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COLLEGE

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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LECTURES ON DIFFERENTIAL GEOMETRY

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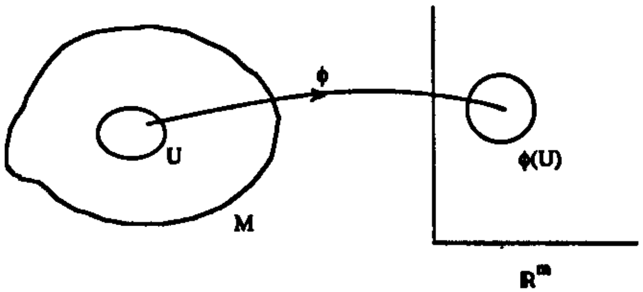
1. Manifolds.

A manifold is a topological space that it is made up from pieces of some especially nice space such as Euclidean space $\mathbb{R}^m$ which are 'glued' together by maps of a particular type. The kind of glue we shall use in this course is smooth or infinitely differentiable maps. Recall that a map $F : X \to Y$ between open subsets $\mathbb{R}^m \supseteq X$ and $\mathbb{R}^n \supseteq Y$ is smooth if the components of F have partial derivatives of all orders. We denote by DF the matrix of first derivatives:

$$DF = [\frac{\partial F^i}{\partial x^j}].$$

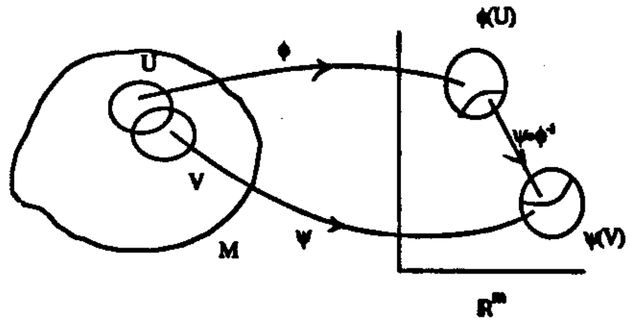
Let M be a Hausdorff topological space. A **coordinate chart** on M is an open subset U of M for which there is a homeomorphism $\phi : U \to \phi(U)$ onto an open subset of $\mathbb{R}^m$. If x is in U we call such a pair (U,ϕ) a **coordinate chart at x**.



We say that two charts (U,ϕ) and (V,ψ) are **compatible** if either $V \cap U = \emptyset$ or the **transition mapping** (i.e. the 'glue' that sticks U and V together)

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is a smooth mapping between open subsets of $\mathbb{R}^m$:



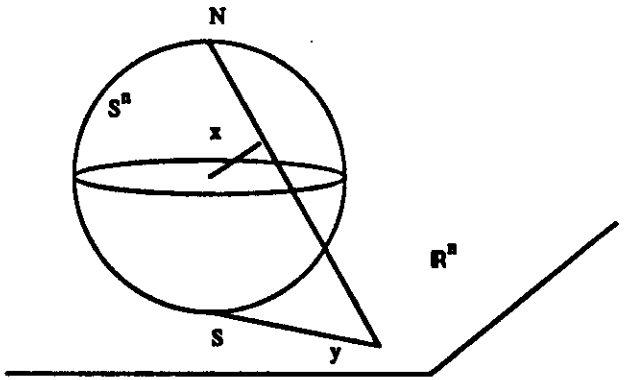
An **atlas** is a family of pairwise compatible charts whose domains cover M. For technical reasons we need the notion of a **maximal atlas** which is an atlas containing all charts compatible with its charts. Any atlas always determines a unique maximal atlas so it suffices just to give an atlas.

A **differentiable structure** on M is determined by specifying a maximal atlas. A pair consisting of a Hausdorff topological space together with a differentiable structure we shall call a **differentiable or smooth manifold**. The dimension of M is then m. We often abuse terminology by refering to the smooth manifold simply as M when it is clear which differentiable structure is meant. Note that it is only necessary to give an atlas whose charts cover M topologically as this automatically determines the maximal atlas of the differentiable structure.

Given a chart (U,ϕ) in the atlas of M, then for each x in U, we write $\phi(x) = (x^1(x),\ldots x^m(x))$ and call the x^i the **coordinate functions on U**.

Example 1.1: $\mathbb{R}^m$ is itself a smooth manifold: We can form a chart by taking the whole of $\mathbb{R}^m$ as the domain together with the identity map and then the maximal atlas containing $(\mathbb{R}^m, id)$. This is called the standard differentiable structure on $\mathbb{R}^m$.

Example 1.2. The Euclidean n-sphere S^n, the set of vectors in Euclidean space of unit length, can be viewed as a smooth manifold as follows: We form a chart by choosing a point and then stereographically projecting from that point onto the opposite tangent plane.



To cover the sphere we need only choose two such charts, say by projecting from the North and South Poles N and S. The open sets $S^n\setminus\{N\}$ and $S^n\setminus\{S\}$ are each diffeomorphic to $\mathbb{R}^n$ and one can check that the transition mapping is smooth.

Example 1.3. Any open set U of a manifold M is a smooth manifold of the same dimension: for (V,ϕ) a coordinate chart in the atlas of M with $U\cap V$ non-empty, then $(U\cap V,\phi|_{U\cap V})$ gives a coordinate chart for U. It is trivial to check that compatible charts for M give compatible charts for U and hence that U inherits an atlas from M.

Example 1.4. Given two manifolds M and N of dimensions m and n respectively, the product space $M\times N$ can be given the structure of a smooth manifold: for any pair of

coordinate charts (U,ϕ) of M and (V,ψ) of N, $(U\times V,\phi\times\psi)$ will be a coordinate chart of $M\times N$. Taking products respects compatibility of charts. The dimension of $M\times N$ is, of course, m+n.

Remark: In the definition of compatibility of charts we could have used some other class of maps, for example those which are just k times differentiable, and we would then have defined the notion of a C^k atlas, and a C^k differentiable manifold. Using just continuous maps (k = 0) yields the notion of a toplogical manifold whilst in the oposite direction we can strengthen the requirement to real analytic transitions giving real analytic or C^ω manifolds. For compatibility we denote smooth manifolds by C^∞. The manifolds of class C^k with $k = 1,\ldots,\infty,\omega$ are essentially all the same so we shall concentrate attention just on the C^∞ case.

2. Smooth functions and smooth mappings.

Once we can give a space a manifold structure, that is once we can think of it locally as $\mathbb{R}^n$, we can try to do things that we can usually carry out locally on $\mathbb{R}^n$. In particular we can differentiate maps and functions as we now show.

Let M and N be smooth manifolds of dimensions m and n respectively. If $f: M \to N$ is a continuous map, we say that it is smooth if, for any pair of coordinate charts (U,ϕ) of M and (V,ψ) of N, $\psi\circ f\circ\phi^{-1}: \phi(f^{-1}(V)\cap U) \to \mathbb{R}^n$ is smooth. Taking $N=\mathbb{R}$ gives the notion of a smooth function. We denote by $C^\infty(M)$ the set of smooth functions on M and by $C^\infty(M,N)$ the set of smooth mappings from M to N.

Example 2.1. The Cartesian coordinate functions $(x^1,\ldots,x^{n+1})$ on $\mathbb{R}^{n+1}$ restrict to smooth functions on S^n and a map $f: M \to S^n$ is smooth if and only if $x^i\circ f$ is smooth for all i.

Example 2.2. For the product of two smooth manifolds with the smooth manifold structure described in Example 1.3, the usual projections $p_1 : M \times N \to M$ and $p_2 : M \times N \to N$ are smooth.

Since a map between open subsets of $\mathbb{R}^m$ is smooth precisely when its components are smooth, it follows that a map into a smooth manifold is smooth whenever its composition with coordinate functions on the range yields smooth functions on the domain.

A bijection between two smooth manifolds whose set-theoretic inverse is also a smooth map is called a **diffeomorphism**. Two manifolds which have a diffeomorphism between them are said to be **diffeomorphic**. This is a global property as locally all manifolds look like $\mathbb{R}^n$. Since a diffeomorphism is necessarily a homeomorphism, topologically distinct spaces such as S^n and $\mathbb{R}^n$ can never be diffeomorphic.

An important feature of working on manifolds is that locally they look like euclidean space where we understand everything. In order to pass from this local situation to objects defined on the whole manifold we frequently have to make use of what are called partitions of unity. To make this notion precise we need a few preliminary ideas. If f is a smooth function on M we define its support supp(f) to be the closure of the set $\{x \in M \mid f(x) \neq 0\}$. A collection of subsets of M is said to be locally finite if any point of M has a neighbourhood which intersects at most a finite number of the subsets.

Definition: A partition of unity on a manifold M is a collection $\{\phi_\alpha\}$ of smooth functions on M such that each ϕ_α only takes values between 0 and 1, the collection of supports $\{supp(\phi_\alpha)\}$ is locally finite and for each point x in M we have

$$\sum_\alpha \phi_\alpha(x) = 1.$$

Note that because of the locally finite condition on the supports, there are only a finite number of the terms of this sum which are non-zero, so no convergence questions are involved.

If we have any open covering of M we say that a partition of unity is subordinate to the covering if for each function ϕ_α in the partition of unity there is an open set U in the covering with $U \supset supp(\phi_\alpha)$.

Theorem.2.1 If M is a smooth manifold and the underlying topological space is paracompact then any covering of M has a subordinate partition of unity.

Lemma If $M \supset U \supset C$ where U is open and C is compact then there is a smooth function f on M only taking values between 0 and 1 which is identically equal to 1 on C and $U \supset$ supp(f).

Proof (Spivak) Take any x in C and any chart (V,ψ) with the closure of V contained in U and $\psi(x)=0$. Then $\psi(V) \supset (-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times\ldots\times(-\varepsilon,\varepsilon)$ for some positive ε. If j denotes the smooth function on $\mathbb{R}$ given by

$$j(t) = \begin{cases} e^{-(x-1)^{-2}-(x+1)^{-2}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

then we set

$$f_x(y) = j(x^1(y)/\varepsilon)\ldots j(x^n(y)/\varepsilon),\ y \in V$$

and extend f_x by zero outside V to give a smooth function f_x on M. Each f_x is non-negative, and strictly positive only on a neighbourhood of x whose closure is contained in

U. The compactness of C allows us to choose a finite number of points $x_1,\dots,x_N$ of C such that those neighbourhoods cover C. Take the sum $f_{x_1}+\dots+f_{x_N}$ which has support in U. It is strictly positive at each point of C, and so bounded below by some positive number δ. If we now compose this function with a function h on ℝ which is zero on $(-\infty,0)$, increases monotonically from 0 to 1 on $[0,\delta]$ and identically equal to 1 on (δ,∞), then we have the conclusion of the Lemma. To see that such a function h exists we can take

$$h(x) = \int_0^x k(t)dt \Big/ \int_0^\delta k(t)dt$$

where

$$k(t) = \begin{cases} e^{-x^{-2} - (x-\delta)^{-2}}, & 0 < x < \delta, \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Theorem 2.1. First we use the paracompactness of M to refine the open covering to one which is locally finite, so without loss of generality we need only show that a locally finite open covering has a subordinate partition of unity. Before handling this case we deal with the case where each open set U in the covering has compact closure. In this case we can find an open subset U'of each U whose closure is contained in U and such that the U' also cover M. Since the closure of U' is a compact set we can aplly the Lemma to find a smooth function f_U, which has support in U and is equal to 1 on the closure of U'. Since the covering is locally finite, so are the supports of the f_U whilst the fact that the U' cover M forces the function

$$f = \sum_U f_U$$

to be everywhere strictly positive. It follows that the functions

$$\phi_U = f_U/f$$

give a partition of unity subordinate to the covering we started with. If the Lemma had been true for sets C which were just closed rather than compact, the argument we have just given would have dealt with the general case. That this is in fact true follows from this special case. For if $M \supset U \supset C$ with U open and C closed then for any x in C we pick a neighbourhood U_x with compact closure and then cover $M\backslash C$ by open subsets V_a with compact closure. If we now apply the first case to a locally finite open refinement $\{U_b\}$ of the open covering $\{U_x,V_a\}$ we obtain a partition of unity ϕ_b. If f is the sum of the ϕ_b where $U_x \supset U_b$ for some x, then the sum is finite in a neighbourhood of each point and so defines a smooth function. It is easy to see that f is identically 1 on C and has support in U. That completes the proof of the theorem.

3. Tangent spaces.

Although a manifold M is locally homeomorphic to $\mathbb{R}^n$, it does not have locally the structure of a vector space over ℝ. However we are going to assign at each point x in M a vector space, the space of all directions at the point x, which is isomorphic to $\mathbb{R}^n$. This is to be thought of as the "linearisation of the manifold" at the particular point and we call it the tangent space of M at x.

Fix a point x in M and let (U,ϕ) be a coordinate chart around x. Let $\gamma : (-\varepsilon,\varepsilon) \to M$ be a smooth mapping with $\gamma(0)=x$. We call γ a smooth curve at x. According to the definition of a smooth mapping $\phi(\gamma(t))$ is a smooth mapping from the interval $(-\varepsilon,\varepsilon)$ into $\mathbb{R}^n$. We can therefore consider the linearisation v of $\phi(\gamma(t))$ at 0:

$$v = \frac{d}{dt}\Big|_{t=0} \phi(\gamma(t))$$

which of course lies in $\mathbb{R}^n$. Then $\phi(\gamma(t)) = \gamma(x)+tv+O(t^2)$.

We define a relation among all curves on M at x as follows: for γ_1 and γ_2 curves at x, we say that $\gamma_1 \sim \gamma_2$ if

$$\frac{d}{dt}\Big|_{t=0} \phi(\gamma_1(t)) = \frac{d}{dt}\Big|_{t=0} \phi(\gamma_2(t))$$

It is easy to check that $\sim$ is an equivalence relation. As with everything that we shall define in terms of local coordinates on a manifold, we need to check that it is independent of the choice of coordinates. For that consider another coordinate chart (V,ψ) around the point x. Using the chain rule, we have

$$\frac{d}{dt}\Big|_{t=0} \psi(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} \psi\circ\phi^{-1}\circ\phi\circ\gamma(t) = D(\psi\circ\phi^{-1})_{\phi(x)} \frac{d}{dt}\Big|_{t=0} \phi(\gamma(t)).$$

Since $D(\psi\circ\phi^{-1})_{\phi(x)}$ is an isomorphism, the two curves are equivalent with respect to (U,ϕ) if and only if they are equivalent with respect to (U,ψ). This shows that the equivalence relation is independent of the choice of coordinates at x, and allows us to define the tangent space of M at x to be the set of equivalence classes of curves at x and denote it by TM_x. The elements of the tangent space are called tangent vectors.

To see that the tangent space can be viewed as vector space, we define a map from $\mathbb{R}^n$ onto TM_x by sending v to the equivalence class of $\phi^{-1}(\phi(x)+tv)$ and use it to define the vector space structure. One can check that this is independent of choice of chart using the linearity of $D(\psi\circ\phi^{-1})_{\phi(x)}$.

Example. The Taylor expansion of any smooth curve with respect to its parameter shows that every smooth curve $\gamma(t)$ in $\mathbb{R}^n$ is equivalent to one of the form $t \to x+tv$, so that we can identify the tangent space at any point x with $\mathbb{R}^n$ itself. More generally, any finite-dimensional real vector space V is a smooth manifold, since we can identify it with $\mathbb{R}^n$ by choosing a basis. The map which sends a vector v to the class of the curve above identifies TV_x with V in a canonical fashion preserving the linear structure.

Example. Consider the unit sphere S^n, the set of points $(x^1,\ldots,x^{n+1})$ in $\mathbb{R}^{n+1}$ with $(x^1)^2+\cdots+(x^{n+1})^2=1$. A curve $\gamma(t)$ on S^n is given by $\gamma(t)=(x^1(t),\ldots,x^{n+1}(t))$ with $x^1(t)^2+\cdots+x^{n+1}(t)^2=1$. Differentiating at 0 we have

$$2x^1(0)x^{1\prime}(0) + \cdots + 2x^{n+1}(0)x^{n+1\prime}(0) = 0.$$

That is $v=\frac{d}{dt}\Big|_{t=0}\gamma(t)$ is perpendicular to x(0). Conversely, given v in $\mathbb{R}^n$ with $v\cdot x=0$ we have that $\frac{x+tv}{\|x+tv\|}$ is a well defined unit vector for small t and that $\frac{d}{dt}\Big|_{t=0}\frac{x+tv}{\|x+tv\|}=v$.

Therefore,

$$TS^n_x = \{v\in\mathbb{R}^{n+1} : v\cdot x = 0\}.$$

It is easy now to see that the tangent vectors defined as above can be made to act on functions. For γ in some equivalence class of curves at x and for $f : U \to \mathbb{R}$ a function defined in a neighborhood of the point x, define $\gamma(0)(f)$ by:

$$\gamma(0)(f)=\frac{d}{dt}\Big|_{t=0} f(\gamma(t)).$$

For any γ representing the same tangent vector we get the same number. Indeed we have

$$\frac{d}{dt}\Big|_{t=0} f(\gamma(t))=\frac{d}{dt}\Big|_{t=0}(f\circ\phi^{-1})(\phi\gamma(t))$$

$$= \sum_{i=1}^{n} \frac{\partial(f\circ\phi^{-1})}{\partial x^i}\frac{d}{dt}\Big|_{t=0} x^i(\gamma(t))$$

which is independent of the representative.

This motivates the second definition of the tangent space at the point x. We say that X is a derivation at x if for each smooth function f defined in a neighborhood U of x we obtain a real number X(f) depending linearly on f and for any two functions f and g the Leibnitz rule holds:

$$X(fg) = X(f)g(x) + f(x)X(g).$$

It is an easy consequence of Leibnitz Rule that $X(f) = 0$ if f is a constant function.

Example. For (U,ϕ) a coordinate chart, a derivation at the point x in U is given by

$$X(f) = \frac{\partial(f\circ\phi^{-1})}{\partial x^i}(\phi(x))$$

for any smooth function f on U. We denote this derivation by

$$\frac{\partial}{\partial x^i}(x)$$

We shall not distinguish this derivation from the usual partial derivative on $\mathbb{R}^n$ since no confusion should arise. The space of derivations forms a vector space by adding and multiplying by scalars in the obvious way. To construct a basis for this vector space we need first the following result.

Lemma: Let V be a star-shaped open neighborhood of 0 in $\mathbb{R}^n$ and let f be a smooth function on V with $f(0) = 0$. Then there exist smooth functions $g_1,\ldots,g_n$ on V such that

$$g_i(0) = \frac{\partial f}{\partial x^i}(0)$$

and

$$f = \sum_{i=1}^{n} x^i g_i$$

in a neighborhood of 0.

Proof: Set

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx)\,dt$$

then

$$f(x) = f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)\,dt = \sum_{i=1}^{n}\int_0^1 \frac{\partial f}{\partial x^i}(tx)\,x^i\,dt.$$

Now, for any f defined in a neighborhood of the point x in the manifold, we apply the lemma to $f - f(x)$ and have that $f = f(x) + \sum_{i=1}^{n} x^i g_i$, where now the x^i's indicate the local coordinates that send x to 0 in $\mathbb{R}^n$. For any derivation X we then have that

$$X(f) = X(f(x)) + \sum_{i=1}^{n} X(x^i g_i) = 0 + \sum_{i=1}^{n} X(x^i) g_i(0) + \sum_{i=1}^{n} x^i(0) X(g_i) =$$

$$= \sum_{i=1}^{n} X(x^i)\frac{\partial f}{\partial x^i}(x).$$

Therefore the $\frac{\partial}{\partial x^i}$'s span the space of derivations. Since $\frac{\partial x^j}{\partial x^i} = \delta^j_i$, they are linearly independent, too. Notice this shows that the dimension of the tangent space as a vector space is the same as the dimension of M as a manifold. This we summarize as

Theorem. If M is a smooth manifold, x a point of M and $x^1,\ldots,x^n$ coordinates on some chart around x then TM_x has a basis

$$\frac{\partial}{\partial x^1}(x),\ldots,\frac{\partial}{\partial x^n}(x).$$

One can easily check that for a curve γ

$$\frac{df(\gamma(t))}{dt}\Big|_{t=0}$$

satisfies the Leibnitz rule and therefore γ defines a derivation which we denote by $\dot{\gamma}(0)$. This is the same tangent vector as we would get using the definition as equivalence classes of curves, and taking the equivalence class of γ at $\gamma(0)$. A converse is also true:

Lemma: For any derivation X at x there exists a smooth curve γ_X through x such that for any function f defined in a neighborhood of x,

$$X(f)=\frac{d}{dt}\Big|_{t=0} f(\gamma_X(t)).$$

Proof: If

$$X=\sum_{i=1}^{n} a^i(x)\frac{\partial}{\partial x^i}(x)$$

define, for sufficiently small t, the curve

$$\gamma(t)=\phi^{-1}(ta^1(x),\dots,ta^n(x)).$$

Then

$$\frac{d}{dt}\Big|_{t=0} f(\gamma(t)) = \frac{d}{dt}\Big|_{t=0}(f\circ\phi^{-1})(ta^1(x),\dots,ta^n(x))$$

$$=\sum_{i=1}^{n}\frac{\partial(f\circ\phi^{-1})}{\partial x^i}(0)a^i(x)$$

$$=X(f).$$

4. Differentials of maps and functions.

A smooth map between manifolds has a linear approximation at each point called its

Jacobian matrix or, in more invariant terms, its differential. Applied to the case where the target is the real numbers $\mathbb{R}$ this gives the notion of the differential of a smooth function which forms the basic idea in the differential calculus.

To make this more precise we can use either of our ways of thinking of tangent vectors. Perhaps the simplest is to consider the action of a smooth mapping $F: M \to N$ on a smooth curve $\gamma: (-\varepsilon,\varepsilon) \to M$ through the point x in M. Then $F\circ\gamma: (-\varepsilon,\varepsilon) \to N$ is a smooth curve through the point $F(x)$ in N. We can then think of F as defining a mapping

$$F_*: TM_x \to TN_{F(x)}$$

by

$$[\gamma] \to [F\circ\gamma],$$

where $[\cdot]$ denotes the equivalence class in the tangent space. We call F_* **the differential of F**. To see that such a mapping is well defined we have to see that it respects the equivalence class of the curve at x. If (V,ψ) and (U,ϕ) are coordinate charts on N and M respectively we have:

$$\frac{d}{dt}\Big|_{t=0}\psi(F\circ\gamma(t)) = \frac{d}{dt}\Big|_{t=0}(\psi\circ F\circ\phi^{-1}\circ\phi)(\gamma(t)) = D(\psi\circ F\circ\phi^{-1})_{\phi(x)}\frac{d}{dt}\Big|_{t=0}\psi((\gamma(t)),$$

and since the Jacobian in the last term is linear, composition with F preserves the equivalence.

Alternatively, in terms of the tangent space as derivations:

$$F_*X(f) = \gamma_{F_*X}(f) = \frac{d}{dt}\Big|_{t=0} f(\gamma_{F_*X}(t))$$

$$=\frac{d}{dt}\Big|_{t=0} f(F(\gamma_X(t))) = \frac{d}{dt}\Big|_{t=0}(f\circ F)(\gamma_X(t)) = X(F\circ f)$$

Therefore $F_*X(f) = X(F\circ f)$ and this can be viewed as an alternative definition. One easily checks that F_* is a linear mapping. As an example, we calculate

$$F_*(\frac{\partial}{\partial x^i})(f) = \frac{\partial}{\partial x^i}(f\circ F) = \frac{\partial}{\partial x^i}(f\circ F\circ\phi^{-1}) = \frac{\partial}{\partial x^i}(f\circ\psi^{-1}\circ\psi\circ F\circ\phi^{-1})$$

$$= \sum_{k=1}^{m} \frac{\partial}{\partial y^k}(f\circ\psi^{-1})\frac{\partial}{\partial x^i}(F\circ\phi^{-1})^k$$

$$= \sum_{k=1}^{m} \frac{\partial}{\partial x^i}(F\circ\phi^{-1})^k \frac{\partial f}{\partial y^k},$$

which shows that the matrix of the linear mapping F_* relative to the bases for the tangent spaces given by coordinate charts in the range and domain is given by:

$$\begin{bmatrix} \frac{\partial(\psi^1\circ F\circ\phi^{-1})}{\partial x^1} & \cdots & \frac{\partial(\psi^1\circ F\circ\phi^{-1})}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial(\psi^m\circ F\circ\phi^{-1})}{\partial x^1} & \cdots & \frac{\partial(\psi^m\circ F\circ\phi^{-1})}{\partial x^n} \end{bmatrix}$$

Example: The differential of the identity mapping Id_M at each point x is the identity Id_{T_xM}.

Consider two maps $G : L \to M$ and $F : M \to N$. If X is a tangent vector to L at x, and f a smooth function on N in a neighbourhood of F(G(x)), then

$$X(f\circ(F\circ G)) = X((f\circ F)\circ G)$$

implies

$$(F\circ G)_* = F_*\circ G_*.$$

Written in coordinates (exercise) this is the **Chain Rule.**

5. Submanifolds

Let $F : M \to N$ be a smooth map between manifolds of dimensions m and n respectively. Take coordinates x^i on a chart U around a point x of M and y^j on an open set V around F(x). As we have seen above, the n×m Jacobian matrix

$$[\frac{\partial(y^i\circ F)}{\partial x^j}]$$

represents the differential of F at x in coordinates. We call the rank of this matrix the **rank** of F at x. x is called a **regular point** for F if the rank is equal to n, the dimension of the range N. All other points of M are are called **critical points.** A point of N is called a **critical value** of F if it is the image of a critical point. All other points of N are called **regular values.** This means that a regular value is either not in the range of F at all or else the image of regular points only.

Example. If f is a smooth function on M, then x is a critical point if and only if the differential vanishes at x: df(x) = 0. A map $F : M \to \mathbb{R}^n$ has 0 as a regular value if and only if $dF^1,\ldots,dF^n$ are linearly independent at each point x where F(x) = 0.

A smooth map $F : M \to N$ is said to be an **immersion at x** in M if its differential F_* is injective on the tangent space TM_x, that is if F has rank m = dim M at x. Of course this can only happen when dim N ≥ dim M. F is called **an immersion** if it is an immersion at each point of M.

A smooth map $F : M \to N$ is called **a submersion at x** if its differential is a surjective map $TM_x \to TN_{F(x)}$. F is called **a submersion** if it is a submersion at each point of M. Clearly a map is a submersion if all its values are regular values.

The above ideas are all local in nature, so we can apply the Implicit Function Theorem and Inverse Function Theorem from differential calculus of Euclidean spaces to yield the following:

Theorem 5.1. (Implicit Function Theorem) Let $F : M \to N$ be a smooth map and x a point of M.

(1) If F is an immersion at x then there are coordinates $x^1,\dots,x^m$, on a neighbourhood U of x in M and coordinates $y^1,\dots,y^n$ on a neighbourhood of F(x) in N such that

$$y^i \circ F|_U = x^i, \ i = 1,\dots,m.$$

(2) If F is a submersion at x, then there are coordinates $x^1,\dots,x^m$, on a neighbourhood U of x in M and coordinates $y^1,\dots,y^n$ on a neighbourhood of F(x) in N such that

$$y^i \circ F|_U = x^i, \ i = 1,\dots,n.$$

(3) If F_* is an isomorphism of TM_x onto $TN_{F(x)}$ then there are neighbourhoods U of X, V of F(x) such that F is a homeomorphism of U onto V and the inverse map from V to U is also differentiable.

An **immersed submanifold** of M is a subset L of M which has its own structure as a manifold, and such that the inclusion map $L \hookrightarrow M$ is an immersion. If in addition the inclusion map is a homeomorphism of L with its image (in other words if the topology of L as a manifold agrees with the rekative topology as a subset of M), we say L is an **embedded submanifold** and call the inclusion map an embedding. An immersed submanifold is called a closed submanifold if it is a closed subset of M.

Closed submanifolds are one way of obtaining a good supply of smooth manifolds.

Proposition 5.2. If $F : M \to N$ is a smooth map and y is a regular value then $F^{-1}(y)$ has

a unique differentiable structure which makes it a closed submanifold of M of dimension m-n.

Proof Exercise.

Example. The map $\mathbb{R}^{n+1} \to \mathbb{R}$ given by

$$(x^1,\dots x^{n+1}) \to (x^1)^2 + (x^2)^2 + \dots + (x^{n+1})^2$$

has 1 as a regular value (exercise) and so the sphere S^n is a closed submanifold of $\mathbb{R}^{n+1}$. You should check that the smooth structure it gets as a closed submanifold of $\mathbb{R}^{n+1}$ agrees with that we gave it using stereographic projection.

Exercise. Construct other examples of closed submanifolds such as hyperboloids in Euclidean spaces, or equatorial spheres in spheres. In every case check that you use a regular value of your smooth map.

6. The tangent bundle

Let TM denote the disjoint union of all the tangent spaces of M. So a point in TM is a tangent vector to M at some point x in M. If X is such a tangent vector we denote by $\pi(X)$ the point x at which the vector is tangent. This gives us a map

$$\pi : TM \to M$$

which we shall call the **projection map**. We want to give TM the structure of a smooth manifold so that this map is smooth. We do this as follows: Take a chart (ϕ,U) on M, and consider $\pi^{-1}(U)$. This is simply the set of all tangent vectors to M at points of U.

We know that the coordinates $x^1,...,x^n$ give rise to a basis $\frac{\partial}{\partial x^i}(x)$ for the tangent space at each point x of U, thus if $X \in \pi^{-1}(U)$, then there are coefficents $a^i(X)$ such that

$$X = \sum_{i=1}^{n} a^i(X)\frac{\partial}{\partial x^i}(\pi(X)).$$

The map

$$X \rightarrow (a^1(X),...,a^n(X),x^1(\pi(X)),...,x^n(\pi(X))) \in \mathbb{R}^{2n}$$

gives $\pi^{-1}(U)$ the structure of a chart. It is an exercise to check that compatible charts on M give compatible charts on TM, and hence that we have made TM into a 2n-dimensional manifold. We call TM the **tangent bundle** of M. It is immediate from the way we have constructed charts that π is a smooth map. Notice that the fibres of π are the tangent spaces of M. TM is an example of a **vector bundle** about which you will learn in the course on fibre bundles.

If $F : M \rightarrow N$ is a smooth map, its differential F_* sends the tangent space to M at x into the tangent space to N at F(x). This means we have a commutative diagram

$$\begin{array}{ccc} & F_* & \\ TM & \rightarrow & TN \\ \pi\downarrow & & \downarrow\pi \\ M & \rightarrow & N \\ & F & \end{array}$$

of smooth maps.

7. Vector fields and integral curves.

We have defined the tangent space at each point of a manifold. We now wish to consider fields of such tangent vectors. These generalise to manifolds the notion of vector valued functions or flows in three dimensions. We say X is a **vector field** on M if X(x) is a vector in the tangent space TM_x for each x in M and if, for any f in $C^\infty(M)$, the function $X(f) : M \rightarrow \mathbb{R}$ that sends x to X(x)(f) is smooth.

Notice that such a vectorfield can also be thought of as a map $X : M \rightarrow TM$ with $\pi \circ X = id_M$. In the terminology of fibre bundles we say X is a **section** of TM. We leave it as an exercise to show that X is smooth as a map precisely when it is smooth in the sense of vectorfields above. We denote by $\mathcal{X}(M)$ the set of all vectorfields on M.

We have two ways of thinking of tangent vectors, either as derivatives on functions or as tangent vectors to curves. We now consider vector fields from these two points of view.

First we look at the action of tangent vectors on functions: If X is a vector field and f a smooth function, then $f \rightarrow (X(f))(x) = X(x)f$ is a derivation at x. From this it easily follows that

$$X(fg) = X(f)g + fX(g)$$

for any pair of smooth functions f and g. Thus X is a derivation of the algebra of smooth functions. The converse is also true: by evaluation at a point x of M each derivation of $C^\infty(M)$ defines a tangent vector at x. Thus the space of vector fields on M coincides with the space of derivations of $C^\infty(M)$.

Our second point of view regards tangent vectors as equivalence classes of curves. If we take a chart (U,ϕ) on M, we know that a tangent vector at a point x in U can be expressed as linear combination of $\frac{\partial}{\partial x^i}(x)$'s. Therefore, a vector field X on U can be expressed as $X(x) = \sum_{i=1}^{n} a^i(x) \frac{\partial}{\partial x^i}(x)$. One easily checks that in order for X to be smooth as defined above, the functions a^i have to be smooth.

We can now ask the question: given a vector field X in U, does there exist a curve γ such that the tangent vector to γ at any point on the curve equals the value of the vector field at that point ? In terms of equations, we want to solve

$$\frac{d}{dt}\gamma(t) = X_{\gamma(t)}$$

on the manifold. Using local coordinates in which $\gamma(t) = (x^1(t),...,x^n(t))$,this is equivalent to solving the ordinary differential equation

$$\frac{d}{dt} x^i(t) = a^i(x^1(t),...,x^n(t)), \qquad i = 1,...,n.$$

Thus a vector field on a manifold is simply an invariant way of writing a sytem of ordinary differential equations. Given an initial condition $\gamma(0)=x_0$, we know from the local theory of ordinary differential equations that a unique solution exists at least for time t close to 0. By piecing together and using the uniqueness of solutions we can find a solution γ on a maximal interval. Such a γ we call an **integral curve** for the vector field X. If for each x_0 the integral curve of X through x_0 is defined for all t in $\mathbb{R}$ we say that the vector field X is **complete**. In what follows we shall always be assuming that the vector fields we are working with are complete.

Assume now that we have a complete vector field X defined on M. Denote by $\gamma(x;t)$ the maximal integral curve. Let $\sigma_t : M \to M$ be defined by $x \to \gamma(x;t)$ for any t in $\mathbb{R}$. Clearly, $\sigma_0 = Id_M$ .Using the uniqueness of the solution of the ordinary differential

equation we see that $\sigma_{t_1} \circ \sigma_{t_2} = \sigma_{t_1+t_2}$. Furthermore, σ_t is jointly smooth in t and x. Since $\sigma_t \circ \sigma_{-t} = \sigma_0 = Id_M$, each σ_t is invertible with inverse σ_{-t}. Thus each σ_t is a diffeomorphism of M Therefore, $\{\sigma_t : t \in \mathbb{R}\}$ is a one-parameter group of diffeomorphisms of M which is called the **flow of X.**

Conversely, suppose that we are given a one-parameter group of diffeomorphisms σ_t of M. We define X(x) at the point x by:

$$X(x) = \frac{d}{dt}\Big|_{t=0} \sigma_t(x).$$

This gives an element of TM_x for each x in M and hence a vector field X on M. Clearly X is smooth, and we have:

$$\sigma_{s*}X(x) = \frac{d}{dt}\Big|_{t=0} \sigma_s(\sigma_t(x)) = \frac{d}{dt}\Big|_{t=0} \sigma_{\sigma+t}(x).$$

This is equal to both $\frac{d}{dt}\big|_{t=0} \sigma_t(\sigma_s(x)) = X(\sigma_s(x))$ and $\frac{d}{ds}\sigma_s(x)$,which shows that $t \to \sigma_t(x)$ is the integral curve for X at x and hence X is complete. We refer to X as the **infinitesimal generator** of the the one-parameter group of diffeomorphisms σ_t . The property $\sigma_{s*}X_x = X_{\sigma_s(x)}$ means that the vector field X is invariant under its own flow.

The Lie algebra of vector fields.

Recall that on $\mathbb{R}^n$ the vector fields $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$ commute on smooth functions. To measure the extent to which two vector fields X and Y on a manifold fail to commute one uses the commutator bracket when they are viewed as operators on the space of smooth fuctions: Define the **Lie bracket** of X and Y by:

$$[X,Y](f) = X(Y(f)) - Y(X(f))$$

then it is easy to see that the result is again a derivation of the smooth functions, and hence a vector field.

In terms of the flows generated by X and Y, we can describe the Lie bracket as follows: let σ_t be the flow of X and define a curve of vector fields Y^t by:

$$Y^t_x = \sigma_{-t} * Y_{\sigma_t(x)}$$

which is the image at time t of the vector field Y when acted on by the flow generated by X. Then for any f in $C^\infty(M)$

$$(\frac{d}{dt}|_{t=0} Y^t)_x(f) = \frac{d}{dt}|_{t=0} Y^t_x(f)$$

$$= \frac{d}{dt}|_{t=0} \sigma_{-t} * Y_{\sigma_t(x)}(f)$$

$$= \frac{d}{dt}|_{t=0} Y_{\sigma_t(x)}(f \circ \sigma_{-t}) =$$

$$= X_x(Y(f)) - Y_x(X(f)) = [X,Y]_x(f) .$$

This is an example of the Lie derivative - the derivative of a geometrical object along the flow of a vector field. If we denote such a Lie derivative by $\pounds_X$ then we have the identity

$$\pounds_X Y = [X,Y].$$

The basic properties of the Lie bracket are:

L1: $[X,Y] = -[Y,X]$;

L2 : $[aX+bY,Z] = a[X,Z] + b[Y,Z], \ \forall a, b \in \mathbb{R}$;

L3: $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$, (Jacobi Identity).

Any vector space over $\mathbb{R}$ with a product satisfying these three properties is called a real Lie algebra.

8. Differential forms.

Let M be a smooth manifold of dimension n. If x is in M, then the tangent space at x, TM_x is an n-dimensional real vector space whose dual space we denote by T^*M_x and call the cotangent space to M at x. Its elements are called covectors. If $x^1,...x^n$ are coordinates on a chart on M and x is a point in the chart, we have a basis $\frac{\partial}{\partial x^1}(x),\dots,\frac{\partial}{\partial x^n}(x)$ for the tangent space TM_x. We denote by $dx^1(x),...,dx^n(x)$ the dual basis for T^*M_x. The collection of all cotangent spaces we denote by T^*M. It can be made into a smooth manifold called the cotangent bundle of M by a method analogous to that used for TM, using the basis described above. As before we have a smooth map $\pi : T^*M \to M$ called the projection map.

A smooth section of this bundle is called a smooth 1-form on M and we denote the space of smooth 1-forms by $\Omega^1(M)$. Thus a 1-form α is the smoothly varying choice of a linear form on the tangent space at each point of M. If $x^1,...x^n$ are coordinates on a chart on M, we can express α in terms of the basis $dx^1(x),...,dx^n(x)$ for T^*M_x:

$$\alpha(x) = \sum_i \alpha_i(x)\, dx^i(x).$$

The smoothness of α is then equivalent to the smoothness of the functions α_i in every chart.

If X is a vector field on M, then for each x in M we can evaluate the linear form $\alpha(x)$ on the tangent vector X(x) to give a number $\alpha(x)(X(x))$ which we denote by $\alpha(X)(x)$. Thus we obtain a function $\alpha(X)$ on M. We have defined a natural bilinear pairing

$$\Omega^1(M)\times X(M) \to C^\infty(M)$$

which clearly has the property that $\alpha(fX) = f\alpha(X)$ for any smooth function f. In fact this is an alternative way of defining vector fields as the next theorem shows

Theorem. 8.1. Every map $X(M) \to C^\infty(M)$ which is linear over $C^\infty(M)$ is given by pointwise evaluation of a 1-form on vectorfields.

Proof Given such a function-linear map A, take a tangent vector X at x and extend it to a vectorfield X' on M. Let $\alpha(x)(X) = A(X')(x)$. If we show this is independent of the extension of X to X' we shall be done as it clearly defines a linear form on the tangent space at x for each x in M. It is enough to show that if X is a vectorfield with a zero at x, then A(X) vanishes at x. Assume first that X vanishes on a neighbourhood U of x, and choose a smooth function f on M which vanishes at x and is identically equal to 1 outside an open set with closure in U. Then $X = fX$, so $A(X) = A(fX) = fA(X)$ vanishes at x. In fact this shows that A(X) vanishes on U. This now allows us to assume that the vectorfield X we are considering vanishes at x, and outside some coordinate neighbourhood U. If $X = \sum_i a^i \frac{\partial}{\partial x^i}$ on U then we can extend the derivatives by zero from some smaller open set and so find globally define vector fields X_i such that X and $\sum_i a^i X_i$ agree on a neighbourhood of x. It follows that A(X) and $\sum_i a^i A(X_i)$ agree at x. But the a^i vanish at x, proving the Theorem.

This theorem allows us to view 1-forms in two different ways: pointwise they are sections of the cotangent bundle, whilst at the same time they are linear functionals on the space of vector fields. We shall choose whichever viewpoint is the most convenient in a particular situation.

We can multiply 1-forms by functions

$$(f\alpha)(X) = f(\alpha(X))$$

since the right-hand-side of this equation is function-linear.

If f is a smooth function on M then the map

$$X \to X(f)$$

is clearly linear in X over the smooth functions and so defines a 1-form which we denote by df:

$$df(X) = X(f),$$

and call the **differential** df of f. Obviously

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

with respect to coordinates.

If we introduce the notation $\Omega^0(M)$ for $C^\infty(M)$ then we have defined a map

$$d : \Omega^0(M) \to \Omega^1(M)$$

with the property

$$d(fg) = (df)g + f(dg).$$

Our next objective is to generalise both the spaces and the map d.

We denote by $\Omega^p(M)$ the space of alternating p-linear maps (linear over $C^\infty(M)$) from vector fields to functions and call the elements of $\Omega^p(M)$ the **p-forms** on M. Thus each p-form α evaluates on p vectorfields $X_1,\ldots,X_p$ to give a function $\alpha(X_1,\ldots,X_p)$ which satisfies:

$$\alpha(X_1,\ldots,X_i,\ldots,X_j,\ldots,X_p) = -\alpha(X_1,\ldots,X_j,\ldots,X_i,\ldots,X_p),\ \forall\, i, j;$$
$$\alpha(X_1,\ldots,fX_i,\ldots,X_p) = f\alpha(X_1,\ldots,X_i,\ldots,X_p)\ \forall\, i, f.$$

We have an analogue of theorem 8.1, namely that if we take the pth exterior powers of each cotangent space $\Lambda^p T^*M_x$ (that is: the alternating p-linear forms on TM_x) then we can form these into a bundle $\Lambda^p T^*M$ whose space of sections is $\Omega^p(M)$.

Elements of $\Omega^p(M)$ can be multiplied by functions as before, but they also admit a more general multiplication called **exterior multiplication** which is defined as follows: given a p-form α and a q-form β then we take p+q vectorfields $X_1,\ldots,X_{p+q}$ and set

$$(\alpha\wedge\beta)(X_1,\dots,X_{p+q}) = \frac{1}{p!q!}\sum_{\sigma\in S_{p+q}} \alpha(X_{\sigma(1)},\dots,X_{\sigma(p)})\beta(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})$$

This is obviously linear over the functions and alternating so yields a p+q form $\alpha\wedge\beta$. We thus have defined the exterior multiplication

$$\Omega^p(M)\times\Omega^q(M) \to \Omega^{p+q}(M).$$

The same formula applies to each fibre also, and it does not matter whether we multiply forms pointwise or as functionals on the vector fields.

If we take coordinates $x^1,\dots,x^n$ then we have 1-forms $dx^1,\dots,dx^n$ which form a basis for the cotangent space at each point. It follows that at the point x we get a basis for the p-covectors at x from $dx^{i_1}(x)\wedge\dots\wedge dx^{i_p}(x)$ for $i_1<\dots<i_p$ which for convenience we denote by $(dx^{i_1}\wedge\dots\wedge dx^{i_p})(x)$. This means that if α is a p-form then on a chart we can expand it as

$$\alpha = \sum_{i_1<\dots<i_p} \alpha_{i_1\dots i_p}\, dx^{i_1}\wedge\dots\wedge dx^{i_p}$$

where $\alpha_{i_1\dots i_p}$ is a smooth function on the chart. We can use this local coordinate expression to extend the definition of d to all p-forms by setting

$$d\alpha = \sum_{i_1<\dots<i_p} d\alpha_{i_1\dots i_p}\wedge dx^{i_1}\wedge\dots\wedge dx^{i_p}.$$

The only problem with this local definition is that it is far from obvious that $d\alpha$ is globally defined. This is most easily seen by giving an alternative global formula for $d\alpha$ and checking the two formulas agree in coordinates. The checking is left to the reader. The global formula is as follows: If α is a p-form then $d\alpha$ is the p+1-form which, for p+1 vectorfields $X_0,\dots,X_p$, is given by

$$(d\alpha)(X_0,\dots,X_p) = \sum_{k=0}^{p} (-1)^k X_k(\alpha(X_0,\dots,\widehat{X_k},\dots,X_p))$$

$$+ \sum_{j<k} (-1)^{j+k}\alpha([X_j,X_k],X_0,\dots,\widehat{X_j},\dots,\widehat{X_k},\dots,X_p),$$

where a hat over an argument indicates it is to be omitted.

It is easy to see from the local formula that d satisfies

$$d(d\alpha) = 0$$

and

$$d(\alpha\wedge\beta) = (d\alpha)\wedge\beta + (-1)^p\alpha\wedge(d\beta)$$

for any p-form α and q-form β. This last property which generalises Leibnitz' Rule to p-forms we refer to as the **graded derivation** property (graded because of the presence of the sign in the second term).

d is in fact completely determined by the properties $d^2 = 0$, being a graded derivation of exterior multiplication and its normalization $df(X) = X(f)$.

We have produced a sequence of maps

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M)$$

with the property that the composition of two successive maps is zero. This is an example of a **complex** called the **de Rham complex** which we have associated to the smooth manifold. When we have such a complex we denote by

$$Z^p(M) = \{\, \alpha \in \Omega^p(M) : d\alpha = 0 \,\}$$ - the closed forms;

$$B^p(M) = \{\, \alpha \in \Omega^p(M) : \alpha = d\beta,\ \beta \in \Omega^{p-1}(M) \,\}$$ - the exact forms.

Then $Z^p(M) \supset B^p(M)$ since $d^2 = 0$ and so we can form the quotient space

$$H^p(M) = Z^p(M) / B^p(M)$$

called the **pth de Rham cohomology group of M**. This appears to be an algebraic object associated to the smooth structure of M, but it is a theorem of de Rham that in fact these groups depend only on the underlying topological space, and in fact coincide with the real Cech cohomology groups of M.

We now consider the behaviour of forms with respect to smooth maps $F : M \to N$. For 0-forms we can simply compose functions f on N to yield functions $f \circ F$ on M which we denote by F^*f. This gives us a map

$$F^* : \Omega^0(N) \to \Omega^0(M)$$

which we now generalise to arbitrary p-forms.

Take a p-form α on N and p vectorfields $X_1,\dots,X_p$ and set

$$(F^*\alpha)(X_1,\dots,X_p)(x) = \alpha(F(x))(F_*X_1(x),\dots,F_*X_p(x)).$$

The result is clearly alternating and p-linear over $C^\infty(M)$ giving

$$F^* : \Omega^p(N) \to \Omega^p(M)$$

for each p, and called the pull-back of forms. It is easy to check that $(F \circ G)^* = G^* \circ F^*$ and that this operation commutes with the previous ones:

$$F^*(\alpha\wedge\beta) = F^*(\alpha)\wedge F^*(\beta),\ F^*(d\alpha) = d(F^*\alpha).$$

Note that this means that F^* sends closed forms to closed forms and exact forms to exact forms, and hence there is an induced map

$$F^* : H^p(N) \to H^p(M),$$

with $(F \circ G)^* = G^* \circ F^*$ on cohomology. In particular any diffeomorphism induces an isomorphism on the de Rham chomology groups.

A particular case is when $M = N$ and we have a 1-parameter group of diffeomorphisms σ_t generated by a vectorfield X. We can pull-back a form α by σ_t and differentiate the resulting curve of forms. We call this the Lie derivative of α with respect to X denoted by $\mathcal{L}_X\alpha$:

$$\mathcal{L}_X\alpha = \frac{d}{dt}\sigma_{-t}{}^*\alpha\,|_0.$$

An easy calculation gives the following formula

$$(\mathcal{L}_X\alpha)(X_1,\dots,X_p) = X(\alpha(X_1,\dots,X_p)) - \sum_k \alpha(X_1,\dots,[X,X_k],\dots,X_p)$$

which makes sense for all vectorfields, not just those which generate global 1-parameter groups. In fact since we differentiate at t=0, we only need a local group for the first definition. It follows from the first definition that $\mathcal{L}_X$ is a derivation of the exterior multiplication:

$$\mathcal{L}_X(\alpha\wedge\beta) = \mathcal{L}_X(\alpha)\wedge\beta + \alpha\wedge(\mathcal{L}_X\beta)$$

(no sign!).

Another way in which a vectorfield X can act on forms which does not involve differentiation at all is interior product i(X) which produces a (p-1)-form from a p-form:

$$(i(X)\alpha)(X_1,\dots,X_{p-1}) = \alpha(X,X_1,\dots,X_{p-1}).$$

This is well-defined, is again a graded derivation of the exterior multiplication and is defined for consistency to be zero on functions.

There is a relationship between the three derivations d, $\mathcal{L}_X$, and i(X) known as Cartan's Identity which follows easily from the definitions:

$$\mathcal{L}_X = d \circ i(X) + i(X) \circ d.$$

Let us use this identity to determine the de Rham chomology of $\mathbb{R}^n$.

Theorem If p>0 then a p-form α on $\mathbb{R}^n$ is closed if and only if it is exact.

Proof We know already that exact implies closed, so it remains to show the converse. For this we consider the 1-parameter group of diffeomorphisms σ_t given by

$$\sigma_t(x) = e^{-t}x.$$

The generator is the vector field X given by

$$X = \sum_i x^i \frac{\partial}{\partial x^i}.$$

Since we have α closed, Cartan's identity implies that

$$\mathcal{L}_X\alpha = d(i(X)\alpha).$$

But then

$$\sigma_s^*\mathcal{L}_X\alpha = \sigma_s^*\frac{d}{dt}\sigma_{-t}^*\alpha|_{t=0}$$

$$= -\frac{d}{ds}\sigma_s^*\alpha.$$

If we now make the substitution $t = e^{-s}$, we obtain the curve of maps

$$\phi_t(x) = tx$$

which have the property that ϕ_0 is the constant map 0 and ϕ_1 is the identity map. In terms of ϕ_t we have

$$t\frac{d}{dt}\phi_t^*\alpha = d\phi_t^*i(X)\alpha$$

which makes sense even for t=0. The term $\phi_t^*i(X)\alpha$ vanishes for t=0 since ϕ_0 is the constant map, consequently $t^{-1}\phi_t^*i(X)\alpha$ exists and is continuous on [0,1]. If we divide across by t and then integrate from 0 to 1 we obtain

$$\alpha = \phi_1^*\alpha - \phi_0^*\alpha = \int_0^1 \frac{d}{dt}\phi_t^*\alpha\, dt = d\int_0^1 t^{-1}\phi_t^*i(X)\alpha\, dt$$

which shows that α is exact as required.

Remark. We should really generalise Cartan's Identity to curves of maps rather than groups of diffeomorphisms, and then we would have an easier version of the above proof by applying the generalization directly to the curve ϕ_t. Exercise: work out this more general version.

Note that $H^0(M)$ consists of the functions f with df = 0, and if M is connected this means that $H^0(M) = \mathbb{R}$. Hence we have the

Corollary $H^p(\mathbb{R}^n) = 0$ for p>0 and $H^0(\mathbb{R}^n) = \mathbb{R}$.

This result has an important local application on any manifold:

Theorem (Poincare Lemma)

If α is a closed p-form (p>0) on a manifold M and x is any point of M then there is a neighbourhood U of x and a (p-1)-form β on U with $\alpha|_U = d\beta$.

Proof Any point x in an n-dimensional manifold M has a neighbourhood U diffeomorphic to $\mathbb{R}^n$, and so the cohomology groups of U are the same as those of $\mathbb{R}^n$, the result follows from the previous Corollary.

9. Integration of Forms on Orientable Manifolds.

After differentiation , we intend to show how integration, too, has its natural generalisation on a manifold. It turns out that this is not as straight forward as diferentiation: to integrate differential forms we need to impose an extra condition on the manifold, that of orientation. It is only after we have defined Riemannian structures that we are able to integrate functions.

We say that the manifold M is **orientable** ifthere exists an atlas such that for any two coordinate charts (U,x) and (V,y) we have $\det\left(\frac{\partial y^j}{\partial x^i}\right)$ positive. Having chosen such an atlas on an orientable manifold, we say that the manifold is **oriented**.

Let us cover an oriented manifold by oriented charts U_i and take a subordinate partition of unity ϕ_i. Consider the n-form

$$\omega = \sum_i \phi_i \, dx_i^1 \wedge \ldots \wedge dx_i^n$$

which clearly is globally defined, and without zeros on M. Moreover, if $X_1,\ldots,X_n$ is any oriented basis for the tangent space at a point x, $\omega(x)(X_1,\ldots,X_n)$ is positive. The existence of such an n-form is in fact characteristic of orientability, for if ω is a nowhere vanishing n-form then we say an ordered basis $X_1,\ldots,X_n$ of the tangent space at a point x is positively (resp. negatively) oriented if $\omega_x(X_1,\ldots,X_n)$ is positive (resp. negative). This applies in particular to the coordinate derivatives defined by charts, and so we construct an oriented atlas. Choosing a zero-free n-form is obviously a slightly stronger condition than choosing an orientation.

Now let ω be a top degree form on the n-dimensional manifold M with compact support contained in a coordinate chart (U,x). Then in terms of the local coordinates we have $\omega = f \, dx^1 \wedge \ldots \wedge dx^n$ for $f: U \to \mathbb{R}$. Define

$$\int_M \omega = \int_{x(U)} (f \circ x^{-1}) \, dx^1 \ldots dx^n,$$

where now $dx_1 \ldots dx_n$ denotes the usual Lebesque measure on $\mathbb{R}^n$. If the support of ω is in another coordinate chart (V,y), too, then $\omega = g \, dy^1 \wedge \ldots \wedge dy^n$ for $g: V \to \mathbb{R}$ with respect to the y local coordinates. Since

$$\det\left(\frac{\partial y^j}{\partial x^i}\right) dx^1 \wedge \ldots \wedge dx^n = dy^1 \wedge \ldots \wedge dy^n$$

Then

$$g(y(x)) \det\left(\frac{\partial y^j}{\partial x^i}\right) dx^1 \wedge \ldots \wedge dx^n = g(y) dy^1 \wedge \ldots \wedge dy^n.$$

Hence

$$g(y(x)) \det\left(\frac{\partial y^j}{\partial x^i}\right) = f(x).$$

The usual change of coordinates formula for the integration on $\mathbb{R}^n$ gives:

$$\int_{x(U)} g(y(x)) \left| \det\left(\frac{\partial y^j}{\partial x^i}\right) \right| dx^1 \ldots dx^n = \int_{y(V)} g(y) dy^1 \ldots dy^n.$$

It is exactly here that we use the orientability assumption. For (U,x) and (V,y) in the oriented atlas we have that the sign of the determinant in the first integral is always positive and therefore this integral is equal to:

$$\int_{x(U)} g(y(x)) \det\left(\frac{\partial y^j}{\partial x^i}\right) dx^1 \ldots dx^n = \int_{x(U)} f(x) \, dx^1 \ldots dx^n .$$

The integral of ω is then well defined.

To define the integral for an arbitrary form, we use a partition of unity subordinate to a covering (U_i,x_i), say ϕ_i. Define then

$$\int_M \omega = \sum_i \int_{U_i} \phi_i \omega .$$

To see that this is independent of the chosen covering, notice that for any other choice of covering (V_j,y_j) and partition of unity ψ_j subordinate to it, $\phi_i \psi_j$ is a partition of unity suboordinate to the cover $U_i \cap V_j$. We then have

$$\sum_i \int_{U_i} \phi_i \omega = \sum_{i,j} \int_{U_i \cap V_j} \phi_i \psi_j \omega = \sum_j \int_{V_j} \psi_j \omega .$$

This completes the procedure of integrating a top degree form on an orientable manifold.

10. Manifolds with boundary and the theorem of Stokes.

An n-dimensional manifold with boundary M is a space on which we have coordinate homeomorphisms $\phi : U \to \phi(U)$ where $\phi(U)$ is either an open set in $\mathbb{R}^n$ (as before) or the upper half region of $\mathbb{R}^n$, $\mathbb{R}^n_+ = \{(x^1,...,x^n) : x^n \geq 0\}$. The compatiblity conditions for the coordinate functions are the same as before. The boundary ∂M of M is the set of all the points in M that are mapped to the boundary $\mathbb{R}^{n-1}$ of $\mathbb{R}^n_+$ and these form an (n-1)-dimensional manifold. If M is C^k then ∂M is C^k, too, for $0 \leq k \leq \omega$. Notice that if $d\alpha$ is a top degree form on the manifold with boundary M then the restriction of α is a top degree form on the boundary ∂M. In this situation we have the following

Theorem: (Stokes Theorem) $\displaystyle\int_M d\omega = \int_{\partial M} \omega$

11. Riemannian metrics

We shall now try to impose on a manifold the notion of distance, in analogy with the Euclidean distance on $\mathbb{R}^n$. For that we shall use inner products and therefore the linearisation of the manifold, the tangent spaces. In addition we shall ask that the whole

procedure is done in a smooth way.

Definition: We say that g is a Riemannian metric on the smooth manifold M if

a) g_x is an inner product on TM_x for all x in M and

b) for any two vector fields X and Y on M the function $x \to g_x(X_x, Y_x)$ is smooth on M.

A manifold M equipped with a Riemannian metric is called a **Riemannian manifold.**

Example 11.1: The manifold $\mathbb{R}^n$ has at each point tangent space $\mathbb{R}^n$ with the usual innner product. With this structure $\mathbb{R}^n$ is a Riemannian manifold.

Example 11.2: The manifold $O(n) = \{ A \text{ in } GL(n): AA^t = I \}$ is Riemannian: For the identity element in O(n) we have the tangent space given by $T_eO(n) = \{A : A + A^t = 0\}$. On this space one can check that the following formula defines an inner product $\langle A , B \rangle = -\operatorname{trace} AB^t$.

To define the inner product for each tangent space, we can use the (smooth) translation mappings L_g to carry around the inner product from the tangent space at the identity to the tangent space at g.

Theorem 11.1 A smooth manifold admits a Riemannian structure if and only if it is paracompact as a topological space.

Sketch Proof: If the manifold is paracompact we can use a partition of unity to build up a Riemannian metric from the obvious local ones. Conversly, if the manifold has a Riemannian metric we can show (using geodesics, see below) that the manifold itself is a metric space and therefore paracompact.

A linear connection on a manifold M is a mapping

$$\nabla: X(M)\times X(M) \to X(M)$$
$$(X,Y) \to \nabla_X Y$$

such that :

1) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$;
2) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$;
3) $\nabla_{fX} Z = f\nabla_X Z$;
4) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

We call ∇_X the covariant derivative of Y in the direction of X. This is a kind of directional derivative. Given local coordinates $(x^1,...,x^n)$ around a point of the manifold, we can calculate everything with respect to the basis $\frac{\partial}{\partial x^i}$.

Since $\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}$ is a vector field there exist functions Γ^k_{ij} such that

$$\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} = \Sigma\,\Gamma^k_{ij}\frac{\partial}{\partial x^k}.$$

We refer to the Γ^k_{ij} 's as the Christoffel symbols for the connection ∇ with respect to the coordinates x^i.

Example: For two vector fields $X = \Sigma X_i \partial x^i$ and $Y = \Sigma Y_i \frac{\partial}{\partial x^i}$ on $\mathbb{R}^n$, define $\nabla_X Y$ by $\nabla_X Y = \Sigma X(Y_i)\frac{\partial}{\partial x^i}$. One verifies that this gives a connection on the

vector fields on $\mathbb{R}^n$.

Let $\gamma:(a,b) \to M$ be a smooth curve in M and $\dot\gamma = \frac{d\gamma}{dt} = \Sigma\frac{\partial\gamma^j}{\partial t}\frac{\partial}{\partial x^i}$ be the velocity vectorfield. We say that a vector field X is parallel along γ if $\nabla_{\dot\gamma} X = 0$. A curve γ is a geodesic if $\nabla_{\dot\gamma}\dot\gamma = 0$.

Theorem: (Parallel transport) Let $\gamma : [a , b] \to M$ be a curve in M. For any tangent vector v at $\gamma(a)$ there exists a unique vector field X parallel along γ with $X_{\gamma(a)} = v$.

Proof: Let U be a coordinate chart around $\gamma(a)$ with coordinates x^i and Christoffel symbols Γ^k_{ij}. Without loss of generality, we can assume that the whole curve γ lies in U, otherwise apply what follows on pieces of it. For a vector field X(t) $= \Sigma X_i(t)\frac{\partial}{\partial x^i}$ along γ to be parallel to γ we must have that $\nabla_{\dot\gamma} X = 0$. This equation is equivalent to

$$\frac{dX_k}{dt} + \sum_{ij} X_i\frac{d\gamma^j}{dt}\Gamma^k_{ij} = 0$$

for all k and for all t. The theory of ordinary differential equations with the initial condition $X(\gamma(a)) = v$ gives the existence and the uniqueness of a solution. Since the initial data is smooth, the solutions X_i will be smooth, too.

Notice that the theorem above gives a mapping from the tangent space at $\gamma(a)$ to the tangent space at $\gamma(t)$ for all t in the interval [a,b]. The linearity of the equation that gives the solutions guarantees that this mapping is linear. We call this linear transformation the parallel transport of v along the curve γ to $\gamma(t)$.

Theorem: (Existence of geodesics.) For any point x in M and any vector v in the tangent space at x there exists a positive number r and a unique geodesic γ defined on $(-r, r)$ with $\gamma(0) = x$ and $\gamma(0) = v$.

Proof: Arguing as in the proof of the previous theorem, the problem is equivalent to finding solutions to the system

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0$$

Once again, the theory of ordinary differential equations provides us with everything asserted in the statement of the theorem.

Given two vector fields X and Y we have by now two ways of creating a new one: the Lie bracket and the covariant differentiation with respect to a connection. To compare the two results, we introduce the notion of the **torsion.** Given a linear connection ∇, define:

$$\mathrm{Tor}_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

We say that the linear connection ∇ **is symmetric or torsion free** if $\mathrm{Tor}_\nabla = 0$. One easily checks the following properties:

1) $\mathrm{Tor}(X,Y) = -\mathrm{Tor}(Y,X)$
2) $\mathrm{Tor}(X+Y,Z) = \mathrm{Tor}(X,Z) + \mathrm{Tor}(Y,Z)$
3) $\mathrm{Tor}(fX,Y) = f\mathrm{Tor}(X,Y)$

As we have often seen, these properties are enough to show that $\mathrm{Tor}(X,Y)_x$ depends only on X_x and Y_x.

Furthermore we define the **curvature** of ∇ to be

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

This is to be understood as follows: for any pair of vector fields X and Y we define a linear transformation on the space of vector fields which on a vector field Z acts by the formula given above. Then R has the following properties:

1) $R(X,Y)Z = -R(Y,X)Z$;
2) $R(fX,Y)Z = fR(X,Y)Z$;
3) $R(X,Y)fZ = fR(X,Y)Z$;
4) Is additive with respect to X,Y and Z.

12. The Levi - Civita connection.

The following theorem provides a link between the metric and the linear connections on the manifold

The Fundamental Theorem of Riemannian Geometry: For any Riemannian metric $g = \langle .,. \rangle$ on the manifold M there exists a unique linear connection ∇ with the following properties:

1) $\mathrm{Tor}_\nabla = 0$,
2) $X(\langle Y,Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

Proof: The following calculation using (1) and (2) alternately establishes uniqueness:

$$\begin{aligned}
\langle \nabla_X Y, Z\rangle &= \langle \nabla_Y X + [X,Y], Z\rangle \\
&= Y(\langle X,Z\rangle) - \langle X, \nabla_Y Z\rangle + \langle [X,Y], Z\rangle \\
&= Y(\langle X,Z\rangle) - \langle X, \nabla_Z Y + [Y,Z]\rangle + \langle [X,Y], Z\rangle \\
&= Y(\langle X,Z\rangle) - Z(\langle X,Y\rangle) + \langle \nabla_Z X, Y\rangle - \langle X, [Y,Z]\rangle + \langle [X,Y], Z\rangle \\
&= Y(\langle X,Z\rangle) - Z(\langle X,Y\rangle) + \langle \nabla_X Z + [X,Z], Y\rangle - \langle X, [Y,Z]\rangle + \langle [X,Y], Z\rangle \\
&= X(\langle Z,Y\rangle) + Y(\langle X,Z\rangle) - Z(\langle X,Y\rangle)
\end{aligned}$$

$$+ <[X,Z],Y> - <X,[Y,Z]> + <[X,Y],Z> - <Z,\nabla_X Y>.$$

Since we have ended up where we started, we can now solve to give

$$<\nabla_X Y,Z> = \frac{1}{2}(X(<Z,Y>) + Y(<X,Z>) - Z(<X,Y>)$$
$$+ <[X,Z],Y> - <X,[Y,Z]> + <[X,Y],Z>)$$

Not only does this formula prove uniqueness, it also provides us with an existence argument, for we can check this formula is function linear in Z, so defines an operation of the vector field X on Y, and then check this satisfies the identities for a covariant derivative. This is completely routine, so left to the reader.

In a chart with coordinates ($x^1,\ldots,x^n$) with the corresponding basis for vector fields

$\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}$ our formula for the connection can be solved to give the Christoffel symbols:

$$\Gamma^k_{ij} = \frac{1}{2}\sum_r g^{kr}\left(\frac{\partial g_{rj}}{\partial x^i} + \frac{\partial g_{ri}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^r}\right)$$

where (g^{ij}) is the inverse matrix of $g = (g_{ij})$.

We refer to the connection in the above theorem as the Levi-Civita connection of the Riemannian manifold (M,g). The Jacobi identity for the Lie bracket, the vanishing of the tensor and an easy computation show that the curvature of any symmetric connection and therefore of the Levi - Civita connection satisfies the Bianchi identity:

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$$

for any vector fields X,Y and Z.

13. Volume element and integration on a Riemannian manifold.

Suppose now that M is a Riemannian manifold that has an orientation as defined in the chapter on integration. Then choose a positively oriented basis which is orthonormal with respect to the Riemannian metric say $X_1,\ldots,X_n$ for the tangent space at each point. If $\varepsilon^1,\ldots,\varepsilon^n$ is the dual basis, consider the n-form $\varepsilon^1\wedge\ldots\wedge\varepsilon^n$.This form is independent of the choice of basis: for if $Y_1,\ldots,Y_n$ is another oriented orthonormal basis as above with corresponding n-form $\delta^1\wedge\ldots\wedge\delta^n$, we have:

$$\varepsilon^1\wedge\ldots\wedge\varepsilon^n = \det(A)\delta^1\wedge\ldots\wedge\delta^n.$$

The transformation A whose determinant appears in this expression is an orthogonal one since it takes an orthonormal basis to an orthonormal basis and therefore the determinant is either +1 or -1. The orientation condition excludes the -1 case. Since the n-form we obtain is independent of the choice of basis, it is globally defined. We call this form the **volume element** of the manifold M and denote it by v_M.

We can now integrate functions on orientable Riemannian manifolds: for a smooth function $f:M\to\mathbb{R}$ and with the extra assumption that the manifold is compact (to avoid infinities) we define:

$$\int_M f = \int_M f\, v_M \ .$$

14. Complex manifolds.

We remarked in the beginning that we could consider different classes of manifolds by taking different gluing together maps, such as smooth or C^k maps. There is another more radical change we can make, namely to replace $\mathbb{R}^n$ by $\mathbb{C}^n$ and consider manifolds M built from open subsets of $\mathbb{C}^n$ and glued together by complex analytic maps. We call such a manifold M a **complex manifold of complex dimension n.** Since $\mathbb{C}^n$ can be viewed as $\mathbb{R}^{2n}$ and complex analytic maps are certainly smooth, a complex manifold of dimension n is at the same time a C^∞ manifold of dimension 2n.

If we take a complex chart on M and denote the (complex) coordinate functions by $(z^1,...,z^n)$ then we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by taking real and imaginary parts. Thus each complex coordinate z^j gives two real coordinates x^j, y^j where $z^j = x^j + iy^j$.

We introduce the conjugate functions

$$\bar{z}^j = x^j - iy^j,$$

which are useful in calculations. If we formally change coordinates from x, y to z, $\bar{z}$ we obtain the following relationships between the tangent vectors:

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \quad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right).$$

We refer to the first of these as **holomorphic derivatives** and the second as **antiholomorphic.** Note that a complex function f on an open set of M is complex analytic or holomorphic precisely when the Cauchy Riemann equations

$$\frac{\partial f}{\partial \bar{z}^j} = 0$$

hold. We use these formulas to define the compex derivatives on the underlying smooth manifold. This means that when we complexify the real tangent space TM_x to M at x we get two subspaces $T'M_x$ spanned by the holomorphic derivatives, and $T''M_x$ spanned by antiholomorphic derivatives.

Exercise: Show that this decomposition is independent of the coordinates chosen.

We call $T'M_x$ the **holomorphic tangent space** and $T''M_x$ the **antiholomorphic tangent space.** It is clear that the splitting

$$(TM_x)^{\mathbb{C}} = T'M_x + T''M_x$$

is a direct sum of two subspaces of equal dimension related by complex conjugation. There is a unique endomorphism J_x of TM_x whose +i eigenspace is given by $T'M_x$ and whose -i eigenspace is $T''M_x$. It can be given in coordinates by

$$J_x\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j}, \quad J_x\frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j}.$$

We call the section J of the endomorphisms of TM the **complex structure** of the manifold. It is clear from the above that $J^2 = -1$, so it gives the action of the complex numbers on the real tangent space. In fact it is essentially equivalent to the complex analytic structure.

If we have a smooth manifold M of dimension 2n we say M has an **almost complex structure** if we are given an endomorphism J_x on each tangent space TM_x which has square -1. We say (M,J) is an **almost complex manifold.** (M,J) is said to be **integrable** if J arises from a complex analytic structure as above. In any case we still have the splitting of the complexified tangent spaces into the +i and -i eigenspaces of J which we still denote by T'M and T''M respectively. The following very deep theorem completely determines when an almost complex structure is integrable:

Theorem. (Newlander-Nirenberg) The almost complex structure J on a smooth manifold M is integrable if and only if T'M is closed under Lie brackets (i.e. whenever two complex vectorfields X and Y on M have their values at every point in T'M then so does their bracket [X,Y]).

If (M,g) is a Riemannian manifold of even dimension, and J is an almost complex structure on M then we say g is **almost Hermitian** if

$$g(JX,JY) = g(X,Y)$$

for any tangent vectors X and Y. This is equivalent to the requirement that

$$g(JX,Y) = -g(JY,X).$$

It follows that

$$\omega(X,Y) = g(JX,Y)$$

defines an alternating and function-linear bilinear form on the vector fields and hence gives a 2-form ω on M. We call ω the **Kaehler form** of the almost Hermitian manifold. If the almost complex structure is integrable then we say (M,g,J) is a **Hermitian manifold**. A **Kaehler manifold** is a Hermitian manifold whose Kaehler form is closed. Kaehler manifolds represent an important class of manifolds with a very rich structure resulting from the interplay between the complex structure and the Riemannian structure.

Theorem. If (M,g,J) is an almost Hermitian manifold then J is covariant constant if and only if J is integrable and the Kaehler 2-form is closed: i.e. M is a Kaehler manifold.

Proof: First see that J covariant constant is equivalent to TM being closed under covariant differentiation. Then the vanishing of the torsion shows that

$$[X,Y] = \nabla_X Y - \nabla_Y X$$

and hence that TM is closed under Lie brackets. The almost complex structure is integrable by the Newlander-Nirenberg Theorem. That the Kaehler form ω is closed follows from the easily proven identity

$$d\omega(X,Y,Z) = (\nabla_X\omega)(Y,Z) - (\nabla_Y\omega)(X,Z) + (\nabla_Z\omega)(X,Y)$$

and the fact that both J and g are covariant constant (and hence also ω).

Returning to the case of complex manifolds, the decomposition of the tangent spaces into holomorphic and antiholomorphic tangents has an analogue for the cotangent bundle. If we choose coordinates z^j as before, then dz^j and $d\bar{z}^j$ give a basis for the complexified cotangent spaces which we refer to as the **holomorphic** and **antiholomorphic differentials**. The spaces they span are invariantly defined, so give a splitting

$$(T^*M_x)^{\mathbb{C}} = \Lambda^{1,0}M_x + \Lambda^{0,1}M_x$$

and a corresponding splitting of the complex 1-forms into

$$\Omega^{1,0}(M) + \Omega^{0,1}(M).$$

A (1,0)-form is a complex 1-form which when written in terms of the complex differentials only involves the holomorphic differentials. Likewise for the (0,1) forms which only involve antiholomorphic differentials. More generally a (p,q)-form is a (p+q)-form which when expressed in complex coordinates involves p holomorphic and q antiholomorphic differentials: i.e. a combination of differentials of the form

$$dz^{i_1}\wedge\ldots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\ldots\wedge d\bar{z}^{j_q}.$$

We thus obtain spaces $\Omega^{p,q}(M)$ of forms of type (p,q) which are spaces of sections of covectors $\Lambda^{p,q}M$ of type (p,q).

If we examine the formula for the exterior derivative in local coordinates, then on functions we have

$$df = \sum_j \frac{\partial f}{\partial z^j}dz^j + \frac{\partial f}{\partial \bar{z}^j}d\bar{z}^j .$$

It follows that the differential of a (p,q)-form α has components of types (p+1,q) and (p,q+1). The component of $d\alpha$ of type (p+1,q) we denote by $\partial\alpha$ and the component of type (q,p+1) we denote by $\bar{\partial}\alpha$. This gives a decomposition of d into two operators d =

$\partial + \bar{\partial}$ where

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M),\ \bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M).$$

From a comparison of types and $d^2 = 0$ we deduce

$$\partial^2 = 0,\ \bar{\partial}^2 = 0,\ \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

In particular we have the Dolbeault or $\bar{\partial}$ complexes

$$\Omega^{p,0}(M) \xrightarrow{\bar{\partial}} \Omega^{p,1}(M) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Omega^{p,n}(M)$$

which give the Dolbeault cohomology groups $H^{p,q}(M)$. There is an analogue of the Poincare Lemma for the $\bar{\partial}$ operator whose proof involves the Cauchy integral formula extended to smooth functions.

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