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The Index Theorem.

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These are preliminary lecture notes, intended only for distribution to participants.

The Index Theorem

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This is a set of notes associated with a short course on the index theorem to be presented at the I.C.T.P. in December 1988; these notes are intended to supplement and provide some additional materials on the lectures; more complete treatments are provided in the references listed at the end. The study of Clifford algebras is intimately connected with the index theorem; the 4 classical elliptic complexes are all most easily described in terms of Clifford algebras and the study of Clifford modules provides a common conceptual framework. Contents:

Section 1: Clifford algebras and spinnors.

Section 2: Spinnors and characteristic classes on manifolds.

Section 3: Elliptic complexes

Section 4: The index theorem.

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Section 1: Clifford algebras

The study of spinors is fundamental to the index theorem as well as to much of mathematical physics. We quote (with slight modifications to the wording) from E.Bolker [B].

> Consider a wrench, which is an object asymmetrical enough so that the result of any proper rotation performed on it is easily recognized. Rotate the wrench through a full 360 degree turn about an axis. Has it returned to the original state? Physical and geometric intuition both say 'yes', yet the calculus of spinors which models the quantum mechanical behavior of neutrons predicts the answer would be 'no' if the wrench were a neutron, or any other Fermion, a particle with half integral spin. More striking still, the predicted answer is 'yes' for two full turns (720 degrees) about the same axis. Consider a wrench (which Dirac would have called by its English name, a spanner, hence a spinor spanner because of the use to which he put it). Attach it by three cords to the walls of the room. When we turn the wrench through 360 the cords become tangled; no tampering can undo that tangle as long as the wrench is fixed. After two full turns, the snarl seems worse but is not. The cords can be untangled. The analogy between the spinor spanner and the neutron suggests that the state of the latter depends not only on its position and momentum but on which of two topologically distinct ways it is tied to its surroundings. A full turn about an axis leaves its position and momentum unchanged but reverses its topological relation to the rest of the universe.

In mathematical language, what is being discussed is the fact $\pi_1(SO(3))=Z_2$; a rotation of 360 is not homotopic to the identity while a rotation of 720 is homotopic to the identity; the strings keep track of the homotopy. If SO(n) is the special orthogonal group for $n\geq3$, then Spin(n) is its universal cover; although a rotation of 360 returns to the identity in SO(n), it does not do so in Spin(n).

Clifford algebras provide a convenient framework for discussing spinors. Let V be a finite dimensional real vector space of dimension m equipped with a positive definite inner product (,); we proceed in a coordinate free language to ease the transition to manifolds later. The exterior algebra $\Lambda(V)$ is the universal algebra generated by V subject to the relations $v\cdot w+w\cdot v=0$. More precisely, let

$$\otimes V=R\oplus V\oplus(V\otimes V)\oplus(V\otimes V\otimes V)\oplus\cdots$$

be the complete tensor algebra. Let I be the 2-sided ideal of $\otimes V$ generated by $\{v\otimes w+w\otimes v\}_{v,w\in V}$. Then $\Lambda(V)=\otimes V/I$. $\Lambda(V)$ inherits a natural inner product from the inner product on V. Let $\{v_i\}$ be an orthonormal basis for V. Let $I=\{1\le i_1<\ldots<i_\nu\le m\}$ be an sequence of integers ordered in increasing order. Let $|I|=\nu$ and define $v_I=v_{i_1}\cdot\cdots\cdot v_{i_\nu}$. Then $\{v_I\}_{|I|=\nu}$ is an orthonormal basis for the space of ν-forms $\Lambda^\nu(V)$ and we may decompose $\Lambda(V)=\oplus_{0\le\nu\le m}\Lambda^\nu(V)$.

$$\dim\{\Lambda^p(V)\}=\binom{p}{m} \text{ and } \dim\{\Lambda(V)\}=2^m.$$

The Clifford algebra Clif(V) is defined similarly. Clif(V) is the universal algebra generated by V subject to the Clifford commutation rules $v*w+w*v=-2(v,w)\cdot 1$. Again we can be more precise. Let J be the 2-sided ideal of $\otimes V$ generated by $\{v\otimes w+w\otimes v+2(v,w)\cdot 1\}$; $\mathrm{Clif}(V)=\otimes V/J$. There is always a question of what sign to adopt in defining Clifford multiplication; as we shall be complexifying for the most part it will not make a difference. Clif(V) inherits a natural inner product from V. If $\{v_i\}$ are an orthonormal basis for V, let $e_I=v_{i_1}*\ldots*v_{i_\nu}$; the $\{e_I\}$ are an orthonormal basis for Clif(V); Clif(V) is the algebra generated by the $\{v_i\}$ subject to the Clifford commutation relations: $v_i*v_j+v_j*v_i=-2\delta_{ij}$ where δ_{ij} is the Kronecker symbol. Dim $\mathrm{Clif}(V)=2^m$.

The exterior algebra inherited a Z-grading since the defining relation $v\cdot w+w\cdot v$ was homogeneous; $\Lambda^p(V)\cdot\Lambda^q(V)\subseteq\Lambda^{p+q}(V)$. Clif(V) inherits a Z_2 grading since the defining relation $v*w+w*v=-2(v,w)\cdot 1$ is quadratic. Let

$$\mathrm{Clif}^{even}(V)=\mathrm{span}\{e_I\}_{|I|=even}, \text{ and } \mathrm{Clif}^{odd}(V)=\mathrm{span}\{e_I\}_{|I|=odd}.$$

Then $\mathrm{Clif}(V)=\mathrm{Clif}^{even}(V)\oplus\mathrm{Clif}^{odd}(V)$.

$$\mathrm{Clif}^{even}(V)*\mathrm{Clif}^{even}(V)\subseteq\mathrm{Clif}^{even}(V),\ \mathrm{Clif}^{odd}(V)*\mathrm{Clif}^{odd}(V)\subseteq\mathrm{Clif}^{even}(V),$$
$$\mathrm{Clif}^{even}(V)*\mathrm{Clif}^{odd}(V)\subseteq\mathrm{Clif}^{odd}(V), \text{ and } \mathrm{Clif}^{odd}(V)*\mathrm{Clif}^{even}(V)\subseteq\mathrm{Clif}^{odd}(V).$$

If $\omega=w_1*\ldots*w_k$, let the transpose ω^t be defined by $\omega^t=w_k*\ldots*w_1$. If $\omega_i\in\mathrm{Clif}^{even}(V)$, then $(\omega_1,\omega_2)=\mathrm{Tr}(\omega_1*\omega_2^t)$ on $\mathrm{Clif}(V))\cdot\dim(\mathrm{Clif}(V))^{-1}$.

Let A be a unital algebra and let c be a linear map from V to A such that $c(v)c(w)+c(w)c(v)=-2(v,w)\cdot 1$. Then c extends to a representation $c:\mathrm{Clif}(V)\rightarrow A$. Equivalently, suppose given matrices $a_i\in A$ satisfying the Clifford commutation relation: $a_ia_j+a_ja_i=-2\delta_{ij}$; such matrices are often called Clifford matrices. If $\{v_i\}$ is an orthonormal basis for V, we can define a representation c of the Clifford algebra by defining $c(v_i)=a_i$. If W is a vector space and if $A=\mathrm{Hom}(W,W)$ is the algebra of linear transformations of W, then c gives W a Clif(V) module structure.

If $v\in V$ and $\omega\in\Lambda(V)$, let $\mathrm{ext}(v)\omega=v\cdot\omega$. Let $\mathrm{int}(v)\omega$ be the dual, interior multiplication. For example, if $v=v_1$ and $\omega=v_I=v_{i_1}\cdot\cdots\cdot v_{i_\nu}$, then

$$\mathrm{ext}(v_1)(v_I)=0 \text{ if } i_1=1, \qquad \mathrm{ext}(v_1)(v_I)=v_{\{1,i_1,\ldots\}} \text{ if } i_1>1$$
$$\mathrm{int}(v_1)(v_I)=v_{\{i_2,\ldots\}} \text{ if } i_1=1, \quad \mathrm{int}(v_1)(v_I)=0 \text{ if } i_1>1.$$

Exterior multiplication by v_1 adds the index 1; interior multiplication by v_1 removes the index 1. Let $c(v)=\mathrm{ext}(v)\text{-}\mathrm{int}(v)$; this is the symbol of the DeRham complex as we shall see presently. It is immediate from this description that $c(v)^2=-|v|^2\cdot I$ so c gives $\Lambda(V)$ a Clif(M) module structure. The map $e\rightarrow c(e)\cdot 1$ defines a linear map from Clif(V) to $\Lambda(V)$; $c(v_{i_1}*\ldots*v_{i_\nu})\cdot 1=v_{i_1}\cdot\cdots\cdot v_{i_\nu}$ so $e\rightarrow c(e)\cdot 1$ sends e_I to v_I; this gives a natural isomorphism between Clif(V) and $\Lambda(V)$ as vector spaces; it does not, of course, preserve the algebra structure; the natural (left) Clif(V) module structures on Clif(V) and $\Lambda(V)$ are isomorphic. Under this isomorphism, $\mathrm{Clif}^{even}(V)$ corresponds to $\Lambda^{even}(V)=\oplus_p\Lambda^{2p}(V)$ while $\mathrm{Clif}^{odd}(V)$ corresponds to $\Lambda^{odd}(V)=\oplus_p\Lambda^{2p+1}(V)$.

If $m=1$, let v be a unit vector in V. Then $\{1,v\}$ is a basis for Clif(V). Since $v^2=-1$, the map $a+bv\rightarrow a+bi$ is an isomorphism from Clif(V) to the complex numbers C. If $m=2$, let $\{v_1,v_2\}$ be an orthonormal basis for V. Then $\{1,v_1,v_2,v_1*v_2\}$ is a basis for Clif(V). Since $v_1^2=v_2^2=(v_1*v_2)^2=-1$, the map $a+bv_1+cv_2+dv_1*v_2\rightarrow a+bi+cj+dk$ is an isomorphism from Clif(V) to the quaternions H. If A is an algebra, let $M_n(A)$ be matrix algebra over A; $M_n(A)$ is the set of nxn matrices with values in A. Let $\mathrm{Clif}(m)=\mathrm{Clif}(R^m)$. Then (see Atiyah-Bott-Shaprio [ABS, page 11 table 1])

Theorem 1.1:

$$Clif(0)=R,\quad Clif(3)=H\oplus H,\quad Clif(6)=M_8(R),$$
$$Clif(1)=C,\quad Clif(4)=M_2(H),\quad Clif(7)=M_8(R)\oplus M_8(R),$$
$$Clif(2)=H,\quad Clif(5)=M_4(C),\quad Clif(8)=M_{16}(R).$$

These algebras are periodic; $Clif(n+8)\cong M_{16}(Clif(n))=Clif(n)\otimes M_{16}(R)$.

Let KU, KO, and KSp be the unitary, real, and symplectic K-theories. The periodicity of the real Clifford algebras with period 8 is at the heart of Bott periodicity for KO and KSp; see Karoubi [K, pages 127 ff]. If we complexify, the picture becomes much simpler; $\mathrm{Clif}(R^m)\otimes C$ has period 2; this is reflected by the fact that KU-theory is periodic with period 2 while KO and KSp are periodic with period 8.

Theorem 1.2: $Clif(2n)\otimes C\cong M_{2^n}(C)$ *and* $Clif(2n+1)\cong M_{2^n}(C)\oplus M_{2^n}(C)$.

The Spin representations

Fix an orientation for V. Let $\{v_1,\ldots,v_m\}$ be an oriented orthonormal basis for V. Let

$$\tau=\tau(V)=\begin{cases} i^n v_1*\ldots*v_m & \text{if } m=2n \text{ is even}\\ i^{n+1}v_1*\ldots*v_m & \text{if } m=2n+1 \text{ is odd}\end{cases}$$

be the normalized orientation class; the normalization is chosen so $\tau^2=1$. Under the correspondence $\omega \to c(\omega)\cdot 1$ which identifies Clif(V) with $\Lambda(V)$, τ corresponds to a suitable complex multiple of the volume element $v_1\cdot\cdots\cdot v_m$.

Theorem 1.3: Let E be a Clif(V)⊗C module.

(a) If $m=2n$, there exists a Clif(V)⊗C module $\Delta(V)$ of dimension 2^n so that $E \cong a\cdot\Delta(V)$ where $a=2^{-n}\cdot dim(E)$.

(b) If $m=2n+1$, there exists two inequivalent Clif(V)⊗C modules of dimension 2^n; E can be decomposed uniquely as a sum of these modules.

Proof: Suppose first $m=2n$ is even. Let $\alpha_1=i\cdot c(v_1)c(v_2)$, ... , $\alpha_n=i\cdot c(v_{2n-1})c(v_{2n})$; $c(\tau)=\alpha_1 * \ldots * \alpha_n$. The α_ν are a commuting family of endomorphisms of E satisfying $\alpha_\nu^2=1$. Let $\epsilon=(\epsilon_1, \ldots, \epsilon_n)$ for $\epsilon_\nu=\pm 1$ be a choice of signs. Let $E(\epsilon)$ be the simultaneous eigenspace of the α_ν: $E(\epsilon)=\{e\in E:\alpha_\nu e=\epsilon_\nu e\}$ for $1\le\nu\le n$. Let $\epsilon_0=(1,\ldots,1)$. $c(v_1)$ anti-commutes with α_1 and commutes with α_ν for $\nu>1$. Consequently $Spin(4)=S^3xS^3$. $E(\epsilon_1, \ldots, \epsilon_n)$ and $E(-\epsilon_1,\epsilon_2, \ldots, \epsilon_n)$. We use multiplication by $c(v_1)$, $c(v_3)$,..., $c(v_{2n-1})$ to define isomorphisms between $E(\epsilon_0)$ and $E(\epsilon)$ for arbitrary ϵ. Let $\{\phi_\mu\}$ be a basis for $E(\epsilon_0)$ where $1\le\mu\le a=\dim E(\epsilon_0)$. Let

$$S_{odd}=\{I:I=1\le i_1<\ldots<i_\nu<2n \text{ and } i_\mu \text{ odd}\} \text{ and } E_\mu=\mathrm{span}\{c(v_I)\cdot\phi_\mu:I\in S\};$$

$\dim(E_\mu)=2^n$. Then $E=\oplus_\mu E_\mu$ so $a\cdot 2^n=\dim(E)$. If $I\in S$, then $c(v_I)\cdot\phi_\mu\in E(\epsilon(I))$. Consequently $\alpha_\nu\cdot E_\mu\subseteq E_\mu$. Since $\{i\cdot v_{2\nu-1} * v_{2\nu}, v_{2\nu-1}\}$ generates Clif(V)⊗C as an algebra, E_μ is a Clif(V) module. The map $c(v_I)\cdot\phi_\mu\to c(v_I)\cdot\phi_1$ extends to a Clif(V) module isomorphism between E_μ and E_1 so $E=a\cdot E_1$; E is irreducible if and only if $\dim(E(\epsilon_0))=1$; any two irreducible Clif(V)⊗C modules must be isomorphic and we let Δ be any such module.

Next let $m=2n+1$ be odd. Then τ commutes with V so τ is central in Clif(V). Decompose $E=E_+\oplus E_-$ into the ± 1 eigenspaces of $c(\tau)$. Let $W=\mathrm{span}\{v_1,\ldots,v_{2n}\}$. Decompose $E_\pm=a_\pm\cdot\Delta(W)$ into irreducible Clif(W)⊗C modules. Since $\{W,\tau\}$ generate Clif(V)⊗C as an algebra and since $c(\tau)=\pm 1$ on $E_\pm$, this decomposition of $E_\pm$ as a Clif(W)⊗C module extends to a decomposition of $E_\pm$ as a Clif(V)⊗C module. ■

Remark: If dim(W) is even and if E is a Clif(W)⊗C module, then there are two different ways to make E a Clif(W⊕R)⊗C module corresponding to whether $c(\tau(W\oplus R))$ is to act as +1 or -1; to distinguish between these two modules we must choose an orientation for V.

We can describe the Spin representation Δ of Theorem 1.3(a) in terms of Clifford multiplication. Let m be even. Let Clif(V)⊗C act on itself by left multiplication. Right and left multiplication commute. Decompose Clif(V)⊗C under right multiplication by $\alpha_1=iv_1 * v_2$, ..., $\alpha_k=iv_{2k-1} * v_{2k}$ into simultaneous right eigenspaces $\Delta_r(\epsilon)$; each $\Delta_r(\epsilon)$ is invariant under left Clifford multiplication and is isomorphic to the spin representation. If $\dim(V)=2$, let $\{u,v\}$ be an orthonormal basis for V. Clif(V)⊗C decomposes as the sum of 2 copies of Δ;

$$\mathrm{Clif}(V)=\mathrm{span}\{u+iv,1-iu*v\}\oplus\mathrm{span}\{u-iv,1+iu*v\}.$$

If $\dim(V)=4$, let $\{u,v,w,x\}$ be an orthonormal basis for V. Clif(V)⊗C decomposes as the sum of 4 copies of Δ; these representation spaces can be taken to be the tensor product of the representation spaces in dimension 2. Clif(V)⊗C=

$$\mathrm{span}\{(u+iv)*(w+ix),(u+iv)*(1-iw*x),(1-iu*v)(w+ix),(1-iu*v)*(1-iw*x)\}\oplus$$
$$\mathrm{span}\{(u-iv)*(w+ix),(u-iv)*(1-iw*x),(1+iu*v)*(w+ix),(1+iu*v)*(1-iw*x)\}\oplus$$
$$\mathrm{span}\{(u+iv)*(w-ix),(u+iv)*(1+iw*x),(1-iuv)*(w-ix),(1-iu*v)*(1+iw*x)\}\oplus$$
$$\mathrm{span}\{(u-iv)*(w-ix),(u-iv)*(1+iw*x),(1+iu*v)*(w-ix),(1+iu*v)*(1+iw*x)\}.$$

Of course, there are many other decompositions possible. We can also discuss the matrices defining Δ. If $\dim(V)=2$, let

$$e_1=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, e_2=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, e_3=\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

These matrices satisfy the Clifford commutation relations. Let $c(a_1v_1+a_2v_2)=a_1e_1+a_2e_2$ define the Spin representation of Clif(2)⊗C on C^2. If $\dim(V)=4$, let $f_1=i\cdot e_1\otimes e_1$, $f_2=i\cdot e_1\otimes e_2$, $f_3=i\cdot e_1\otimes e_3$, $f_4=e_2\otimes 1$; these matrices satisfy the Clifford commutation relation so $c(a_\nu v_\nu)=\Sigma_\nu a_\nu v_\nu$ defines the spin representation Δ of Clif(4)⊗C on C^4.

Suppose $m=2n$ is even. Then the orientation τ anti-commutes with $\mathrm{Clif}^{odd}(V)$ but commutes with $\mathrm{Clif}^{even}(V)$. If we restrict Δ to $\mathrm{Clif}^{even}(V)$, it is no longer irreducible, but decomposes into two representations $\Delta^\pm$ called the half-spin representations corresponding to the ± 1 eigenvalues of τ. If $m=2$ and $\{u,v\}$ is an oriented orthonormal basis for V, then:

$$\Delta^+\cong\mathrm{span}\{u+iv\} \text{ and } \Delta^-\cong\mathrm{span}\{1-iuv\}.$$

If $m=4$ and if $\{u,v,w,x\}$ is an oriented orthonormal basis for V, then:

$$\Delta^+\cong\mathrm{span}\{(u+iv)(w+ix),(1-iuv)(1-iwx)\}$$
$$\Delta^-\cong\mathrm{span}\{(u+iv)(1-iwx),(1-iuv)(w+ix)\}.$$

More generally if V_i are even dimensional, then:

$$\Delta^+(V_1\oplus V_2)\cong\{\Delta^+(V_1)\cdot\Delta^+(V_2)\}\oplus\{\Delta^-(V_1)\cdot\Delta^-(V_2)\}$$
$$\Delta^-(V_1\oplus V_2)\cong\{\Delta^+(V_1)\cdot\Delta^-(V_2)\}\oplus\{\Delta^+(V_1)\cdot\Delta^-(V_2)\}.$$

It is convenient to write this as a formal difference:

Theorem 1.4: Let $m=2n$. The spin representation Δ decomposes into 2 representations $\Delta^\pm$ of $Clif^{even}(V)$⊗C; $c(\tau)=\pm 1$ on $\Delta^\pm$. If V_i are even dimensional,

$$\Delta^+(V_1\oplus V_2)-\Delta^-(V_1\oplus V_2)\cong\{\Delta^+(V_1)-\Delta^-(V_1)\}\otimes\{\Delta^+(V_2)-\Delta^-(V_2)\}.$$

The groups Pin(-), Spin(-), and Spin^c(—).

Let m be arbitrary. There are Lie groups which underly the algebras we have described above. Let O(V) be the orthogonal group of linear transformations of V which preserve the inner product; SO(V) is the subgroup of orientation preserving transformations. If $V=R^n$, then

$$O(n)=\{n\times n \text{ matrices } A \text{ so } A\cdot A^t=I\} \text{ and } SO(n)=\{A\in O(R^n):\det(A)=1\}.$$

Let

$$\mathrm{Pin}(V)=\{\omega\in \mathrm{Clif}(V):\ \omega=w_1*\ldots*w_j \text{ for } w_j\in V \text{ and } |w_j|=1\} \text{ and}$$

$$\mathrm{Spin}(V)=\mathrm{Pin}(V)\cap \mathrm{Clif}^{even}(V)$$

$$=\{\omega\in \mathrm{Clif}(V):\omega=w_1*\ldots*w_{2j} \text{ for } w_j\in V \text{ and } |w_j|=1\}.$$

If $\omega\in\mathrm{Pin}(V)$, then $\omega^t\in\mathrm{Pin}(V)$. Since $\omega\omega^t=(-1)^j$, Pin(V) forms a group under Clifford multiplication; Spin(V) is a normal subgroup. If $\omega\in\mathrm{Pin}(V)$, then $|\omega|=1$ so Pin(V) is contained in the unit sphere of Clif(V). If $v\in V$ and $\omega\in\mathrm{Pin}(V)$, define $\rho(\omega)\cdot v=\omega*v*\omega^t$. Let v_1 be a unit vector and embed v_1 in an orthonormal basis $\{v_1,\ldots,v_m\}$ for V. Then

$$\rho(v_1)\cdot v_1=v_1*v_1*v_1=-v_1 \text{ and } \rho(v_1)\cdot v_\nu=v_1*v_\nu*v_1=-v_1*v_1*v_\nu=v_\nu \text{ for } \nu>1.$$

$\rho(v_1)$ is reflection in the hyperplane defined by v_1; if $\omega=w_1*\ldots*w_j$, then $\rho(\omega)$ is a product of reflections in the hyperplanes defined by the w_j so $\rho(\omega)\in O(V)$. If $\omega\in\mathrm{Spin}(V)$, the number of hyperplane reflections is even so $\rho(\omega)\in SO(V)$. Since every element of O(V) is the product of hyperplane reflections, ρ defines a surjective map from Pin(V) to O(V) and from Spin(V) to SO(V). If $\rho(\omega)=I$ and $\omega\in\mathrm{Spin}(V)$, then $\omega^{-1}=\omega^t$ so $\omega*v=v*\omega$ for all $v\in V$. This implies $\omega\in\mathrm{Center}(\mathrm{Clif}(V))\cap\mathrm{Clif}^{even}(V)$ so ω is scalar. As $|\omega|=1$, $\omega=\pm1$. Therefore ρ defines a short exact sequence:

$$1\to Z_2\to \mathrm{Spin}(V)\xrightarrow{\rho} SO(V)\to 1.$$

If $\dim(V)\geq3$, then the fundamental group $\pi_1(SO(V))=Z_2$. Since Spin(V) is connected, this sequence shows Spin(V) is simply connected so Spin(V) is the universal cover of SO(V). If $\dim(V)=2$, then $SO(V)=\mathrm{Spin}(V)=U(1)$ is the circle and the map $\rho:\mathrm{Spin}(V)\to SO(V)$ is the double cover $z\to z^2$.

Theorem 1.5: If $\dim(V)\geq3$, Spin(V) is the universal cover of SO(V).

Remark: O(V) is not connected so the universal cover is not uniquely defined; one must decide how to multiply the components. Pin(V) is one universal cover of O(V); the other universal cover can be defined by choosing the opposite sign for the Clifford algebra.

Let $\mathrm{Spin}(n)=\mathrm{Spin}(R^n)$. We can describe Spin(3) and Spin(4) in terms of other classical groups. If we identify R^4 with the unit quaternions H, then the unit sphere S^3 inherits a natural group structure. There are two other equivalent ways of viewing S^3 as a group which will be useful. If $z\in S^3$ and $y\in H$, let $\rho(z)\cdot y=zy\bar z$ define a linear transformation of H. Since $|zy\bar z|=|y|$, $\rho(z)\in SO(H)$. Embed R^3 in H as the purely imaginary vectors; $R^3=i\cdot R+j\cdot R+k\cdot R$. Since $\rho(z)\cdot 1=1$, $\rho(z)$ preserves R^3 so $\rho(z)\in SO(R^3)=SO(3)$. The map $z\to\rho(z)$ is a 2-fold cover from S^3 to SO(3) so $S^3=\mathrm{Spin}(3)$ and $SO(3)=RP^3$ is real projective space of dimension 3. Let

$$e_1=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},\ e_2=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\ e_3=\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix};$$

$e_1^2=e_2^2=e_3^2=-1$ and $e_1e_2=e_3$. Let $\alpha=\alpha_0+i\alpha_1$ and $\beta=\beta_0+i\beta_1$ satisfy $|\alpha|^2+|\beta|^2=1$. Let

$$x=\begin{pmatrix} \alpha & \beta\\ -\bar\beta & \bar\alpha\end{pmatrix}=\alpha_0+\alpha_1e_1+\beta_0e_2+\beta_1e_3\in SU(2).$$

The map $x\to\alpha_0+i\alpha_1+j\beta_0+k\beta_1$ identifies SU(2) with S^3.

We can give another description of the identification of SO(3) with RP^3 which is closely related to the spinor spanner. Real projective 3-space RP^3 is the 3-sphere S^3 with antipodal points identified. Let D^3 be the unit 3-disk; we also regard D^3 as the upper hemisphere of S^3. RP^3 is D^3 with antipodal points on the boundary of D^3 identified. Let (r,ξ) be spherical coordinates on $D^3\subset R^3$; if $r\in[0,1]$ and $\xi\in S^2$, then the corresponding point of D^3 is $r\xi$. Let $A(r,\xi)\in SO(3)$ be rotation in R^3 through an angle of πr about the axis ξ. $A(0,\xi)=I$ and $A(1,\xi)=A(1,-\xi)$ so the map $(r,\xi)\to A(r,\xi)$ extends to a map from RP^3 to SO(3). Conversely let $A\in SO(3)$. Since $\det(\lambda I-A)=\lambda^3+\ldots-1$, A has a non-trivial positive eigenvalue; as $A\in SO(3)$, $\lambda=1$. Let $\xi(A)$ be a corresponding unit eigenvector; A is a rotation in the plane perpendicular to $\xi(A)$ about some angle θ. By replacing $\xi(A)$ by $-\xi(A)$ if need be, we can suppose $\theta=\pi r$ for $0\leq r\leq1$. If $r=0$, then $A=I$; if $r=1$, then the roles of $\xi(A)$ and $\xi(-A)$ are the same. This shows the correspondence between (r,ξ) and $A(r,\xi)$ is a bijective correspondence between RP^3 and SO(3); this correspondence is a diffeomorphism. These are the Euler angles.

We can also define Spin(4) in terms of quaternions. Let $f(z,w)y=zy\bar w$ for $z,w\in S^3$ and $y\in R^4=H$. Since $|f(z,w)y|=|y|$, $f(z,w)\in SO(4)$; the kernel of f is $\pm(1,1)\in S^3\times S^3$. This shows $S^3\times S^3$ is the universal cover of SO(4) so $\mathrm{Spin}(4)=S^3\times S^3$. $Z_2\oplus Z_2=\{(\pm1,\pm1)\}$ is a subgroup of Spin(4) and

$$\mathrm{Spin}(4)/Z_2\oplus Z_2=SO(4)/\{\pm1\}=RP^3\times RP^3.$$

This is one of the salient features of 4-dimensional geometry which fails in all other dimensions; Spin(4) is not a simple group. Equivalently, the lie algebra of SO(4) is decomposable.

We complexify. Let $Spin^c(V)=Spin(V)xU(1)/(g,\lambda)=(-g,-\lambda)$; formally speaking we just allow complex coefficients in our spinors. $Spin^c(V)$ acts naturally on the complexification of the Clifford algebra $Clif(V)\otimes C$. The map $(g,\lambda)\rightarrow(\rho(g),\lambda^2)$ gives a double cover $Spin^c(V)\rightarrow SO(V)xU(1)$. We can embed $Spin(V)\subseteq Spin^c(V)$ by $g\rightarrow(g,1)$. $Spin(3)=SU(2)$ so $Spin^c(3)=SU(2)xU(1)/Z_2$. The map $(g,\lambda)\rightarrow\lambda g$ is a surjective group homomorphism from $SU(2)xU(1)$ to $U(2)$; the kernel is $\{(1,1),(-1,-1)\}$ so this map descends to identify $Spin^c(3)$ with $U(2)$.

Theorem 1.6: $SO(3)=RP^3$, $Spin(3)=SU(2)=S^3$, $Spin^c(3)=U(2)$, *and* $Spin(4)=S^3xS^3$.

Section 2: Spinors and characteristic classes on manifolds.

Let M be a compact manifold of dimension m without boundary. The characteristic classes play a crucial role in this subject. We first discuss the Stieffel-Whitney classes; these are Z_2 characteristic classes associated with real vector bundles. The first and second Stiefel-Whitney classes measure the obstruction to a bundle being orientable and having a Spin structure. Next we discuss the Chern classes; these are elements of the DeRham cohomology of M given in terms of curvature which are associated with a complex vector bundle. Finally, we discuss the Pontrjagin and Euler classes; these are elements of DeRham cohomology associated with real vector bundles.

Stieffel-Whitney Classes

Let V be a real vector bundle over M. Using a partition of unity, we can put a smooth inner product on V. Let $\{U_a\}$ be a cover of M by geodesically convex balls. Let s_a be local orthonormal frames for V over U_a; on the overlap $U_a\cap U_\beta$, let $s_a=g_{a\beta}s_\beta$ where $g_{a\beta}:U_a\cap U_\beta\rightarrow O$; the $\{g_{a\beta}\}$ are the transition functions and satisfy the cocycle condition: $g_{a\beta}g_{\beta\gamma}=g_{a\gamma}$ and $g_{aa}=1$. The $g_{a\beta}$ are the "glueing" functions which determine how to construct the bundle. Let P_O^V be the bundle of orthonormal frames to V; P_O^V is a principal O bundle with transition functions $g_{a\beta}$.

V is orientable if we can choose the local orientations consistently or equivalently if we can choose the $\{s_a\}$ so that $g_{a\beta}\in SO$. This is called reducing the structure group; equivalently this means the principal bundle P_O^V has two components. There is an obstruction; not every vector bundle is orientable; the Mobius strip over the circle and the tangent bundle of RP^2 are not orientable.

The first Stieffel-Whitney class w_1 is the obstruction to orientability. Fix a connection ∇ on V. If γ is a closed loop in M, let $w_1(V)[\gamma]=1$ if parallel translation around γ preserves the orientation of a frame and let $w_1(V)[\gamma]=-1$ if parallel translation reverses the orientation; $w_1(V)$ takes values in the group $Z_2=\{\pm1\}$; $w_1(V)[\gamma]$ is independent of the connection ∇ chosen. If γ_t is a 1-parameter family of such loops, then parallel translation is continuous and hence $w_1(V)[\gamma_t]$ is independent of t. If $\gamma=\gamma_1\gamma_2$ is the composite of two loops, then $w_1(V)[\gamma_1\gamma_2]=w_1(V)[\gamma_1]\cdot w_1(V)[\gamma_1]$. Thus the first Stieffel-Whitney class $w_1(V)$ is a representation from the fundamental group $\pi_1(M)$ to Z_2; $w_1(V)\in H^1(M;Z_2)$. V is orientable if and only if $w_1(V)=0$—i.e. parallel translation around any closed loop preserves the orientation of a frame. Let L be the Mobius line bundle over the circle; $L=[0,2\pi]\times R$ where we identify $(0,t)=(2\pi,-t)$ to put in a half twist. Then going once around the circle reverses the orientation so $w_1(L)$ is the non-trivial representation of $\pi_1(S^1)=Z$ to Z_2; $w_1(L)$ is the non-trivial element of $H^1(S^1;Z_2)=Z_2$. If V is a real line bundle, then V is trivial if and only V is orientable or equivalently if $w_1(V)=0$. The first Stieffel-Whitney class classifies real line bundles. The higher Stieffel-Whitney classes are generalizations of w_1; w_2 is the obstruction to a Spin structure as we shall see shortly. The total Stieffel-Whitney class of a real vector bundle is characterized by the following axioms; we refer to Husemoller [Hu, page 234ff] for further details.

Theorem 2.1: If V is a real vector bundle of dimension v, then $w(V)=1+w_1(V)+...+w_v(V)$ *where the* $w_i\in H^i(M;Z_2)$.

(a) If $f:M\rightarrow N$, *then* $f^*(w(V))=w(f^*(V))$. *(naturality).*

(b) $w(V\oplus W)=w(V)\cdot w(W)$. *(additivity).*

(c) $w_1(L)$ *generates* $H^1(S^1;Z_2)=Z_2$. *(normalization).*

Remark: Although it is convenient to write Z_2 multiplicatively, expressing (b) gets a bit tricky. Let F_2 be the field with two elements; the additive group of F_2 is isomorphic to the multiplicative group Z_2. If we regard $w\in H^*(M;F_2)$, then cup product gives a map from $H^i(M;F_2)\otimes H^j(M;F_2)$ to $H^{i+j}(M;F_2)$. Property (b) becomes $w_i(V\oplus W)=\Sigma_{j+k=i}\, w_j(V)\cup w_k(W)$. Therefore $w_1(V_1\oplus V_2)=w_1(V_1)+w_1(V_2)$; if $w_1(V_i)=0$ then $w_2(V_1\oplus V_2)=w_2(V_1)+w_2(V_2)$.

Unfortunately, the description of w_1 given above in terms of parallel translation does not generalize to discuss the obstruction to spin structures. It is convenient, therefore, to give a definition of w_1 in terms of Cech cohomology. A Cech k-cochain is a function $f(\alpha_0,\ldots,\alpha_k)\in Z_2=\{\pm1\}$ defined for j+1 tuples of indices where $U_{\alpha_0}\cap\cdots\cap U_{\alpha_k}\neq\emptyset$. We assume f is totally symmetric-i.e. $f(\alpha_{\sigma(0)},\ldots,\alpha_{\sigma(k)})=f(\alpha_0,\ldots,\alpha_k)$ for any permutation σ. One of the many advantages of Z_2 coefficients is that there is no need to worry about signs. Let $Cech^k(M;Z_2)$ be the multiplicative group of all such functions. The coboundary $\delta:Cech^k(M;Z_2)\rightarrow Cech^{k+1}(M;Z_2)$ is defined by:

$$(\delta f)(\alpha_0,\ldots,\alpha_{k+1})=\Pi_\nu\, f(\alpha_0,\ldots,\alpha_{\nu-1},\alpha_{\nu+1},\ldots,\alpha_{k+1})$$

if $U_{\alpha_0}\cap...U_{\alpha_{k+1}}\neq\emptyset$; otherwise $\delta f(\alpha_0,\ldots,\alpha_{k+1})$ isn't defined. Let $H^k(M;Z_2)=\{\ker(\delta_k)\}/\mathrm{image}(\delta_{k+1})\}$; these groups are independent of the particular cover of M by small geodesic balls which was chosen. Let V be a real vector bundle over M and let $\{s_\alpha\}$ be local orthonormal frames for V over U_α. Let $s_\alpha=g_{\alpha\beta}s_\beta$ and let $f(\alpha,\beta)=\det(g_{\alpha\beta})=\pm1$. Since $\det(g_{\alpha\beta})=\det(g_{\beta\alpha}^{-1})=\det(g_{\beta\alpha})$, $f\in Cech^1(M;Z_2)$. We compute

$$\delta f(\alpha,\beta,\gamma)=\det(g_{\alpha\beta})\det(g_{\beta\gamma})\det(g_{\alpha\gamma})=\det(g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha})=1$$

so $f\in\ker(\delta_2)$. If we change the choice of frames and let $\tilde s_\alpha=\epsilon_\alpha s_\alpha$, then the

corresponding transition functions $\tilde g_{\alpha\beta}$ are defined by $\tilde g_{\alpha\beta}=\epsilon_\alpha g_{\alpha\beta}\epsilon_\beta^{-1}$ so $\tilde f(\alpha,\beta)=f(\alpha,\beta)\cdot\det(\epsilon_\alpha)\cdot\det(\epsilon_\beta)$. This shows f and $\tilde f$ differ by a co-boundary. Consequently the cohomology class $[f]\in H^1(M;Z_2)$ is independent of the particular local frames chosen. If $f(\alpha,\beta)=1$ for all α,β with $U_\alpha\cap U_\beta\neq\emptyset$, then $\det(g_{\alpha\beta})=1$ so V is orientable. More generally, if the cohomology class of f is trivial, then we can find signs $\epsilon(\alpha)\in Z_2=\{\pm1\}$ so that $f(\alpha,\beta)=\epsilon(\alpha)\epsilon(\beta)$ for $U_\alpha\cap U_\beta\neq\emptyset$. We reverse the orientation of s_α if $\epsilon_\alpha=-1$ to create a new system of local frames with consistent local orientations; this shows V is orientable if and only if $w_1(V)=0$.

Consider the Mobius bundle. Let $\theta\rightarrow e^{i\theta}$ parametrize the circle for $\theta\in[0,2\pi]$. Let $U_1=(0,\pi)$, $U_2=(2\pi/3,5\pi/3)$, $U_3=(4\pi/3,7\pi/3)$ and let s_i be sections to the Mobius bundle over the U_i. We may identify $s_1=s_2$ over $U_1\cap U_2=(2\pi/3,\pi)$ and $s_2=s_3$ over $U_2\cap U_3=(4\pi/3,5\pi/3)$. However, to take into account the twist, we must identify $s_3=-s_1$ over $U_1\cap U_3$ (recalling that θ is periodic with period 2π). Consequently $f_{12}=1$, $f_{23}=1$, and $f_{13}=-1$; since $U_1\cap U_2\cap U_3=\{\emptyset\}$, $Cech^\nu(S^1;Z_2)=\{0\}$ for $\nu\geq2$ and the coboundary map δ is trivial on $Cech^2$. If $\epsilon_1\epsilon_2=1$, $\epsilon_1\epsilon_2=1$, and $\epsilon_1\epsilon_3=-1$ then $(\epsilon_1\epsilon_2\epsilon_3)^2=1$; this shows f is not a coboundary so w_1 is non-trivial in $H^1(S^1;Z_2)=Z_2$.

Spin Structures

We suppose V orientable henceforth; let P_{SO}^V be the bundle of oriented orthonormal frames; P_{SO}^V is a principal SO bundle and is one of the components of P_O^V. A spin structure on V is a lifting of the transition functions from SO to Spin. We suppose given $h_{\alpha\beta}:U_\alpha\cap U_\beta\rightarrow Spin$ so that $\rho(h_{\alpha\beta})=g_{\alpha\beta}$ and $h_{\alpha\beta}h_{\beta\gamma}=h_{\alpha\gamma}$. This is equivalent to constructing a principal Spin bundle P_{Spin}^V together with a double cover $\rho:P_{Spin}^V\rightarrow P_{SO}^V$. We describe this obstruction in terms of Cech cohomology. Let $h_{\alpha\beta}$ be any lifting of $g_{\alpha\beta}$ to Spin; there are always two possible liftings $\pm h_{\alpha\beta}$ and there is nothing (locally) to prefer about one as opposed to the other. Since $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}=1$, $h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}=\epsilon(\alpha,\beta,\gamma)\in\ker(\rho)$; this implies $\epsilon(\alpha,\beta,\gamma)=\pm1$. ϵ is a Cech cocycle and $[\epsilon]\in H^2(M;Z_2)$ is independent of the choices made. Furthermore $h_{\alpha\beta}$ defines a spin structure if and only if ϵ is trivial; an oriented bundle V admits a Spin structure if and only $w_2(V)=0$. V admits a $Spin^c$ structure if instead of lifting the transition functions from SO to Spin they are lifted from SO to $Spin^c$.

Theorem 2.2: Let V be an oriented real vector bundle.

(a) V admits a Spin structure if and only if $w_2(V)=0$. *Inequivalent Spin structures are parametrized by* $H^1(M;Z_2)$ *or equivalently by real line bundles over M.*

(b) V admits a $Spin^c$ *structure if and only if* $w_2(V)$ *can be lifted from* $H^2(M;Z_2)$ *to* $H^2(M;Z)$; *inequivalent* $Spin^c$ *structures are parametrized by* $H^2(M;Z)$ *or equivalently by complex line bundles over M.*

Remark: If V_1 is Spin, then $V_1\oplus V_2$ is Spin if and only if V_2 is Spin; similarly if V_1

is $Spin^c$, then $V_1 \oplus V_2$ is $Spin^c$ if and only if V_2 is $Spin^c$. We note any Spin bundle is $Spin^c$; the converse need not be true.

There is an important category of examples. If W is a complex vector bundle, let W_r be the underlying real vector bundle.

Theorem 2.3: If W is a complex vector bundle, let W_r be the underlying real vector bundle. W_r admits a canonical $Spin^c$ structure.

Proof: The mod 2 reduction of the first Chern class of W is the second Stieffel-Whitney class of W. As this topological proof is a bit unsatisfying, we also give a geometric proof. Let $\gamma: U(k) \rightarrow SO(2k) \times U(1)$ be induced from the inclusion $U(k) \rightarrow SO(2k)$ and $\det: U(k) \rightarrow U(1)$. We wish to lift γ to a map $\hat{\gamma}$ from U(k) to $Spin^c(2k)$; this will give any complex bundle a canonical $Spin^c$ structure. Let $g \in U(k)$ and choose an orthonormal basis $\{e_\nu\}$ for C^k so $g(e_\nu) = e^{i\theta_\nu} \cdot e_\nu$. $\gamma(g)$ is orthogonal rotation through an angle θ_ν on the 2-dimensional real subspace spanned by e_ν and $f_\nu = ie_\nu$. Define

$$\hat{\gamma}(g) = \Pi_\nu \{\cos(\theta_\nu/2) + \sin(\theta_\nu/2) e_\nu f_\nu\} e^{i\theta_\nu/2} \in Spin^c(2k).$$

There is always an indeterminacy involved in choosing the angle θ_ν; replacing θ_ν by $\theta_\nu + 2\pi$ changes the sign of both $\{\cos(\theta_\nu/2) + \sin(\theta_\nu/2) e_\nu f_\nu\}$ and $e^{i\theta_\nu/2}$ so the product is independent of the choices made. It is essential to have the additional complex factor to correct for this indeterminacy; we can not lift U(k) to Spin(2k).

We say that M is Spin if M is oriented and if the tangent bundle T(M) admits a Spin structure. Similarly we say that M is $Spin^c$ if M is oriented and if T(M) admits a $Spin^c$ structure. If M is 1 or 3 dimensional, then T(M) is trivial so M is always Spin; we omit these cases from the examples below:

Example 1: The sphere $S^n = \{x \in R^{n+1}: |x| = 1\}$ is always orientable and Spin.

Example 2: Real projective space RP^n is the set of real lines through the origin in R^{n+1}. By identifying a point of S^n with the line it defines, we may identify $RP^n = S^n/Z_2$ with the sphere modulo the antipodal action. RP^n is orientable if $n = 2k-1$ is odd. It is Spin if k is even; it is $Spin^c$ but not Spin if k is odd.

Example 3: Complex projective space CP^n is the set of complex lines through the origin in C^{n+1}. By identifying a point of S^{2n-1} with the line it defines, we may identify $CP^n = S^{2n+1}/S^1$. CP^n is always $Spin^c$; it is Spin if n is odd.

Example 4: Any holomorphic manifold is always $Spin^c$.

Example 5: Embed the cyclic group Z_n diagonally in U(k). Let $M = S^{2k-1}/Z_n$ be a lens space. If n is odd, M is Spin. If n is even, M is Spin if and only if k is even. M is always $Spin^c$.

Example 6: Any orientable 2-dimensional manifold is Spin.

Example 7: The product of Spin/$Spin^c$ manifolds is Spin/$Spin^c$.

Example 8: The connected sum of Spin/$Spin^c$ manifolds is Spin/$Spin^c$.

Chern Classes

The Stieffel-Whitney classes were Z_2 classes. The Chern classes are Z classes; by taking coefficients in C, we can regard the Chern classes in terms of curvature as elements of the DeRham cohomology of M. Let $\Lambda^p(M) = \Lambda^p(T^*M)$ be the bundle of exterior p-forms and let $\Lambda(M) = \oplus_p \Lambda^p(M)$ be the exterior algebra of M. Exterior differentiation is a map d from $C^\infty(\Lambda^p(M))$ to $C^\infty(\Lambda^{p+1}(M))$; $d^2 = 0$. Let $H^p_{DeR}(M) = \ker(d_p)/\mathrm{image}(d_{p-1})$ be the DeRham cohomology; $H^p_{DeR}(M) = H^p(M;C)$ by the DeRham theorem. If M is integrable, $\omega \rightarrow \int_M \omega$ is an isomorphism from $H^m_{DeR}(M)$ to C.

Let V be a complex vector bundle and let ∇ be a connection on V. There are many definitions of a connection (see for example [EGH]), we choose the following as it is perhaps the most conceptual. We wish to generalize the notion of the total directional derivative. A connection ∇ on V is a first order partial differential operator $\nabla: C^\infty(V) \rightarrow C^\infty(V \otimes T^*M)$ which satisfies Leibnitz's rule $\nabla(f \cdot s) = f \cdot \nabla(s) + s \otimes df$. There is a natural extension of ∇ to the exterior algebra:

$$\nabla: C^\infty(V \otimes \Lambda^p(T^*M)) \rightarrow C^\infty(V \otimes \Lambda^{p+1}(T^*M))$$

defined by $\nabla(s \otimes \omega) = \nabla(s) \otimes \omega + s \otimes d\omega$. In contrast to ordinary exterior differentiation, ∇^2 need not be 0. However $\nabla^2(fs) = f\nabla^2(s)$ so ∇^2 is a 0^{th} order differential operator; it is a 2-form valued endomorphism of V. ∇^2 is the curvature Ω of the connection ∇. If ω is the connection 1-form of ∇, then $\Omega = d\omega + \omega \cdot \omega$. We say ∇ is Riemannian if it behaves properly with respect to the (Hermitian) inner product-

$$(\nabla s_1, s_2) + (s_1, \nabla s_2) = d(s_1, s_2).$$

We restrict to such connections henceforth. Relative to a local orthonormal frame, the curvature is skew-symmetric; $\Omega + \Omega^* = 0$. We can always embed V in the trivial bundle $1^\nu = M \times C^\nu$. Let s be a local frame for V; we can differentiate the components of s term by term. Let π_V be orthogonal projection on V. Then we can define a connection ∇ by projecting ordinary differentiation back to V; $\nabla(s) = \pi_V(ds)$. The curvature of this connection is $\Omega = \pi_V d\pi_V d\pi_V$. If V is the tangent bundle of M and if the embedding of V in a trivial bundle arises from an embedding of M in R^ν, then ∇ is the Levi-Civita connection.

We define the total Chern form as

$$c(\Omega) = \det(I + i \cdot \Omega/2\pi) = 1 + c_1(\Omega) + c_2(\Omega) + \ldots$$

where the individual Chern forms $c_i(\Omega)$ are forms of degree 2i. For example:

$$c_1 = (i/2\pi)\,\mathrm{Tr}(\Omega),$$

$$c_2 = (1/8\pi^2) \cdot \{\mathrm{Tr}(\Omega \cdot \Omega) - \mathrm{Tr}(\Omega) \cdot \mathrm{Tr}(\Omega)\},$$

$$c_3=(1/48\pi^3)\cdot\{-2\mathrm{Tr}(\Omega\cdot\Omega\cdot\Omega)+3\mathrm{Tr}(\Omega\cdot\Omega)\cdot\mathrm{Tr}(\Omega)-\mathrm{Tr}(\Omega)\cdot\mathrm{Tr}(\Omega)\cdot\mathrm{Tr}(\Omega)\}.$$

$c(\Omega)$ is a closed differential form; let $[c(\Omega)]\in H^{even}_{DeR}(M;C)$ be the corresponding element of cohomology. $[c(\Omega)]$ is independent of the particular connection chosen; let $c(V)=[c(\Omega)]$ be the total Chern class of the bundle.

We illustrate this with an example on the Riemann sphere. Let $CP^1=S^3/S^1$ be the set of complex lines through the origin in C^2. If $0\neq v\in C^2$, let $<v>\in CP^1$ be the line determined by v. The map $z\rightarrow<z,1>$ is an embedding of C in CP^1. $CP^1-C=\{<1,0>\}$ so CP^1 is the Riemann sphere $S^2=C\cup\{\infty\}$. Let L be the classifying bundle over CP^1;

$$L=\{(<x>,\lambda)\in CP^1\times C^2:\lambda\in<x>\}\subset CP^1\times C^2.$$

L is a sub-bundle of the trivial bundle 2-plane bundle. Let $s(z)=(<z,1>,(z,1))$ be the canonical section to L over C; s is a meromorphic section with a simple pole at ∞. Let π_L be orthogonal projection from 1^2 to L. Then

$$\nabla(s)=\pi_L\{(1,0)\}\otimes dz=(1+|z|^2)^{-1}s\otimes\bar{z}dz,\quad \Omega=2i(1+x^2+y^2)^{-2}dx\cdot dy,$$
$$\nabla^2(s)=(1+|z|^2)^{-2}s\otimes d\bar{z}\cdot dz,\quad c_1(\Omega)=-\pi^{-1}(1+x^2+y^2)^{-1}dx\cdot dy.$$

Therefore

$$\int_{CP^1}c_1(L)={}_0\int^{\infty}{}_0\int^{2\pi}-\pi^{-1}r(1+r^2)^{-1}drd\theta=-1.$$

The Chern class can be characterized by the following properties; see Husemoller [Hu, page 234 ff]

Theorem 2.4: If V is a real vector bundle of dimension v, then $c(V)=1+c_1(V)+\ldots+c_v(V)$ where the $c_i\in H^i(M;C)$.

(a) If $f:M\rightarrow N$, then $f^*(c(V))=c(f^*(V))$. (naturality)

(b) $c(V\oplus W)=c(V)\cdot c(W)$. (additivity)

(c) $\int_{CP^1}c_1(L)=-1$. (normalization).

The Chern class converts direct sum into multiplication. The Chern character ch converts direct sum into addition and tensor product into multiplication; ch extends to a ring isomorphism between $KU(M)\otimes C$ (the complex K-theory with coefficients in C) and $H^{even}_{DeR}(M;C)$. Modulo torsion, complex K-theory and cohomology are the same. Define:

$$ch(\Omega)=\mathrm{Tr}\{e^{i/2\pi\cdot\Omega}\}=\Sigma_\nu(1/\nu!)\cdot(i/2\pi)^\nu\cdot\mathrm{Tr}(\Omega^\nu);$$

$ch(\Omega)$ is a closed differential form and $ch(V)=[ch(\Omega)]\in H^{even}(M;C)$;

$$ch(V\oplus W)=ch(V)+ch(W)\text{ and }ch(V\otimes W)=ch(V)\cdot ch(W).$$

We can express the Chern character in terms of the Chern class:

$$ch_0(V)=\dim(V),$$
$$ch_1(V)=c_1(V)=(i/2\pi)\mathrm{Tr}(\Omega),\text{ and}$$

$$ch_2(V)=-c_2(V)+c_1^2(V)/2=(-1/8\pi)\mathrm{Tr}(\Omega^2).$$

We compute the Chern character for the instanton (or possibly the anti-instanton) bundle over S^4. Let $e(x)=\Sigma_\nu x_\nu e_\nu$ be a representation of $Clif(R^5)\otimes C$ on C^4 as in Theorem 1.3; we choose the normalization so $e(\tau)=i^3e_1e_2e_3e_4e_5=1$ or equivalently $\mathrm{Tr}(e_1e_2e_3e_4e_5)=4i$. If $x\in S^4$, then $e(x)^2=-1$. Let

$$\Pi_\pm(x)=\{(x,w)\in S^4\times C^4:\ e(x)\cdot w=\pm i\cdot w\}$$

be the bundle of $\pm i$ eigenvalues over S^4. Let $\pi(x)=(1-i\cdot e(x))/2$ be orthogonal projection from the trivial bundle $S^4\times C^4$ to $\Pi_+(x)$. The curvature Ω of the canonical connection on Π_+ is $\pi\cdot d\pi\cdot d\pi$. Let $dvol_\theta$ be the spherical volume element on S^4. Since this construction is equivariant with respect to the action of SO(5), $\mathrm{Tr}(\Omega^2)$ is a constant multiple of $dvol_\theta$. We evaluate the multiple by computing at the north pole $N=(1,0,0,0,0)$:

$$\pi(N)=(1-ie_1)/2$$
$$d\pi(N)=(e_2dx_2+e_3dx_3+e_4dx_4+e_5dx_5)/2$$
$$\Omega(N)=\pi(N)\cdot d\pi(N)\cdot d\pi(N)=(1-ie_1)/2\cdot\Sigma_{2\leq i,j\leq 5}(e_ie_jdx_i\cdot dx_j)/4$$
$$\Omega^2(N)=\{(1-ie_1)/2\}\cdot 4!\cdot(1/16)\cdot e_2e_3e_4e_5\cdot dx_2\cdot dx_3\cdot dx_4\cdot dx_5$$
$$\mathrm{Tr}(\Omega^2)(N)=(1/2)\cdot 4!\cdot(1/16)\cdot(-i)\cdot\mathrm{Tr}(e_1e_2e_3e_4e_5)dvol_\theta=3\cdot dvol_\theta.$$

The volume of S^{2n} is $\pi^n\cdot 2^{2n+1}\cdot n!/(2n)!$ so $vol(S^4)=8\pi^2/3$ and

$$\int_{S^4}ch_2(\Pi_+)=3\cdot(i/2\pi)^2\cdot(1/2)\cdot vol(S^4)=-1.$$

Theorem 2.5: Let $\{e_i\}_{1\leq i\leq 5}$ be 4x4 Clifford matrices so $i\cdot e_1e_2e_3e_4e_5=Id_4$. Let Π_+ be the bundle of $+i$ eigenvalues of $e(x)=\Sigma_\nu e_\nu x_\nu$ over S^4. Then $\int_{S^4}ch_2(\Pi_+)=-1$.

Remark: If e is a representation of $Clif(R^{2n+1})\otimes C$ on C^{2^n}, then $\int_{S^{2n}}ch_n(\Pi_+(e))=\pm 1$; the particular sign depends on which of the two inequivalent irreducible representations were chosen. Modulo a possible sign convention, this the Bott class.

The Chern character is defined by the exponential function; there are other characteristic classes which appear in the index theorem which are defined using other generating functions. Let $x=(x_1,\ldots)$ be a collection of indeterminates. Let $\{s_\nu(x)\}$ be the elementary symmetric functions; $\Pi_\nu(1+x_\nu)=1+s_1(x)+\ldots$. Let $f(x)$ be a symmetric polynomial. We can express $f(x)=F(s_1(x),\ldots)$ in terms of the elementary symmetric functions. If $A=\mathrm{diag}(x_1,\ldots)$, then $c(A)=\det(1+A)=1+s_1(x)+\ldots$; we omit the factors of $i/2\pi$ in the interests of notational convenience. We define $f(\Omega)=F(c_1(\Omega),\ldots)$ by substitution. For example if $f(x)=\Sigma_\nu e^{x_\nu}$, then $f(\Omega)=ch(\Omega)$. Define the Todd class by:

$$Td(x)=\Pi_\nu\, x_\nu/(1-e^{-x_\nu})=1+Td_1(x)+\ldots$$

Then:

$$Td_1 = c_1/2\ ,\ Td_2 = (c_2 + c_1^2)/12\ ,\ Td_3 = c_1c_2/24.$$

Pontrjagin Classes

If V is a real vector bundle with a Riemannian connection Ω, the total Pontrjagin class is defined by $p(\Omega) = \det(1 + (1/2\pi)\cdot\Omega)$. This is a closed differential form and defines an element $[p(\Omega)]$ of DeRham cohomology. Let $p(V) = [p(\Omega)] \in H_{DeR}(M)$; this is independent of the particular connection chosen. Since $\Omega + \Omega^t = 0$, the forms of odd degree vanish and we can express $p(\Omega) = 1 + p_1(\Omega) + \ldots + p_\nu(\Omega)$ for $\nu = [\dim(V)/2]$ the greatest integer in $\dim(V)/2$ and $p_j(\Omega) \in \Lambda^{4j}(M)$. For example,

$$p_1(V) = -1/(8\pi^2)\cdot Tr(\Omega\cdot\Omega).$$

If $V \otimes C$ is the complexification of V, then $p_j(V) = (-1)^j c_{2j}(V \otimes C)$. Let $p(M) = p(T(M))$ be the total Pontrjagin class of M. We can define some additional characteristic classes by means of formal power series. Let

$$L(x) = \Pi_\nu\ x_\nu/\tanh(x_\nu) = 1 + L_1(x) + L_2(x) + \ldots \text{ and}$$

$$\hat{A}(x) = \Pi_\nu\ (x_\nu/2)/\sinh((x_\nu/2)) = 1 + \hat{A}_1(x) + \ldots.$$

Then:

$$L_1 = p_1/3, \qquad L_2 = (7p_2 - p_1^2)/45,$$

$$\hat{A}_1 = -p_1/24, \quad \hat{A}_2 = (-4p_2 + 7p_1^2)/5760.$$

We define $L(M)$ and $\hat{A}(M)$ by taking $V = T(M)$.

Euler Class

There is one final characteristic class which will play an important role in our analysis. While a real-anti-symmetric matrix A of shape 2nx2n cannot be diagonalized, it can be put in block diagonal form with 2x2 blocks of the form:

$$\begin{pmatrix} 0 & x_\nu \\ -x_\nu & 0 \end{pmatrix}$$

The top Pontrjagin class $p_n(A)$ is the square of the product of the x_a; $p_n(A) = (x_1 \ldots x_n)^2$. The Euler class $e(A) = x_1 \ldots x_n$ is the square root of p_n. If V is an oriented vector bundle of dimension 2n, the Euler class $e(V) \in H^{2n}(M;C)$ is a well defined characteristic class satisfying $e(V)^2 = p_n(V)$. If W is a complex vector bundle and if V is the underlying real oriented vector bundle, then $e(V) = c_n(W)$. For example

$$e(V) = 1/(2\pi)\cdot\Omega_{12} \text{ if } \dim(V) = 2.$$

If $\epsilon(abcd)$ is the totally anti-symmetric tensor, then

$$e(V) = 1/(32\pi^2)\ \epsilon_{abcd}\Omega_{ab}\Omega_{cd} \ \dim(V) = 4.$$

If we reverse the orientation of M, then $e(V)$ changes sign. Let $e(M) = e(T(M))$. If we reverse the local orientation of M, then $e(M)$ changes sign. Consequently $e(M)$ is a distribution or measure rather than an m-form. $e(M)$ is the integrand of the Chern-Gauss-Bonnet theorem. Let R_{ijkl} be the components of the Levi-Civita connection with respect to some orthonormal local frame field; we adopt the convention that $R_{1212} = -1$ on the standard sphere S^2. Let dvol be the Riemannian volume element on M. Then:

$$e(M) = (-1/2\pi)\cdot R_{1212}\cdot \text{dvol if } m = 2;$$

$$e(M) = (1/128\pi^2)\cdot\epsilon(abcd)\epsilon(uvwx)R_{abuv}R_{cdwx}\text{dvol if } m = 4.$$

Section 3: Elliptic complexes

Let M be a compact Riemannian manifold without boundary of dimension m. Let V and W be be vector bundles over M and let $A:C^\infty(V)\to C^\infty(W)$ be a first order partial differential operator. If we choose a system of local coordinates $x=(x_1,\ldots,x_m)$ for M and local frames s_V and s_W for V and W, then we can expand $A=\Sigma_{1\le\nu\le m} a^\nu(x)\partial/\partial x_\nu + b$ where $\{a^\nu(x),b\}$ are smooth matrix valued functions of x. The leading symbol of A is defined by formally substituting ξ_ν for $\partial/\partial x_\nu$;

$$\sigma(A)(x,\xi)=\Sigma_{1\le\nu\le m} a^\nu(x)\cdot\xi_\nu.$$

The leading symbol is sometimes defined with factors of i to make formulas involving the Fourier transform and adjoint more elegant; we delete these factors in the interests of simplicity here. If we change the local frame for V and W, then $\sigma(A)(x,\xi)$ transforms like a tensor; the 0^{th} order part transforms according to a more complicated rule since we must differentiate the rule for the change of frame. Consequently it is natural to regard $\sigma(A)(x,\xi)$ as a section to the bundle Hom(V,W). If we change the system of local coordinates, $\sigma(A)(x,\xi)$ transforms like a co-vector. More precisely, let $\omega\in T^*(M)$ be a cotangent vector. Expand $\omega=\Sigma_\nu \xi_\nu dx^\nu$ and define

$$\sigma(A)(\omega)_X=\Sigma_{1\le\nu\le m} \xi_\nu\cdot a^\nu(x).$$

Let y_ν be a different set of local coordinates. Since $dy^\mu=\Sigma_\nu\partial y^\mu/dx_\nu\cdot dx^\nu$,

$$\omega=\Sigma_\mu\varsigma_\mu dy^\mu=\Sigma_{\nu,\mu}\varsigma_\mu\partial y^\mu/dx_\nu\cdot dx^\nu=\Sigma_\nu\xi_\nu dx^\nu.$$

Therefore $\xi_\nu=\Sigma_\mu\varsigma_\mu dy^\mu/dx_\nu$. Dually $\partial/\partial x_\nu=\Sigma_\mu\partial y^\mu/\partial x_\nu\cdot\partial/\partial y_\mu$ so that

$$A=\Sigma_\nu a^\nu\cdot\partial/\partial x_\nu=\Sigma_{\nu,\mu} a^\nu\partial y^\mu/\partial x_\nu\cdot\partial/\partial y_\mu=\Sigma_\mu\{\Sigma_\nu a^\nu\partial y^\mu/\partial x_\nu\}\partial/\partial y_\mu.$$

Therefore the leading symbol in the coordinate system Y is given by.

$$\sigma(A)(\omega)_Y=i\cdot\Sigma_{\nu,\mu} a^\nu\partial y^\mu/\partial x_\nu\varsigma_\mu=i\cdot\Sigma_\nu a^\nu\cdot\Sigma_\mu\partial y^\mu/\partial x_\nu\varsigma_\mu=i\cdot\Sigma_\nu a^\nu\xi_\nu=\sigma(A)(\omega)_X.$$

This shows the leading symbol of A is a well defined map $\sigma(A):T^*(M)\to\mathrm{Hom}(V,W)$.

More generally, let $\alpha=(\alpha_1,\ldots,\alpha_m)$ for $\alpha_\nu\in N$ be a multi-index. Let

$$|\alpha|=\alpha_1+\ldots+\alpha_m,\ \xi^\alpha=\Pi_\nu\xi_\nu^{\alpha_\nu},\text{ and } d_x^\alpha=\Pi_\nu(\partial^{\alpha_\nu}/\partial x_{\alpha_\nu}).$$

Let $P=\Sigma_{|\alpha|\le k} p_\alpha(x)d_x^\alpha:C^\infty(V)\to C^\infty(W)$ be a k^{th} order partial differential operator (PDO). Let $S^k(M)$ be the k^{th} symmetric power of the cotangent space. Then

$$\sigma(P)=\Sigma_{|\alpha|=k} p_\alpha(x)\xi^\alpha:S(T^*M)\to\mathrm{Hom}(V,W)$$

is a symmetric polynomial of order k on $T^*(M)$ taking values in Hom(V,W). Fix a

fiber metric on V and W and let $P^*:C^\infty(W)\to C^\infty(V)$ be the adjoint. Then modulo lower order terms $P^*=(-1)^k\cdot\Sigma_{|\alpha|=k} p_\alpha(x)^* d_x^\alpha$ so $\sigma(P^*)=(-1)^k\sigma(P)^*$. If we can compose the PDO's P and Q, then $\sigma(PQ)=\sigma(P)\sigma(Q)$. We say P is elliptic if $\sigma(P)(\omega)$ is invertible for $\omega\ne0$; the class of elliptic operators is closed under adjoint and composition.

Spectral theory of self-adjoint PDO's.

A spectral resolution $\{\phi_n,\lambda_n\}$ of a self-adjoint operator $P:C^\infty(V)\to C^\infty(V)$ is a complete orthonormal basis $\{\phi_n\}$ for $L^2(V)$ of eigenfunctions ϕ_n corresponding to the eigenvalues λ_n. We order the eigenfunctions so $|\lambda_1|\le|\lambda_2|,\ldots$. We refer to Gilkey [G, Lemma 1.6.2] for:

Theorem 3.1: Let P be a self-adjoint elliptic PDO.

(a) There exists a spectral resolution $\{\phi_n,\lambda_n\}$ of P.

(b) The $\{\phi_n\}$ are smooth functions of P.

(c) There exists $\delta>0$, $\epsilon>0$, and $n_0>0$ so that if $n\ge n_0$, then $|\lambda_n|\ge\epsilon\cdot n^\delta$.

(d) $\dim\ker(P)<\infty$. $L^2(V)=\ker(P)\oplus\mathrm{image}(P)$.

The spectral resolution of $-\partial^2/\partial\theta^2$ on S^1 is $\{e^{in\theta},n^2\}_{n\in Z}$. The eigenfunctions $e^{in\theta}$ are smooth and the eigenvalues grow quadratically. We generalize the notion of Fourier series as follows. Let $\{\phi_n,\lambda_n\}$ be a spectral resolution of a self-adjoint elliptic PDO. Expand $f\in C^\infty(V)$ in the form $f=\Sigma_n a_n(f)\phi_n$ where $a_n(f)=\int_M(f,\phi_n)d\mathrm{vol}$. This series converges absolutely and can be differentiated term by term arbitrarily often. The generalized Fourier coefficients $a_n(f)$ decay faster than any power of n.

Given a symbol $a:T^*M\to\mathrm{Hom}(V,W)$ and a connection ∇ on V, we can define a canonical differential operator A with leading symbol a. Let $a=\Sigma_\nu a^\nu\cdot\xi_\nu$ and define $A=\Sigma_\nu a^\nu(x)\nabla_{\partial/\partial x_\nu}$. ∇ is a first order PDO from $C^\infty(V)$ to $C^\infty(V)\otimes T^*M$ and a is linear map from $V\otimes T^*M$ to W; A is the composition $a\cdot\nabla$. d is the canonical operator associated with ext and the Levi-Civita connection. δ is the canonical operator associated with -int and the Levi-Civita connection.

*Poincare Duality and the Hodge * operator*

Let d be exterior differentiation. $d(\Sigma_I f_I dx^I)=\Sigma_\nu\partial f_I/\partial x_\nu\cdot dx^\nu\cdot dx^I$ so

$$\sigma(d)(\omega)\{\Sigma_I f_I dx^I\}=\Sigma_{\nu,I}\xi_\nu dx^\nu\cdot f_I dx^I=\Sigma_I\omega\cdot f_I dx^I=\mathrm{ext}(\omega)\{\Sigma_I f_I dx^I\}.$$

This shows the leading symbol of exterior differentiation is exterior multiplication; dually the leading symbol of the adjoint δ is minus interior multiplication. The leading symbol of $d+\delta$ is Clifford multiplication $c(\omega)=\mathrm{ext}(\omega)-\mathrm{int}(\omega)$. Since $c(\xi)^2=-|\xi|^2$, $c(\xi)$ is invertible for $\xi\ne0$ so $d+\delta$ is elliptic. Let $\Delta=(d+\delta)^2$ be the Laplacian; $\sigma(\Delta)(x,\xi)=-|\xi|^2$ so Δ is elliptic. We expand $L^2(\Lambda^p)$ in an orthogonal direct sum using Theorem 3.1:

$$L^2(\Lambda^p)=\ker(\Delta_p)\oplus\text{image}(\Delta_p)=\ker(\Delta_p)\oplus\text{image}(d_{p-1})\oplus\text{image}(\delta_{p-1});$$

$\ker(\Delta_p)=\ker(d_p)\cap\ker(\delta_{p-1})$. Let $\omega\in\ker(\Delta_p)$ be a harmonic p-form. Since $d\omega=0$, ω determines an element $[\omega]$ of the DeRham cohomology groups $H^p_{Der}(M)$. Conversely, let $[\theta]\in H^p_{Der}(M)$. Expand $\theta=\omega+d\omega_1+\delta\omega_2$. Since $d\theta=0$, $d\delta\omega_2=0$ so $0=(d\delta\omega_2,\omega_2)=(\delta\omega_2,\delta\omega_2)$ so $\delta\omega_2=0$ and $\theta=\omega+d\omega_1$. This shows the map $\ker(\Delta_p)\rightarrow H^p_{Der}(M)$ is surjective. Since the sum is an orthogonal decomposition, $\ker(\Delta_p)\cap\text{image}(d_{p-1})=\{0\}$. This proves:

Theorem 3.2: $\ker(\Delta_p)\cong H^p_{Der}(M)=H^p(M)$.

Let $\text{Clif}(M)\otimes C$ be the complexification of the Clifford algebra on the cotangent space T(M). Suppose M is oriented and let $\tau\in\text{Clif}(M)\otimes C$ be the normalized orientation form. If $\{v_1,\dots,v_m\}$ is a local oriented orthonormal frame for $T^*(M)$, then $\tau=\epsilon\cdot v_1*\dots*v_m$ where ϵ is a suitable power of i. Clifford multiplication $c(\tau)$ is a map from Λ^p to Λ^{m-p}.

Lemma 3.3:

(a) $(d+\delta)c(\tau)=(-1)^{m+1}c(\tau)(d+\delta)$.

(b) $\Delta_p c(\tau)=c(\tau)\Delta_{m-p}$.

Proof: If $\omega\in T^*M$, then $c(\omega)c(\tau)=(-1)^{m+1}c(\tau)c(\omega)$. Since $c(\omega)$ is the leading symbol of $d+\delta$, $\sigma\{(d+\delta)c(\tau)-(-1)^{m+1}c(\tau)(d+\delta)\}=0$ so $(d+\delta)c(\tau)=(-1)^{m+1}c(\tau)(d+\delta)+B$ where B is a 0^{th} order operator. Fix a point P of M; we wish to show B vanishes at P. Choose a system of local coordinates centered at P and let $ds^2=\Sigma_{j,k}\, g_{jk}dx^j\cdot dx^k$ be the Riemannian metric on M relative to the system of coordinates. We can express d, δ, and $c(\tau)$ functorially with respect to this coordinate system; this shows B is linear expression in the first derivatives of the metric with coefficients which are smooth universal expression of the g_{jk}'s. If we choose geodesic polar coordinates centered at P, then all the first derivatives of the metric vanish. This proves (a); (b) follows from (a). ■

Since $c(\tau)$ intertwines Δ_p and Δ_{m-p}, $c(\tau)$ is an isomorphism from $H^p(M)$ to $H^{m-p}(M)$; modulo suitable factors of i, $c(\tau)$ is Poincare duality.

$$\chi(M)=\Sigma_p(-1)^p\dim H^p(M)=\Sigma_p(-1)^p\dim H^{m-p}(M)=(-1)^m\chi(M)$$

so $\chi(M)=0$ if m is odd. It is possible to describe Poincare duality in terms of the Hodge star operator. Let $\{v_1,\dots,v_m\}$ be a local orthonormal frame for $T^*(M)$. Let $\text{dvol}=v_1\cdot\,\cdots\,\cdot v_m$ be the unnormalized Riemannian volume element; $c(\tau)\cdot 1=\epsilon\cdot\text{dvol}$ where ϵ is a suitable power of i. Let $(\cdot,\cdot)$ be the Riemannian inner product on the exterior algebra $\Lambda(M)$. If $\alpha,\beta\in\Lambda^p(M)$, define the Hodge star duality operator $*:\Lambda^p(M)\rightarrow\Lambda^{m-p}(M)$ by $(\alpha,\beta)\cdot\text{dvol}=\alpha\cdot*\beta$. If $I\cup J=\{1,\dots,m\}$, then $*(v_I)=\epsilon\cdot v_J$ where ϵ is the sign of the permutation $\binom{1,\dots,m}{i_1,\dots,i_p,j_1,\dots,j_{m-p}}$; $v_I\cdot v_J=\epsilon\cdot\text{dvol}$. The whole difficulty with working with the Hodge $*$ operator is keeping track of the signs. The proof

of (b,c,d) is an algebraic exercise left to the reader:

Lemma 3.4:

(a) $c(\tau)$ *and* $*$ *are isomorphisms between* $H^p(M)=\ker(\Delta_p)$ *and* $H^{m-p}(M)=\ker(\Delta_{m-p})$.

(a) $*^2=(-1)^{p(m-p)}$

(b) $\delta=(-1)^{mp+m+1}*d*$

(c) If m is even, then $c(\tau)=i^{m/2+p(p-1)}*_p$.

Elliptic Complexes

We say that (V,A) is an elliptic complex over M if: (a) $V=\{V_p\}_{0\le p\le n}$ *is a finite collection of vector bundles over M.*

(b) $A=\{A_p\}_{0\le p<n}$ where $A_p:C^\infty(V_p)\rightarrow C^\infty(V_{p+1})$ are 1^{st} order PDOs.

(c) $A_p\cdot A_{p-1}=0$.

(d) If $\omega\in T^*(M)$ and $\omega\neq0$, then $\ker\sigma(A_p)(\omega)=\text{image}\,\sigma(A_{p-1})(\omega)$.

(d) is the assumption the complex is exact on the symbol level and is a nondegeneracy condition. If (a,b,c) hold, then the following conditions are equivalent:

(d) If $\omega\in T^*(M)$ and $\omega\neq0$, then $\ker\sigma(A_p)(\omega)=\text{image}\,\sigma(A_{p-1})(\omega)$.

(d1) $A+A^*:C^\infty(\oplus_i V_i)\rightarrow C^\infty(\oplus_i V_i)$ is elliptic

(d2) the associated Laplacian $\Delta=(A+A^*)^2$ is elliptic.

Since $A^2=(A^*)^2=0$, $\Delta=\oplus_p$ where $\Delta_p=A_{p-1}A^*_{p-1}\oplus A^*_pA_p:C^\infty(V_p)\rightarrow C^\infty(V_p)$ is a self-adjoint elliptic 2^{nd} order PDO.

The Hodge decomposition theorem is the fundamental tool used to study elliptic complexes; it follows easily from Theorem 3.1.

Theorem 3.5: (Hodge Decomposition Theorem). Let (V,A) be an elliptic complex.

(a) There is an orthogonal direct sum decomposition

$$L^2(V^p)=\text{image}(A^{p-1})\oplus\text{image}(A^{p*})\oplus\ker(\Delta^p).$$

(b) $\ker(\Delta^p)$ *is a finite dimensional subset of* $C^\infty(V^p)$.

(c) Let $H^p(V,A)=\ker(A_p)/\text{image}(A_{p-1})$. $H^p(V,A)\cong\ker(\Delta_p)$.

Let $\text{index}(V,A)=\Sigma_p(-1)^p\cdot\dim H^p(V,A)=\Sigma_p(-1)^p\cdot\dim\ker(\Delta_p)$; the index is constant under perturbations as we shall see in Theorem 4.2. The Atiyah-Singer index theorem provides a formula for index(V,A) in terms of topological information about the bundles V and the leading symbol of A. The index vanishes if m is odd and is multiplicative under finite coverings. We can always "roll up" the complex to define a 2-term elliptic complex with the same index. Let $V_{even}=\oplus_p V_{2p}$ and $V_{odd}=\oplus_p V_{2p+1}$. Then $A+A^*:C^\infty(V_{even})\rightarrow C^\infty(V_{odd})$ is a 2-term elliptic complex and $\text{index}(V,A)=\text{index}(\{V_{even},V_{odd}\},A+A^*)$.

Let (V,A) be an elliptic complex over M and let (W,B) be an elliptic complex over N. We form $(V\otimes W, A\otimes B)$ over $M\times N$ by defining $(V\otimes W)_\nu = \oplus_{a+b=\nu} V_a \otimes V_b$ and $(A\otimes B)_\nu = \oplus_{a+b=\nu}(A_a \otimes 1_b + (-1)^a 1_a \otimes B_b)$; the $(-1)^a$ sign is inserted so that $(A\otimes B)^2 = 0$. This is an elliptic complex over $M\times N$ and $\mathrm{index}((V\otimes W, A\otimes B)) = \mathrm{index}(V,A)\cdot\mathrm{index}(W,B)$.

The DeRham Complex

The DeRham complex $\{\Lambda^{even}-\Lambda^{odd}, d+\delta\}$ is defined by taking

$$V_0 = \oplus_p \Lambda^{2p}(M),\ V_1 = \oplus_p \Lambda^{2p+1}(M),\ \text{and } A = (d+\delta).$$

Conditions (a,b,c) are immediate; since Clifford multiplication $c(\omega)$ is non-singular for $\omega\neq 0$, the complex is elliptic. This complex is also often regarded as $\{d: C^\infty(\Lambda^p) \to C^\infty(\Lambda^{p+1})\}$; we have rolled up this complex to form a 2-term complex. The associated cohomology can be regarded as a formal difference $H^{even}(M) - H^{odd}(M)$. The Laplacian is the Bochner Laplacian $\Delta_p = (d^*d + dd^*)_p$; we can identify the topological cohomology groups $H^p(M;C)$ with the DeRham cohomology groups $H^p_{Der}(M)$. These groups can be identified with $\ker(\Delta_p)$ by the Hodge decomposition theorem. Therefore:

$$\mathrm{index}(\Lambda^{even}-\Lambda^{odd}, d+\delta) = \dim H^{even}(M) - \dim H^{odd}(M) = \Sigma(-1)^p \dim H^p(M) = \chi(M).$$

This is a homotopy invariant of M. The DeRham complex is multiplicative with respect to products

$$(\Lambda^{even}(M\times N) - \Lambda^{odd}(M\times N), d+\delta) = (\Lambda^{even}(M) - \Lambda^{odd}(M), d+\delta)\otimes(\Lambda^{even}(N) - \Lambda^{odd}(N), d+\delta).$$

The Euler characteristic is a combinatorial invariant. For example, if M is a 2-dimensional polyhedra, then $\chi(M) = \#\text{vertices} - \#\text{edges} + \#\text{faces}$.

Let S^m be the sphere of dimension m, let $RP^m = S^m/Z_2$ be real projective space, and let CP^n be complex projective space. If M and N are manifolds of dimension m, the connected sum $M\#N$ is defined by punching out a disk from M and from N and glueing them together along the common boundary; we perform this glueing in such a fashion that if M and N are oriented, $M\#N$ inherits an orientation which agrees with that on M and on N away from the disks which have been punched out.

Example 0: $\chi(M) = 0$ if $m \equiv 1 \bmod 2$.

Example 1: $\chi(S^{2n}) = 2$.

Example 2: $\chi(RP^{2n}) = 1$.

Example 3: $\chi(CP^n) = n+1$.

Example 4: $\chi(M\times N) = \chi(M)\chi(N)$.

Example 5: $\chi(M\#N) = \chi(M) + \chi(N) - \chi(S^m)$

Example 6: If $F \to M \to N$ is a finite cover, $\chi(M) = \chi(N)\cdot|F|$.

The Signature Complex:

Let M be oriented and even dimensional. The signature complex $\{\Lambda^+ - \Lambda^-, d+\delta\}$ is defined by decomposing the exterior algebra into the ± 1 eigenspaces of $c(\tau)$. Since m is even, $c(\tau)$ anti-commutes with $d+\delta$ so $(d+\delta): C^\infty(\Lambda^+(M)) \to C^\infty(\Lambda^-(M))$. Let $\mathrm{sign}(M) = \mathrm{index}\{\Lambda^+ - \Lambda^-, d+\delta\}$. Let $\Delta^\pm$ be the Laplacians on $C^\infty(\Lambda^\pm(M))$ and let $H^\pm(M) = \ker(\Delta^\pm(M))$. This gives a decomposition of the cohomology $H^*(M) = H^+(M)\oplus H^-(M)$ similar to the decomposition $H^*(M) = H^{even}(M)\oplus H^{odd}(M)$ given by the DeRham complex. If we reverse the orientation of M, the signature changes sign. If $m=2$, $\tau = iv_1 * v_2$ so

$$\Lambda^+(M) = \mathrm{span}\{1 + iv_1\cdot v_2,\ v_1 + iv_2\} \text{ and } \Lambda^-(M) = \mathrm{span}\{1 - iv_1\cdot v_2,\ v_1 - iv_2\}$$

If $m=4$, $\tau = -v_1 * v_2 * v_3 * v_4$ so

$$\Lambda^+(M) = \mathrm{span}\{(1 - v_1\cdot v_2\cdot v_3\cdot v_4)\}$$
$$\oplus\,\mathrm{span}\{v_1 - v_2\cdot v_3\cdot v_4,\ v_2 + v_1\cdot v_3\cdot v_4,\ v_3 - v_1\cdot v_2\cdot v_4,\ v_4 + v_1\cdot v_2\cdot v_3\}$$
$$\oplus\,\mathrm{span}\{v_1\cdot v_2 + v_3\cdot v_4,\ v_1\cdot v_3 - v_2\cdot v_4,\ v_1\cdot v_4 + v_2\cdot v_3\}.$$
$$\Lambda^-(M) = \mathrm{span}\{(1 + v_1\cdot v_2\cdot v_3\cdot v_4)\}$$
$$\oplus\,\mathrm{span}\{v_1 + v_2\cdot v_3\cdot v_4,\ v_2 - v_1\cdot v_3\cdot v_4,\ v_3 + v_1\cdot v_2\cdot v_4,\ v_4 - v_1\cdot v_2\cdot v_3\}$$
$$\oplus\,\mathrm{span}\{v_1\cdot v_2 - v_3\cdot v_4,\ v_1\cdot v_3 + v_2\cdot v_4,\ v_1\cdot v_4 - v_2\cdot v_3\}.$$

Since $\tau(V_1\oplus V_2) = \tau(V_1)\tau(V_2)$, the signature complex is multiplicative with respect to products;

$$(\Lambda^+(M\times N) - \Lambda^-(M\times N), d+\delta) = (\Lambda^+(M) - \Lambda^-(M), d+\delta)\otimes(\Lambda^+(N) - \Lambda^-(N), d+\delta).$$

As with the DeRham complex, it is possible to give a topological interpretation of sign(M). If $m \equiv 2 \bmod 4$, then $c(\tau)$ is purely imaginary; complex conjugation defines an isomorphism from $\Lambda^+(M)$ to $\Lambda^-(M)$ and from $H^+(M)$ to $H^-(M)$ so $\mathrm{sign}(M) = 0$. Let $m \equiv 0 \bmod 4$. If $p < 2k$, then $\theta \to \theta \pm c(\tau)\theta \in \Lambda^\pm(M)$ is an isomorphism from $H^p(M)$ to $H^\pm(M)\cap(H^p(M)\oplus H^{m-p}(M))$. Consequently, the dimension of these harmonic forms plays no role in the computation of the index since these terms cancel in pairs. We focus our attention on the middle dimension. Decompose $\Lambda^{2k} = \Lambda^{2k,+}\oplus\Lambda^{2k,-}$ and $H^{2k}(M) = H^{2k,+}(M)\oplus H^{2k,-}(M)$. Since $c(\tau) = i^{2k+(2k)(2k-1)} *_{2k}$, $c(\tau)$ and $*_{2k}$ agree in the middle dimension. There is a natural symmetric bilinear form on $H^{2k}(M)$. The index form $I(\alpha_1,\alpha_2)$ is defined by $\int_M \alpha_1\cdot\alpha_2$; this is a non-singular symmetric bilinear form. $I(-,-)$ is the evaluation of the cup product of two cohomology classes on the top dimensional cycle. so $I(-,-)$ can be defined topologically and is a homotopy invariant of M. Let $0\neq\alpha\in H^{2k}(M)$ be a harmonic 2k form. If $*\alpha = \alpha$, then $I(\alpha,\alpha) > 0$; if $*\alpha = -\alpha$, then $I(\alpha,\alpha) < 0$. This shows I is positive definite on $H^+(M)$ and negative definite on $H^-(M)$; sign(M) is the index of this quadratic form.

Example 0: $\mathrm{sign}(M)=0$ if $m\equiv 2 \bmod 4$.

Example 1: $\mathrm{sign}(S^{4n})=0$.

Example 2: $\mathrm{sign}(CP^{n})=1$.

Example 3: $\mathrm{sign}(M\times N)=\mathrm{sign}(M)\mathrm{sign}(N)$.

Example 4: $\mathrm{sign}(M\# N)=\mathrm{sign}(M)+\mathrm{sign}(N)-\mathrm{sign}(S^{m})=\mathrm{sign}(M)+\mathrm{sign}(N)$.

Example 5: If $F\to M\to N$ is a finite cover, $\mathrm{sign}(M)=\mathrm{sign}(N)\cdot|F|$.

Elliptic complexes given by Clifford modules.

Let m be even and let M be oriented. We can generalize the signature complex to the category of Clifford modules. Let E be a $\mathrm{Clif}(M)\otimes C$ module. We assume given a linear map c from $T^*(M)$ to End(E) so that $c(\omega)^2=-|\omega|$ for $\omega\in T^*(M)$. Let ∇ be a connection on E. We can covariantly differentiate the Clifford module structure. Let ∇ be the Levi-Civita connection on $\mathrm{Clif}(M)\otimes C$ and define the covariant derivative of the Clifford module structure by

$$(\nabla c)(\omega)\cdot e=\nabla\{c(\omega)\cdot e\}-c(\nabla\omega)\cdot e-c(\omega)\cdot\nabla e.$$

We choose a connection so $\nabla c=0$; we will show in section 4 such connections always exist. We postpone until that time a study of the local geometry and the curvature of such a connection. For example, if $E=\Lambda(M)\otimes C$ is the complexified exterior algebra with the canonical $\mathrm{Clif}(M)\otimes C$ module structure, then this structure is covariant constant with respect to the Levi-Civita connection.

Decompose $E=E^+\oplus E^-$ into the ± 1 eigenvalues of $c(\tau)$. Since $\nabla(\tau)=0$, ∇ restricts to connections $\nabla^\pm$ on $E^\pm$. Clifford multiplication anti-commutes with $c(\tau)$ and defines symbols $c^+:T^*(M)\to\mathrm{Hom}(E^+,E^-)$ and $c^-:T^*(M)\to\mathrm{Hom}(E^-,E^+)$. Let $A^+:C^\infty(E^+)\to C^\infty(E^-)$ and $A^-:C^\infty(E^-)\to C^\infty(E^+)$ be the operators canonically associated with $c^\pm$ by the connections $\nabla^\pm$; if $\{x_i\}$ is a local system of coordinates on M, then $A^\pm=\Sigma_\nu c(dx^\nu)\cdot\nabla^\pm_{\partial/\partial x_\nu}$ are operators of Dirac type; A^- is the adjoint of A^+ and the associated Laplacians $\Delta^\pm$ are elliptic with leading symbol $-|\xi|^2\cdot\mathrm{Id}^\pm$. Let (E^+-E^-,A) be the elliptic complex $A^+:C^\infty(E^+)\to C^\infty(E^-)$; the index is independent of the particular connection chosen. If E(M) and F(N) are Clif(M) and Clif(N) modules over M and N, we can give $E(M)\otimes F(N)$ a $\mathrm{Clif}(M\times N)$ module structure by taking into account the signs involved; this construction is multiplicative

$$((E(M)\otimes F(N))^+-(E(M)\otimes F(N))^-,A)=(E^+(M)-E^-(M),A)\otimes(E^+(N)-E^-(N),A).$$

If E is a $\mathrm{Clif}(M)\otimes C$ module over M and if W is a coefficient bundle, we give $E\otimes W$ the tensor product $\mathrm{Clif}(M)\otimes C$ module structure.

Signature complex: $\Lambda(M)$ inherits a natural Clif(M) module structure by defining $c(\omega)=\mathrm{ext}(\omega)-\mathrm{int}(\omega)$; the elliptic complex of Dirac type associated with this module structure is the signature complex. More generally, let W be a coefficient bundle over M. Let $c(\omega)\otimes 1$ define a Clif(M) module structure on $\Lambda(M)\otimes W$. This is the signature complex with coefficients in W and we denote the index by sign(M,W). In contrast to the case $W=1$, there exist examples where sign(M,W) is non-trivial if $m\equiv 2 \bmod 4$.

DeRham Complex: $c(\tau)$ preserves the decomposition of $\Lambda(M)$ into forms of even/odd degrees. This yields a 4-fold decomposition

$$\Lambda(M)=\Lambda^{even,+}(M)\oplus\Lambda^{odd,+}(M)\oplus\Lambda^{even,-}(M)\oplus\Lambda^{odd,-}(M).$$

For example if $m=2$,

$$\Lambda^{even,+}(M)=\mathrm{span}\{1+iv_1\cdot v_2\},\ \Lambda^{odd,+}(M)=\mathrm{span}\{v_1+iv_2\},$$

$$\Lambda^{even,-}(M)=\mathrm{span}\{1-iv_1\cdot v_2\},\ \Lambda^{odd,-}(M)=\mathrm{span}\{v_1-iv_2\};$$

if $m=4$,

$$\Lambda^{even,+}(M)=\mathrm{span}\{1-v_1\cdot v_2\cdot v_3\cdot v_4,\ v_1\cdot v_2+v_3\cdot v_4,\ v_1\cdot v_3-v_2\cdot v_4,\ v_1\cdot v_4+v_2\cdot v_3\},$$

$$\Lambda^{odd,+}(M)=\mathrm{span}\{v_1-v_2\cdot v_3\cdot v_4,\ v_2+v_1\cdot v_3\cdot v_4,\ v_3-v_1\cdot v_2\cdot v_4, v_4+v_1\cdot v_2\cdot v_3\},$$

$$\Lambda^{even,-}(M)=\mathrm{span}\{1+v_1\cdot v_2\cdot v_3\cdot v_4,\ v_1\cdot v_2-v_3\cdot v_4,\ v_1\cdot v_3+v_2\cdot v_4,\ v_1\cdot v_4-v_2\cdot v_3\},$$

$$\Lambda^{odd,-}(M)=\mathrm{span}\{v_1+v_2\cdot v_3\cdot v_4,\ v_2-v_1\cdot v_3\cdot v_4,\ v_3+v_1\cdot v_2\cdot v_4,\ v_4-v_1\cdot v_2\cdot v_3\}.$$

Let $E_1=\{\Lambda^{even,+}(M)\oplus\Lambda^{odd,-}(M),d+\delta\}$ and $E_2=\{\Lambda^{odd,+}(M)\oplus\Lambda^{even,-}(M),d+\delta\}$ be elliptic complexes defined by the Clifford module structure.

Theorem 3.6:

(a) $E_1+E_2=\{\Lambda^+-\Lambda^-,d+\delta\}$ *so* $\mathrm{index}(E_1)+\mathrm{index}(E_2)=\mathrm{sign}(M)$

(b) $E_1-E_2=\{\Lambda^{even}-\Lambda^{odd},d+\delta\}$ *so* $\mathrm{index}(E_1,c)-\mathrm{index}(E_2,c)=\chi(M)$.

Proof: This is immediate from the definition. We shall see in section 4 that this decomposition of the DeRham and signature complexes into finer pieces plays a crucial role in the computation of the dimension of the Yang-Mills moduli space.

Remark: As an immediate consequence we see $\mathrm{sign}(M)\equiv\chi(M) \bmod 2$.

The Dolbeault Complex

The Dolbeault complex is a complex analogue of the DeRham complex. Let M be a holomorphic manifold of complex dimension m and corresponding real dimension 2m. Let $z=(z_1,\ldots,z_m)$ be a system of local holomorphic coordinates on M. If $z_\nu=x_\nu+iy_\nu$, define:

$$dz^\nu=dx^\nu+idy^\nu,\qquad \partial/\partial z_\nu=(\partial/\partial x_\nu-i\partial/\partial y_\nu)/2,$$

$$d\bar{z}^\nu=dx^\nu-idy^\nu,\qquad \partial/\partial\bar{z}_\nu=(\partial/\partial x_\nu+i\partial/\partial y_\nu)/2,$$

$$\Lambda^{1,0}(M)=\mathrm{span}\{dz^\nu\}\subset T^*(M)\otimes C,\qquad T^{1,0}(M)=\mathrm{span}\{\partial/\partial z_\nu\}\subset T(M)\otimes C,$$

$$\Lambda^{0,1}(M)=\mathrm{span}\{d\bar{z}^\nu\}\subset T^*(M)\otimes C,\qquad T^{0,1}(M)=\mathrm{span}\{\partial/\partial\bar{z}_\nu\}\subset T(M)\otimes C.$$

The decomposition $T(M)\otimes C=T^{1,0}(M)\oplus T^{0,1}(M)$ and $T^*(M)\otimes C=\Lambda^{1,0}(M)\oplus\Lambda^{0,1}(M)$ is independent of the particular coordinate system chosen. We choose a Riemannian metric on M so that this splitting is orthogonal. Let

$$\Lambda^{p,q}(M)=\operatorname{span}\{dz^I\cdot d\bar z^J\}_{|I|=p,\,|J|=q}$$

so $\Lambda^j(M)\otimes C=\oplus_{p+q=j}\Lambda^{p,q}(M)$. Decompose exterior differentiation $d=d^{1,0}+d^{0,1}$ where $d^{1,0}:C^\infty(\Lambda^{p,q}(M))\to C^\infty(\Lambda^{p+1,q}(M))$ and $d^{0,1}:C^\infty(\Lambda^{p,q}(M))\to C^\infty(\Lambda^{p,q+1}(M))$ are defined by:

$$d^{1,0}\{\Sigma_{I,J}f_{I,J}dz^I\cdot d\bar z^J\}=\Sigma_{I,J}\ \partial f/\partial z_\nu\cdot dz^\nu\cdot dz^I\cdot d\bar z^J \text{ and}$$

$$d^{0,1}\{\Sigma_{I,J}f_{I,J}dz^I\cdot d\bar z^J\}=\Sigma_{I,J}\ \partial f/\partial\bar z_\nu\cdot d\bar z^\nu\cdot dz^I\cdot d\bar z^J.$$

f is holomorphic if and only if $d^{0,1}f=0$. $\{d^{1,0}\}^2=\{d^{0,1}\}^2=0$ and $d^{0,1}$ is the complex conjugate of $d^{0,1}$. These operators are also often denoted by $\partial=d^{1,0}$ and $\bar\partial=d^{0,1}$. If $\omega\in T^*(M)$, we can decompose $\omega=\omega_1+\bar\omega_1$ for $\omega_1\in\Lambda^{1,0}$ and $\bar\omega_1\in\Lambda^{0,1}$. More specifically, if $\omega=\Sigma_\nu\ \xi_\nu dx^\nu+\varsigma_\nu dy^\nu$, then:

$$\omega_1=\Sigma_\nu(\xi_\nu-i\varsigma_\nu)/2\cdot(dx^\nu+idy^\nu) \text{ and } \bar\omega_1=\Sigma_\nu(\xi_\nu+i\varsigma_\nu)/2\cdot(dx^\nu-idy^\nu).$$

Extend ext and int to be complex linear. Let $\delta^{1,0}$ and $\delta^{0,1}$ be the adjoints of $d^{1,0}$ and $d^{0,1}$ so $\delta=\delta^{1,0}+\delta^{0,1}$. Then

$$\sigma(d^{0,1})(\omega)=\operatorname{ext}(\bar\omega_1)=\Sigma_\nu\ (\xi_\nu+i\varsigma_\nu)/2\cdot\operatorname{ext}(dx^\nu-idy^\nu)$$

$$\sigma(\delta^{0,1})(\omega)=-\operatorname{int}(\omega_1)=-\Sigma_\nu(\xi_\nu-i\varsigma_\nu)/2\cdot\operatorname{int}(dx^\nu-idy^\nu)$$

$$\sigma(d^{0,1}+\delta^{0,1})^2(\omega)=-|\omega|^2/2.$$

Let $c_{Dol}(\omega)=\sqrt2\cdot(\operatorname{ext}(\bar\omega_1)-\operatorname{int}(\omega_1))$. Since $c_{Dol}(\omega)^2=-|\omega|^2$, this defines a $Clif(M)\otimes C$ module structure on $\Lambda^{0,*}(M)=\oplus_q\Lambda^{0,q}(M)$. Let $dx_1\cdot dy_1\cdot\cdots\cdot dx_m\cdot dy_\nu$ define the canonical orientation of M. Let $\Lambda^{0,\pm}(M)$ be the decomposition of $\Lambda^{0,*}(M)$ into the ±1 eigenvalues of $c(\tau)$ and let

$$\Lambda^{0,even}(M)=\oplus_p\Lambda^{0,2p}(M) \text{ and } \Lambda^{0,odd}(M)=\oplus_p\Lambda^{0,2p+1}(M).$$

We wish to show $\Lambda^{0,\pm}(M)=\Lambda^{0,even/odd}(M)$; as this is a purely local question we may study this question for $C^m=R^{2m}$. Suppose m=1. Let $\{v_1,v_2\}$ be an oriented orthonormal basis for R^2 so that v_1+iv_2 spans $\Lambda^{1,0}(C)$ and v_1-iv_2 spans $\Lambda^{0,1}(C)$. Then $v_1=(v_1+iv_2)/2+(v_1-iv_2)/2$ and $v_2=(v_2-iv_1)/2+(v_2+iv_1)/2$ is the decomposition of v_i into $\Lambda^{1,0}\oplus\Lambda^{0,1}$.

$$c_{Dol}(v_1)=\sqrt2\cdot\{\operatorname{ext}(v_1-iv_2)-\operatorname{int}(v_1+iv_2)\}/2,$$

$$c_{Dol}(v_2)=\sqrt2\cdot\{\operatorname{ext}(v_2+iv_1)-\operatorname{int}(v_2-iv_1)\}/2,$$

$$c_{Dol}(\tau)=i\cdot c_{Dol}(v_1)\cdot c_{Dol}(v_2)$$

$$=\{\operatorname{ext}(v_1-iv_2)-\operatorname{int}(v_1+iv_2)\}\cdot\{\operatorname{ext}(v_2+iv_1)-\operatorname{int}(v_2-iv_1)\}/2,$$

$$c_{Dol}(\tau)\{1\}=i\cdot(\operatorname{ext}(v_1-iv_2)-\operatorname{int}(v_1+iv_2))\{(v_2+iv_1)\}/2=1, \text{ and}$$

$$c_{Dol}(\tau)\{v_1-iv_2\}=i(\operatorname{ext}(v_1-iv_2)-\operatorname{int}(v_1+iv_2))\{i\}=-(v_1-iv_2).$$

This shows that $\Lambda^{0,0}(C)=\Lambda^{0,+}(C)$ and $\Lambda^{0,1}(C)=\Lambda^{0,-}(C)$. We now proceed by induction. Let $C^m=C\oplus C^{m-1}$. Then

$$\Lambda^{0,even}(C^m)=\Lambda^{0,even}(C)\otimes\Lambda^{0,even}(C^{m-1})\oplus\Lambda^{0,odd}(C)\otimes\Lambda^{0,odd}(C^{m-1})$$

$$\Lambda^{0,+}(C^m)=\Lambda^{0,+}(C)\otimes\Lambda^{0,+}(C^{m-1})\oplus\Lambda^{0,-}(C)\otimes\Lambda^{0,-}(C^{m-1})$$

$$\Lambda^{0,odd}(C^m)=\Lambda^{0,even}(C)\otimes\Lambda^{0,odd}(C^{m-1})\oplus\Lambda^{0,odd}(C)\otimes\Lambda^{0,even}(C^{m-1})$$

$$\Lambda^{0,-}(C^m)=\Lambda^{0,+}(C)\otimes\Lambda^{0,-}(C^{m-1})\oplus\Lambda^{0,-}(C)\otimes\Lambda^{0,+}(C^{m-1}).$$

This identifies the bundles of the Dolbeault complex with $\Lambda^{0,even}(M)$ and $\Lambda^{0,odd}(M)$. There are many possible natural connections on $\Lambda^{0,*}(M)$; all these connections will coincide if M is Kaehler but this is not true in general; not all of these connections are compatible with the Clifford module structure. Fortunately, the index is not sensitive to lower order perturbations so we can take the operator of the Dolbeault complex to be

$$d^{0,1}+\delta^{0,1}:C^\infty(\Lambda^{0,even}(M))\to C^\infty(\Lambda^{0,odd}(M));$$

this is the rolled up version; equivalently $\{d^{0,1}:C^\infty(\Lambda^{0,p})\to C^\infty(\Lambda^{0,p+1})\}$ defines the Dolbeault complex. The normalizing constant of $\sqrt2$ plays no role nor do lower order perturbations. The index is the *arithmetic genus* ag(M). We remark that it is not necessary to assume a holomorphic structure; this invariant can be defined in the category of almost complex manifolds. If W is a coefficient bundle, let $(\Lambda^{0,even}-\Lambda^{0,odd})\otimes W$ be a short hand notation for the Dolbeault complex;

$$\operatorname{index}\{(\Lambda^{0,even}-\Lambda^{0,odd})\otimes W\}=ag(M,W).$$

We refer to Gilkey [G, Corollary 3.5.7] for the following result which relates the arithmetic genus, the Euler characteristic, and the signature in complex dimensions 1,2 (and corresponding real dimensions 2,4).

Theorem 3.7:

(a) if m=1, then $ag(M)=\chi(M)/2$.

(b) if m=2, then $ag(M)=\{\chi(M)+sign(M)\}/4$.

Remark: We noted previously that $\chi(M)\equiv sign(M)$ mod 2; (b) provides an additional integrality result. For example, if we try to compute $ag(S^4)$, we would deduce $ag(S^4)=\{2+0\}/4$. Since this isn't an integer, S^4 does not admit an almost complex structure. Similarly $ag(CP^2\#CP^2)=\{3+3-2+2\}/4$ is not an integer so $CP^2\#CP^2$ does not admit an almost complex structure. The index theorem can be used to prove other non-existence results.

Example 1: m=1, let $g=\dim H^{0,1}(M)$. $\chi(M)=2-2g$ and $ag(M)=1-g$.

Example 2: $ag(CP^n)=1$.

Example 3: $ag(M\times N)=ag(M)\cdot ag(N)$.

Example 4: If $F \to M \to N$ is a finite cover, $ag(M) = ag(N) \cdot |F|$.

The Spin^c complex

The Spin^c complex includes the DeRham, the signature, and the Dolbeault complexes within a common framework. Let M be an even dimensional Spin^c manifold; the lift of w_2 from $H^2(M;Z_2)$ to $H^2(M;Z)$ can be thought of analogous to an orientation. Let $\Delta_c(M)$ be the bundle associated to the fundamental representation of Spin^c described in Theorem 1.3. $\Delta_c(M)$ inherits a natural $Clif(M)\otimes C$ module structure. The Spin^c complex $\{\Delta^+-\Delta^-, A\}$ is the associated elliptic complex of Dirac type $A_c^+: C^\infty(\Delta_c^+(M)) \to C^\infty(\Delta_c^-(M))$. If W is a coefficient bundle, we can twist this complex by taking coefficients in W.

If M is holomorphic, we may give M the Spin^c structure of Theorem 1.3. We refer to Gilkey [G, Lemma 3.5.4] for:

Theorem 3.8: $index\{\Delta_c^+ - \Delta_c^-, A_c\} = index\{\Lambda^{0,even} - \Lambda^{0,odd}, \bar\partial^{0,1} + \delta^{0,1}\}$

Remark: We must always choose a connection which is compatible with the Clifford module structure. If M is not Kaehler, there are many different connections so the operators do not in general agree even when adjusted by a suitable normalizing constant. The Spin^c complex is a generalization of the Dolbeault complex to non-complex manifolds.

If M is Spin, let $\Delta(M)$ be the Spin bundle; and let $\{\Delta^+-\Delta^-, A\}$ be the Spin complex $A^+: C^\infty(\Delta^+(M)) \to C^\infty(\Delta^-(M))$. We can describe the other classical elliptic complexes in terms of the Spin complex.

Theorem 3.9:

(a) *If M is holomorphic, then M is Spin if and only if there exists a square root* L_1 *of* $\Lambda^{0,m}(M)$. $index\{\Lambda^{0,even} - \Lambda^{0,odd}, \bar\partial^{0,1} + \delta^{0,1}\} = index\{(\Delta^+ - \Delta^-)\otimes L_1, A\}$

(b) *If M is Spin,* $index\{\Lambda^{even} - \Lambda^{odd}, d+\delta\} = (-1)^m\, index\{(\Delta^+ - \Delta^-)\otimes(\Delta^+ - \Delta^-), A\}$

(c) *If M is Spin,* $index\{\Lambda^+ - \Lambda^-, d+\delta\} = index\{(\Delta^+ - \Delta^-)\otimes(\Delta^+ + \Delta^-), A\}$

Remark: These isomorphisms preserve the $Clif(M)\otimes C$ module structure. In (b,c), the isomorphisms preserve the operators involved; in (a), the isomorphism preserve the operator modulo a suitable normalization and modulo lower order terms. The spin bundle Δ is the square root of the exterior algebra; $\Delta(M)\otimes\Delta(M) \cong \Lambda(M)$. If we take the tensor product of $\Delta(M)$ with itself, we permit spinors to act on both the left and the right; this gives the representation ρ. As with any square root, there is always a choice of signs. In this instance, the choice of signs manifests itself in the fact that inequivalent spin structures on M are parametrized by real line bundles; this indeterminacy does not change the index. Changing a Spin^c structure can change the index.

It is worth exploring these isomorphisms a bit more. Let $\{u,v\}$ be an oriented orthonormal basis for R^2. As left $Clif(R^2)\otimes C$ modules, we can take

$$\Delta_L^+ = span\{1 + iu*v\} \text{ and } \Delta_L^- = span\{u - iv\}$$

as representatives for $\Delta^\pm$ embedded in $Clif(R^2)\otimes C$. We must take the transpose to turn $Clif(R^2)\otimes C$ under right Clifford multiplication into a left $Clif(R^2)\otimes C$ module; thus $c(\tau)$ corresponds to right multiplication by $i\cdot v*u$ so

$$\Delta_R^+ = span\{u + iv\} \text{ and } \Delta_R^- = span\{1 + iu*v\}.$$

Then

$$\{\Delta_L^+\}*\{\Delta_R^+\} = span\{(1+iu*v)(u+iv)\} = span\{u+iv\}$$
$$\{\Delta_L+\}*\{\Delta_R^-\} = span\{(1+iu*v)(1+iu*v)\} = span\{1+iu*v\}$$
$$\{\Delta_L^-\}*\{\Delta_R^+\} = span\{(u-iv)(u+iv) = span\{1-iu*v\}$$
$$\{\Delta_L^-\}*\{\Delta_R^-\} = span\{(u-iv)*(1+iu*v)\} = span\{u-iv\}.$$

Clifford multiplication defines a vector space isomorphism between:

$$\Delta_L^+\otimes\Delta_R^+ = \Lambda^{odd,+}(M),\ \Delta_L^+\otimes\Delta_R^- = \Lambda^{even,+}(M)$$
$$\Delta_L^-\otimes\Delta_R^+ = \Lambda^{even,-}(M),\ \Delta_L^-\otimes\Delta_R^- = \Lambda^{odd,-}(M).$$

Furthermore, since the action of SO corresponds to the product of left action by $Clif(M)\otimes C$ and right action by the transpose, this isomorphism is true in terms of representations spaces. We use product formulas to conclude this decomposition holds (with suitable sign changes) in for general m. The relationship with the Dolbeault complex is derived similarly, $\sqrt{\Lambda^{0,m}}$ enters from the definition of the lifting of the transition functions from $U(m)$ to Spin$^c(2m)$.

Section 4: The index theorem

In this section, we will give a brief discussion of the heat equation proof of the Atiyah-Singer index theorem and we will discuss the formulae for the index of the classical elliptic complexes. We will discuss the Yang-Mills complex. We conclude with a statement of the Atiyah-Singer index theorem in general.

We begin by reviewing some results concerning the heat equation and refer to Gilkey [G, chapter 1] for further details. Let M be a compact Riemannian manifold of dimension m without boundary and dvol(x) be the Riemannian element of volume on M. Let $P: C^\infty(V) \to C^\infty(V)$ be a self-adjoint elliptic 2^{nd} order PDO such that $\sigma(P)(\omega)$ is negative definite for $0 \neq \omega \in T^*(M)$. For example, if P is the Laplacian, then $P = -\sum_{\mu,\nu} g^{\mu\nu}\partial^2/\partial x^\mu dx^\nu +$ lower order terms so $\sigma(P)(\omega) = -|\omega|^2 \cdot I_V$ is negative definite. Let $\{\phi_n, \lambda_n\}$ be a spectral resolution of $L^2(P)$. Since the symbol of P is negative definite, only a finite number of eigenvalues of P are negative. The fundamental solution of the heat equation e^{-tP} is an infinitely smoothing operator of trace class for $t > 0$; $Tr_{L^2}(e^{-tP}) = \sum_n e^{-t\lambda_n}$. If n is large, then $\lambda_n \geq \epsilon \cdot n^\delta$ so this series converges absolutely to an analytic function. The asymptotic behavior as $t \to 0^+$ is given as follows:

Theorem 4.1: Let P be as above. There exist local invariants $a_n(x,P)$ in the jets of the total symbol of P so that

$$Tr(e^{-tP}) \sim \sum_{n=0}^{\infty} t^{(2n-m)/2} \int_M a_n(x,P)\, dvol(x).$$

Let $A: C^\infty(V_0) \to C^\infty(V_1)$ be a 2-term elliptic complex; we can always roll up a longer elliptic complex. Let $\Delta_0 = A^*A$ and let $\Delta_1 = AA^*$ be the associated Laplacians. Let $a_n(x,A) = a_n(x,\Delta_0) - a_n(x,\Delta_1)$ and $a_n(A) = \int_M a_n(x,A)dvol(x)$.

Theorem 4.2:

(a) $Tr(e^{-t\Delta_0}) - Tr(e^{-t\Delta_1}) = index(A)$.

(b) $index(A) = 0$ if m is odd; $index(A) = a_{m/2}(A)$ if m is even.

(c) If A(t) is a smooth 1-parameter family, $index(A_t)$ is independent of t.

(d) If $F \to M_1 \to M$ is a finite smooth covering and if $A_1: C^\infty(V_1) \to C^\infty(V_2)$ is the corresponding operator over M_1, then $index(A_1) = |F| \cdot index(A)$.

(e) $sign(M_1 \# M_2) = sign(M_1) + sign(M_2)$ and $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^m)$.

Proof: Let $E(\lambda,P) = \{\phi \in L^2(V): P\phi = \lambda\phi\}$; $E(\lambda,P)$ is a finite dimensional subspace of $C^\infty(V)$. Then $Tr(e^{-tP}) = \sum_\lambda e^{-t\lambda} \cdot Dim\, E(\lambda,P)$. Since $A\Delta_0 = \Delta_1 A = AA^*A$, A defines a linear map A_λ from $E(\lambda,\Delta_0)$ to $E(\lambda,\Delta_1)$. Similarly A^* defines a linear map A_λ^* from $E(\lambda,\Delta_1)$ to $E(\lambda,\Delta_0)$. Since $A_\lambda^* A_\lambda = \lambda \cdot I_{E(\lambda,\Delta_0)}$ and $A_\lambda A_\lambda^* = \lambda \cdot I_{E(\lambda,\Delta_1)}$, $\dim E(\lambda,\Delta_0) = \dim E(\lambda,\Delta_1)$ for $\lambda \neq 0$. Therefore

$$Tr(e^{-t\Delta_0}) - Tr(e^{-t\Delta_1}) = \sum_\lambda e^{-t\lambda} \cdot \{\dim E(\lambda,\Delta_0) - \dim E(\lambda,\Delta_1)\} =$$

$$\dim E(0,\Delta_0) - \dim E(0,\Delta_1) = index(A)$$

which proves (a). We use the asymptotic series of Theorem 4.1 to compute:

$$index(A) = Tr(e^{-t\Delta_0}) - Tr(e^{-t\Delta_1}) \sim \sum_n t^{(2n-m)/2} a_n(A).$$

Since index(A) is independent of the parameter t, only the constant term in the asymptotic series is non-zero; this proves (b). If A_t is a smooth 1-parameter family of elliptic PDOs, then $a_n(A_t) = \int_M a_n(x,A_t)dvol(x)$ is a smooth function of t since $a_n(x,A_t)$ is given by a local formula. This shows $index(A_t)$ is smooth in t. Since $index(A_t)$ is integer valued, it is constant. This proves (c). Since integration is multiplicative under finite coverings, (d) follows. We form $M_1 \# M_2$ by punching out two disks; consequently we can decompose $M_1 \cup M_2$ set theoretically as $(M_1 \# M_2) \cup (S^m)$. If we choose the metrics on the removed disks to be product near the boundary, then this is also a metric decomposition. We compute the Euler characteristic or signature as the index of an elliptic complex; this is given by integrating a local formula. Integration is additive with respect to such decompositions. This shows

$$\chi(M_1 \# M_2) + \chi(S^m) = \chi(M_1) + \chi(M_2) \text{ and } sign(M_1 \# M_2) + sign(S^m) = sign(M_1) + sign(M_2);$$

(e) follows since $sign(S^m) = 0$. ∎

The local geometry of Clifford modules

Let m be even. Let $\{v_1,\dots,v_m\}$ be a local orthonormal frame for M. Let ${}^g\nabla$ be the Levi-Civita connection extended to act on tensors of all types. We adopt the Einstein convention of summing over repeated indices and ignore the usual distinction between raised and lowered indices since we are dealing with orthonormal frames. Let ${}^g\nabla(v_j) = {}^g\Gamma_{ijk} v_k \otimes v_i$. The spin bundle Δ always exists locally. The ambiguity in defining Δ globally is a Z_2 ambiguity; the global obstruction plays no role in the local theory. Δ inherits a natural connection ${}^\Delta\nabla$ called the Spin connection. The connection 1-form of ${}^\Delta\nabla$ is $(1/4) \cdot c(v_j)c(v_k) {}^g\Gamma_{ijk} \otimes v_i$.

Let (E, c_E) be a $Clif(M) \otimes \mathbf{C}$ module. A fiber metric $(,)_E$ on E is said to be compatible if $c_E(\omega)$ is unitary for $\omega \in T^*(M)$ and $|\omega| = 1$. A connection ${}^E\nabla$ on E is said to be compatible if ${}^E\nabla$ is Riemannian with respect to a compatible fiber metric $(,)_E$ and if ${}^E\nabla(c_E) = 0$.

Theorem 4.3: Let (E, c_E) be a $Clif(M) \otimes \mathbf{C}$ module.

(a) There exist compatible fiber metrics $(,)_E$.

(b) There exist compatible connections ${}^E\nabla$.

(c) Let $(,)_E$ and ${}^E\nabla$ be compatible. We can decompose $E = \Delta \otimes E_0$ locally so $c_E = c_\Delta \otimes 1$, $(,)_E = (,)_\Delta \otimes (,)_{E_0}$, and ${}^E\nabla \cong {}^\Delta\nabla \otimes 1 + 1 \otimes {}^{E_0}\nabla$.

Proof: Since the convex combination of compatible metrics or connections is compatible connection. This proves (b). Let ${}^{E}\nabla$ be compatible. The Spin connection ${}^{\Delta}\nabla\otimes 1$ on $\Delta\otimes E_0$ is compatible. Let $\Theta={}^{E}\nabla-{}^{\Delta}\nabla\otimes 1$; Θ is an endomorphism of $\Delta\otimes E_0$ which commutes with the Clif(M)$\otimes$C module structure. Consequently $\Theta=1\otimes\theta$ where $\theta\in \mathrm{End}(E_0)$. Let ${}^{E_0}\nabla$ have connection 1-form θ, then ${}^{E}\nabla\approx{}^{\Delta}\nabla\otimes 1+1\otimes{}^{E_0}\nabla$. ■

Elliptic complexes given by Clifford modules

Let (E,c) be a Clif(M)$\otimes$C module. Let (E^+-E^-,A) be the elliptic complex of Dirac type discussed in section 3. If we reverse the orientation, we interchange the roles of E^+ and E^- so the local invariants $a_n(x,A)$ of Theorem 4.2 change sign. Consequently $a_n(x,A)\in C^\infty(\Lambda^m(M))$ are m-form valued rather than scalar valued; this is a crucial point. The invariants $a_n(x,A)$ are local invariants of the derivatives of the total symbol of A. By Theorem 4.3, E is locally isomorphic to a tensor product $\Delta\otimes E_0$. Consequently $a_n(x,A)$ can be expressed in terms of the jets of the metric and the derivatives of the connection 1-form on E_0. It then follows from the analysis described in Gilkey [G] that $a_n(x,A)=0$ for $2n<m$ while $a_{m/2}(x,A)$ is a characteristic form. The normalizing constants can be determined using the method of universal examples. Let $\hat{A}$ be the A-roof genus described previously. Then $a_{m/2}(x,A)=\Sigma_{2a+4b=m}\, ch_a(E_0)\cdot\hat{A}_b$. The Chern character $ch(\Delta)$ is a well defined characteristic form even if M is not Spin; $ch_\nu(\Delta)=0$ for ν—odd. As the constant term is non-zero, $ch(\Delta)^{-1}$ is also a well defined characteristic form. For example:

$ch(\Delta)=2$	$ch(\Delta)^{-1}=(1/2)$	$(m=2)$,
$ch(\Delta)=4\cdot(1+p_1/8)$	$ch(\Delta)^{-1}=(1/4)\cdot\{1-p_1/8\}$	$(m=4)$.

Since $ch(E)=ch(\Delta)\cdot ch(E_0)$, $a_n(x,A)=\{ch(\Delta(M))^{-1}\cdot\hat{A}\cdot ch(E)\}_m$ where we take the form of top degree. This is globally defined and independent of the choices made in the decomposition of Theorem 4.3. This proves:

Theorem 4.5: Let E be a $Clif(T^*M)$ *module and let* $A: C^\infty(E^+)\to C^\infty(E^-)$.

(a) $a_n(x,A)=0$ *for* $2n<m$.

(b) $a_{m/2}(x,A)=\hat{A}\cdot ch(\Delta)^{-1}\cdot ch(E)$.

(c) $index\{E^+-E^-,A\}=\int_M \hat{A}\cdot ch(\Delta)^{-1}\cdot ch(E)$.

Remark: This shows the index is independent of the particular Clif(M)$\otimes$C module structure chosen for E.

Classical Elliptic Complexes

We could useTheorem 4.5 to compute the index of the 4 classical elliptic complexes. However it is just as easy to note that there must exist universal formulas involving the appropriate characteristic classes and then use the method of universal examples to evaluate these formulas. We refer to Gilkey [G] for the derivation of the following formulas:

DeRham Complex: $\chi(M)=\int_M e(M)$ where $e(M)$ is the Euler form;

$$e(M)=-R_{1212}/2\pi\cdot dvol \text{ if } m=2$$

$$e(M)=\epsilon(abcd)\epsilon(uvwx)R_{abuv}R_{cdwx}/128\pi^2\cdot dvol \text{ if } m=4.$$

If W is a coefficient bundle and if $\chi(M,W)$ is the index of the DeRham complex with coefficients in W, then $\chi(M,W)=\chi(M)\cdot\text{index}(W)$ so W plays an inessential role in this example. If $m=4$,

$$\chi(M^4,W)=\chi(M^4)\cdot\dim(W).$$

Signature Complex: $\text{Sign}(M)=\int_M L_k(M)$ for $m=4k$ where L_k is the Hirzebruch polynomial. $L_1(M)=p_1/3$ if $m=4$. If W is a coefficient bundle and if $\text{sign}(M,W)$ is the index of the signature complex with coefficients in W, then

$$\text{sign}(M,W)=\Sigma_{2a+4b=m}\int_M 2^a ch_a(W)\cdot L_b(M).$$

If $m=4$,

$$\text{sign}(M^4,W)=\dim(W)\text{sign}(M)+\int_M\{2c_1^2-4c_2\}(W).$$

Dolbeault Complex: $ag(M)=\int_M Td_k(M)$ for $m=2k$ where Td is the Todd polynomial. If $c(M)$ is the Chern class of the complex tangent bundle $T^{1,0}(M)$, then $Td_1=c_1/2$ and $Td_2=(c_2+c_1^2)/12$. If W is a coefficient bundle and if $ag(M,W)$ is the index of the Dolbeault complex with coefficients in W, then

$$ag(M,W)=\Sigma_{2a+2b=m}\int_M ch_a(W)\cdot Td_b(M).$$

Spin Complex: $\text{index}(\Delta^+-\Delta^-,A)=\int_M\hat{A}_k(M)$ for $m=4k$. $\hat{A}_1=-p_1/24$. If we take coefficients in a bundle W,

$$\text{index}((\Delta^+-\Delta^-)\otimes W)=\Sigma_{2a+4b=m}\int_M ch_a(W)\cdot\hat{A}_{b(M)}$$

In dimension 4, this takes the form:

$$\text{index}((\Delta^+-\Delta^-)\otimes W)=-\text{sign}(M^4)\cdot\dim(W)/8+\int_M\{c_1^2(W)-2c_2(W)\}/2.$$

In particular, if M is Spin, then $\text{sign}(M^4)\equiv 0 \mod 8$.

Yang-Mills Complex

If $m=4$, there is a natural elliptic complex which appears in Yang-Mills theory. Let $\pi:\Lambda^2(M)\to\Lambda^{2,\pm}(M)$ be projection on the ±1 eigenvalue of the Hodge $*$ operator. Let $\{(\Lambda^0\oplus\Lambda^{2,-}-\Lambda^1\},YM)$ be the complex:

$$0\to C^\infty(W)\xrightarrow{\nabla}C^\infty(\Lambda^1(M)\otimes W)\xrightarrow{\pi\nabla}C^\infty(\Lambda^{2,-}(M)\otimes W)\to 0.$$

This plays a crucial role in the description of the moduli space in the neighborhood of a self-dual connection. We have already studied this elliptic complex in another guise. We supress the role of W for the moment. Let $\{v_i\}$ be a local oriented orthonormal frame for $T^*(M)$ and let $\tau=i^2v_1*v_2*v_3*v_4$ be the normalized orientation; the ±1 eigen spaces of $c(\tau)$ on Λ^2 are the bundles $\Lambda^{2,\pm}$. Let $\Phi_\pm(\theta)=\{\theta\pm c(\tau)\theta\}$ be projection from $\Lambda(M)\otimes C$ to $\Lambda^\pm(M)$. This defines isomorphisms:

$$\Phi_\pm:\Lambda^0(M)\otimes C\to\{(\Lambda^0(M)\oplus\Lambda^4(M))\otimes C\}^\pm \text{ and } \Phi_\pm:\Lambda^1(M)\otimes C\to\Lambda^{odd,+}(M).$$

Consequently $\Lambda^0(M)\oplus\Lambda^{2,-}(M)\cong\Lambda^{even,-}(M)$ and $\Lambda^1(M)\cong\Lambda^{odd,+}(M)$. We wish to show the YM operator corresponds to $d+\delta$ under these isomorphisms. To do this, it suffices to compute on the symbol level since the 0^{th} order terms are controlled by naturality. Let $\omega\in T^*(M)$. The YM operator from $\Lambda^1(M)$ to $\Lambda^0(M)\oplus\Lambda^{2,-}(M)$ is $\delta\oplus(1-c(\tau)/2)\cdot d$ so the symbol is $-\text{int}(\omega)\oplus(1-c(\tau))/2\cdot\text{ext}(\omega)$. The corresponding operator from $C^\infty(\Lambda^1(M))$ to $C^\infty(\Lambda^{even,-}(M))$ is defined by composing with $\Phi_-:\Lambda^0\oplus\Lambda^{2,-}\to\Lambda^{even,-}$ and has symbol

$$(1-c(\tau))/2(-\text{int}(\omega)\oplus\text{ext}(\omega))=(1-c(\tau))/2\cdot c(\omega)=c(\omega)(1+c(\omega))/2.$$

The operator from $C^\infty(\Lambda^{odd,+}(M))$ to $C^\infty(\Lambda^{even,-}(M))$ is defined by composing with Φ_+^{-1}:

$$u=\{\theta+c(\tau)\theta\}/2\to c(\omega)\cdot(1+c(\tau))/2\cdot\theta=c(\omega)\cdot u.$$

This proves:

Lemma 4.6: $\text{index}\{(\Lambda^0(M)\oplus\Lambda^{2,-}(M))-\Lambda^1(M),YM\}=\text{index}\{\Lambda^{even,-}(M)-\Lambda^{odd,+}(M),d+\delta\}$.

Let W be a coefficient bundle. Let

$$E_1=\{(\Lambda^{even,+}(M)-\Lambda^{odd,-}(M))\otimes W,A\}, \text{ and } E_2=\{(\Lambda^{odd,+}(M)-\Lambda^{even,-}(M))\otimes W,A\}.$$

Then:

$$\chi(M,W)=\text{index}(E_1)-\text{index}(E_2) \text{ and } \text{sign}(M,W)=\text{index}(E_1)+\text{index}(E_2)$$

We solve ththis system of equations to see $\text{index}(E_2)=\{\chi(M,W)-\text{sign}(M,W)\}/2\}$.

Theorem 4.7: $\text{index}((\Lambda^0\oplus\Lambda^{2,-}-\Lambda^1)\otimes W,YM)=\dim(W)\cdot\{\chi(M)-\text{sign}(M)\}/2+\int_M(2c_2-c_1^2)(W)$.

Suppose that M is simply connected so $H^1(M)=0$; this shows $H^3(M)=0$ by Poincare duality. Suppose the index form on M is positive definite so

$$\chi(M)-\text{sign}(M)=\dim H^0(M)+\dim H^4(M)+\dim H^2(M)-\dim H^2(M)=2.$$

Then $\text{index}(YM,W)=\dim(W)+2c_2(W)$.

Atiyah Singer Index Theorem

The Atiyah-Singer index theorem contains these as special cases. Let $A:C^\infty(V_0)\to C^\infty(V_1)$ be an elliptic complex where A is allowed to be a pseudo-differential operator. Let $a:T^*(M)\to Hom(V_0,V_1)$ be the symbol of A; this is assumed to be invertible for $\omega\neq 0$ and homogeneous of some non-negative degree. Let S(M) be the unit sphere bundle of $T^*(M)$ and let $\Sigma(M)$ be the unit sphere bundle of $T^*(M)\oplus R$. If $(\omega,t)\in\Sigma(M)$, let

$$\Sigma(a)=\begin{pmatrix} t\cdot I_{V_0} & a^*(\omega) \\ a(\omega) & -t\cdot I_{V_1}\end{pmatrix}:\Sigma(M)\to Hom(V_0\oplus V_1,V_0\oplus V_1).$$

$\Sigma(a)$ is self adjoint and invertible. In analogy with the Bott bundle discussed in section 2, let $\Pi_+(\Sigma(a))$ be the sub-bundle of $V_0\oplus V_1$ over $\Sigma(M)$ of eigenvectors of $\Sigma(a)$ corresponding to positive eigenvalues. This bundle encodes all the topological information necessary to describe the index theorem.

Let $D_\pm(M)=\{(\omega,t):t\geq 0 \text{ or } t\leq 0\}$ be the upper and lower hemispheres of $\Sigma(M)$. Projection on V_0 gives an isomorphism between $\Pi_+(\Sigma(a))$ and V_0 over $D_+(M)$; similarly projection on V_1 gives an isomorphism between $\Pi_-(\Sigma(a))$ and V_1 over $D_-(M)$. On the equator $S(M)=D_+(M)\cap D_-(M)$, these two isomorphisms are related by the original symbol $a(\omega):S(M)\to Hom(V_0,V_1)$; thus $\Pi_+(\Sigma(a))$ is defined by the clutching data (V_0,V_1,a).

Let Todd(M) be the real Tod class of M; Todd(M)=Todd(T(M)$\otimes$C). There exists a suitable choice of orientation for the bundle $\Sigma(M)$; we refer to Atiyah-Singer [AS] for details. With this choice of orientation,

Theorem 4.8: (Index theorem) $index(V_0-V_1,A)=\int_{\Sigma(M)} Todd(M)\cdot ch(\Pi_+(\Sigma(a)))$.

Remark: There are non trivial index problems in odd dimensions since we have permitted A to be a pseudo-differential operator.

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C O L L E G E

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

(21 November - 16 December 1988)

Addendum (Page 33)
to
THE INDEX THEOREM.

P. Gilkey

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compatible, we can use a partition of unity to establish the global existence once the corresponding local existence is established. We use the construction of Theorem 1.3. Let $\{v_1,\ldots,v_m\}$ be a local oriented orthonormal frame for $T^*(M)$. Let $\alpha_\nu = i\cdot c_E(v_{2\nu-1})c_E(v_{2\nu})$ and let $E_0=\{e\in E:\alpha_\nu\cdot e=e \text{ for } 1\le\nu\le m/2\}$ be the simultaneous eigenspace. Clifford multiplication gives an isomorphism between E and $\Delta\otimes E_0$ which preserves the module structure: $c_E=c_\Delta\otimes 1$; this proves the first assertion of (c). $\Delta\subseteq \mathrm{Clif}(-)\otimes C$ inherits a natural compatible metric $(,)_\Delta$; as Δ is irreducible, $(,)_\Delta$ is unique up to scale. The tensor product of this metric with an arbitrary fiber metric on E_0 is a compatible metric on E; we patch together these metrics using a partition of unity to construct a global compatible metric on E. This proves (a). Fix a compatible metric on E. Let $\{\phi_a\}$ be an orthonormal frame for E_0 and decompose $E=\oplus_a \Delta\otimes\phi_a$. Let $S_{odd}=\{I=(i_1,\ldots):i_\nu \text{ are odd}\}$. Then $\Delta\otimes\phi_a=\mathrm{span}\{c_E(v_I)\cdot\phi_a\}_{I\in S_{odd}}$; the $\{c_E(v_I)\cdot\phi_a\}$ are unit vectors. Since $c_E(v_I)\cdot\phi_a$ and $c_E(v_J)\cdot\phi_b$ belong to different eigenspaces of the α_ν for $I\neq J$, they are orthogonal. Similarly $c_E(v_I)\cdot\phi_a$ and $c_E(v_I)\cdot\phi_b$ are orthogonal for $a\neq b$. Consequently $\{c_E(v_I)\cdot\phi_a\}_{I\in S_{odd}}$ is an orthonormal frame e for E. This proves $(,)_E=(,)_\Delta\otimes(,)_{E_0}$ and establishes the second assertion of (c).

Let ${}^E\nabla$ be a connection on E and let ${}^E\nabla\cdot e=\sum_i {}^E\Gamma_i(e)\otimes v_i$. Let $[x,y]=xy-yx$.

Sublemma 4.4: The following conditions are equivalent:

(a) ${}^E\nabla$ *is compatible*

(b) ${}^E\Gamma_i+{}^E\Gamma_i^*=0$ *for all i and* $[{}^E\Gamma_i,c_E(v_j)]=\Gamma_{ijk}\,c_E(v_k)$ *for all i,j.*

Proof: Since the frame $e=\{c_E(v_I)\cdot\phi_a\}$ is orthonormal, ${}^E\nabla$ is compatible with the metric $(,)_E$ if and only if ${}^E\Gamma_i+{}^E\Gamma_i^*=0$. Let $c_j=c_\Delta(v_j)$ so $c_E(v_j)=c_j\otimes 1$. Since these are constant matrices,

$$({}^E\nabla c_E)(v_j)\cdot e=\{{}^E\Gamma_i\cdot c_E(v_j)-c_E(\nabla_{v_i}v_j)-c_E(v_j){}^E\Gamma_i\}(e)\otimes v_i$$
$$=\{[{}^E\Gamma_i,c_E(v_j)]-\Gamma_{ijk}c_E(v_k)\}\cdot e$$

This vanishes if ${}^E\nabla$ is compatible. Conversely, if $[{}^E\Gamma_i,c_E(v_j)]-\Gamma_{ijk}c_E(v_k)=0$ for all i,j then ${}^E\nabla c_E(v_j)=0$ and by linearity $({}^E\nabla c_E)(\omega)=0$ for $\omega\in T^*(M)$. It then follows ${}^E\nabla c_E=0$ on $\mathrm{Clif}(T^*M)\otimes C$ so the connection is compatible. This proves the sublemma.

Let ${}^E\Gamma_i=(1/4)\cdot{}^g\Gamma_{ijk}c_E(v_j)c_E(v_k)$. Since

$$\{c_E(v_j)c_E(v_k)\}^*=c_E(v_k)c_E(v_j)=-c_E(v_j)c_E(v_k),$$

$({}^E\Gamma_i)^*=-{}^E\Gamma_i$ so ${}^E\nabla$ is Riemannian. We compute

$$[{}^E\Gamma_i,c_E(v_j)]=(1/4)\cdot{}^E\Gamma_{iab}[c_E(v_a)c_E(v_b),c_E(v_j)].$$

Since $[c_E(v_a)c_E(v_b),c_E(v_j)]=0$ for $j\notin\{a,b\}$ we may suppose $j=a$ or $j=b$. These two cases are symmetric so $[{}^E\Gamma_i,c_E(v_j)]=\Gamma_{ijb}c_E(v_b)$ and ${}^E\nabla$ is compatible. We patch together such connections using a partition of unity to construct a global