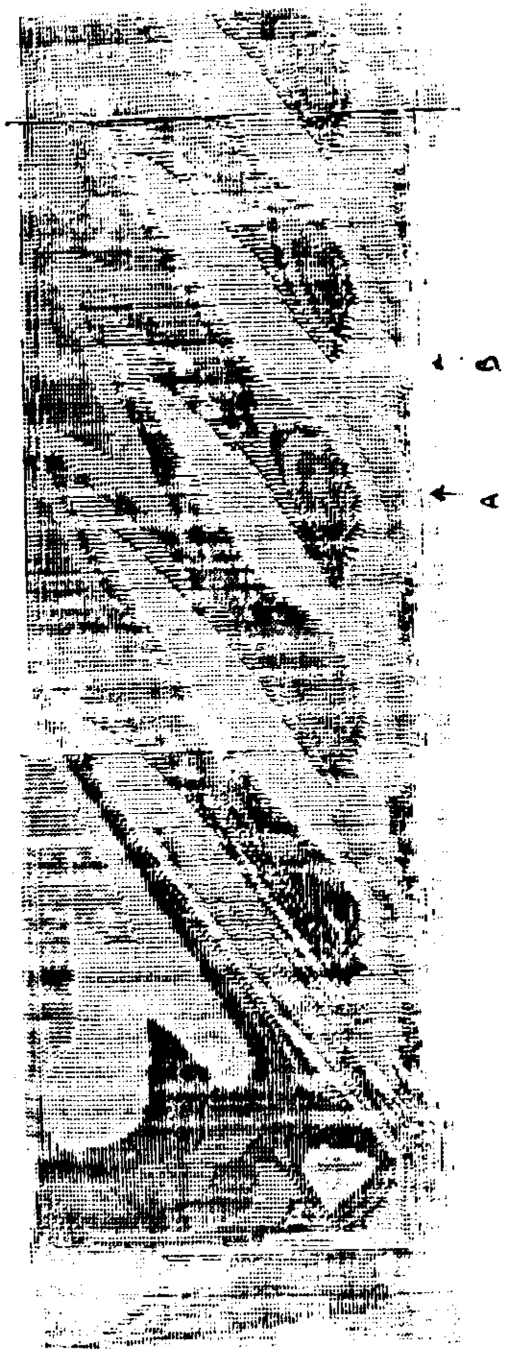
For time Predictable Model






P(E)
0
Emax
0
Emax
LARGE
LARGE




B
A