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C O L L E G E

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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YANG-MILLS FIELDS AND 4-MANIFOLDS.

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First let me say that it is a great pleasure to be here at the ICTP and to have the opportunity to speak to such an audience.

The subject of my talk, and one of the principle themes in the later part of this college, is the application of the theory of Yang-Mills fields to the study of the geometry and topology of 4-dimensional manifolds. We mathematicians are used to the idea of mathematics being used in physics but it is important to point out that here we are dealing with the opposite process, ideas from physics being used in mathematics, and what is more used to prove very striking new theorems. The theory of 4-manifolds has advanced dramatically in the last 8 years and it is safe to say that the key new ingredient is the introduction of Yang-Mills theory. My present aim is to survey some of the high points and key ideas in the subject; of course in order to do this it is necessary to leave out many important and difficult technical points. Detailed treatment of many of these points may be found in Donaldson's papers, the books *"Instantons and Four-Manifolds"* by Freed and Uhlenbeck, *"The Theory of Gauge Fields in Four Dimensions"* by Lawson and *"Geometry of Gauge Fields"* by Atiyah. I should say at the outset that most of the material I will describe is due to Simon DONALDSON.

Naturally, I must begin by saying something about the theory of 4-manifolds and mention the problems one wishes to solve. Throughout I will assume, unless it is explicitly stated otherwise, that we are working with a smooth, simply connected, closed (i.e. compact and no boundary), oriented manifold. I will often be necessary to assume that we have chosen a Riemannian metric on X, but I will try to make it clear precisely where I am using the metric. Associated to X is a basic invariant, its ***intersection form*** Q_X. This is a symmetric, unimodular (i.e. determinant ±1) bilinear form defined on the free abelian group $H(X) = H^2(X;\mathbb{Z})$. More concretely it should be thought of as a symmetric, unimodular matrix of integers: here are two important examples.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$E_8: \quad \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & 0 & -1 \\ & & & & -1 & 2 & -1 & 0 \\ & & & & 0 & -1 & 2 & 0 \\ & & & & -1 & 0 & 0 & 2 \end{pmatrix}$$

I will simply use the term ***form*** for a symmetric unimodular form defined over the integers.

The intersection form of a 4-manifold X measures the way that 2 dimensional sub-manifolds of X intersect. In purely topological terms it is the bilinear form

$$Q_X(\alpha,\beta) = \langle \alpha \cup \beta, [X] \rangle$$

defined, using the cup product $\cup$, on H(X). A lot of work has been done on the algebraic classification of these forms Q. First they are divided up into two basic classes.

DEFINITE FORMS. Positive definite means $Q(\alpha,\alpha) \geq 0$, and $Q(\alpha,\alpha) = 0$ if and only if $\alpha = 0$ and Q is negative definite if -Q is positive definite. Geometrically we can always change the orientation of the 4-manifold to change negative definite into positive definite so we will assume the form is positive definite. The study of the definite forms involves some hard number theory, and though a lot is known these definite forms have not been classified. A great deal of information is contained in Serre's book *"A course in Arithmetic"* or the book by Milnor and Husemoller *"Symmetric bilinear forms'.*

INDEFINITE FORMS. Here there is a classification due to Hasse-Minkowski. There are two types

type II	even forms:	i.e. $Q(\alpha,\alpha)$ is even, or in terms of the matrix, the diagonal entries are even.
type I	odd forms:	the others.

Indefinite forms are classified by three invariants

type
rank = size of matrix
signature = number of positive eigenvalues - number of negative eigenvalues when the matrix is diagonalised over $\mathbb{R}$.

The classification of indefinite forms goes as follows: If Q is an indefinite form then either Q or -Q is one of the following list

$$\left.\begin{array}{ll} \text{type I} & n(1)+m(-1) \\ \text{type II} & nE_8+m\begin{pmatrix}0&1\\1&0\end{pmatrix}\end{array}\right\} \; n,m\geq 0$$

and two forms are isomorphic if and only if they have the same type, rank and signature.

Now let us return to the theory of 4-manifolds and ask the natural first question:

REALISATION QUESTION. Is every form the intersection form of a smooth 4-manifold.

Theorem 1. Suppose that X is a smooth 4-manifold and Q_X is positive definite; then Q_X is isomorphic (over the integers) to $x_1^2+\ldots+x_r^2$ where $r = \text{rank } H(X)$.

Therefore smooth 4-manifolds see none of the complications of positive definite quadratic forms over the integers. Let us now turn our attention to the indefinite forms. It is easy to realise the type I indefinite forms since the intersection form of

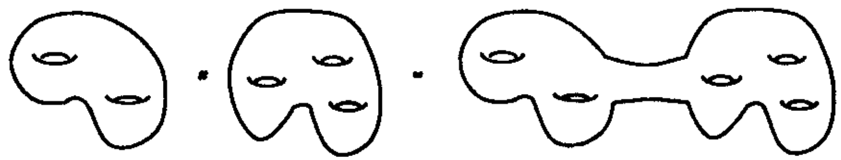
$$nP^2 \# m\overline{P}^2$$

is just $n(1)+m(-1)$. I should explain the notation:

P^2 is $\mathbb{C}P^2$ is oriented so that its intersection form is 1,

$\overline{P}^2$ is $\mathbb{C}P^2$ with reversed orientation so its intersection form is -1

is the connected sum operation which is most easily explained by drawing a picture



Now we turn to the type II forms. The intersection form of $S^2{\times}S^2$ is

$$\begin{pmatrix}0&1\\0&1\end{pmatrix}$$

but in general, for type II, life is not so simple.

If Q is a form, define $b^-(Q)$ to be the number of negative terms which appear when Q is diagonalised over $\mathbb{R}$. So if $n,m\geq 0$

$$b^-(nE_8 + m\begin{pmatrix}0&1\\0&1\end{pmatrix}) = m .$$

If X is a smooth 4-manifold then we define $b^-(X)$ to be $b^-(Q_X)$. Next note that X is a spin manifold if and only if Q_X has type II, this gives us a geometrical way of viewing the condition that the intersection form has type II. The next theorem gives some information on the realisation of type II indefinite forms.

Theorem 2. Suppose X is a smooth spin 4-manifold with indefinite intersection form:

(a) If $b^-(X) = 1$, then $Q_X = \begin{pmatrix}0&1\\0&1\end{pmatrix}$

(b) If $b^-(X) = 2$, then $Q_X = \begin{pmatrix}0&1\\0&1\end{pmatrix} \oplus \begin{pmatrix}0&1\\0&1\end{pmatrix}$

However here things stop. The K3 surface is defined to be

$$K3 = \{ [z_0:z_1:z_2:z_3] \in \mathbb{CP}^3 \mid z_0^1 + z_1^4 + z_2^4 + z_3^4 = 0\}$$

and a calculation using characteristic classes shows that

$$Q_{K3} = E_8 \oplus E_8 \oplus \begin{pmatrix}0&1\\1&1\end{pmatrix} \oplus \begin{pmatrix}0&1\\1&0\end{pmatrix} \oplus \begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

This shows that

$$2E_8 + 3\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

occurs as the intersection form of a smooth 4-manifold. In fact it is still unknown what possible indefinite type II forms occur as the intersection forms of 4-manifolds, but there is a well-known conjecture.

CONJECTURE Suppose X is a smooth spin 4-manifold with indefinite intersection form, then

$$Q_X \cong n(Q_{K3}) + m\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad \text{where } n,m \geq 0.$$

Finally let me point out that the theory of topological 4-manifolds is dramatically different: According to Freedman ***every form is the intersection form of a topological 4-manifold.*** This difference leads to a very surprising conclusion. The only way to reconcile the difference between Donaldson's results and Freedman's results is that there must exist a fake $\mathbb{R}^4$; by this I mean a smooth 4-manifold X which is homeomorphic to $\mathbb{R}^4$ but not diffeomorphic to $\mathbb{R}^4$. It is known that in every other dimension there cannot exist a fake $\mathbb{R}^n$.

Now let me turn to the methods Donaldson uses to prove these results, so I must now do a complete turn around and start to discuss ***Yang-Mills theory.*** For simplicity I will describe the Yang-Mills equations on $\mathbb{R}^4$ equipped with its usual Euclidean metric. I will assume that the gauge group or structure group of the theory is SU(2) the group of 2×2 unitary matrices with determinant 1 (i.e $AA^* = A^*A = 1$, $\det A = 1$). Its Lie algebra su(2) is the space of skew adjoint (i.e. $A^* = -A$) matrices with trace zero.

Then an ***SU(2) connection*** is simply a 1-form on $\mathbb{R}^4$ with values in su(2) i.e.

$$A = \sum_{\mu=1}^{4} A_\mu dx_\mu \qquad\qquad A_\mu : \mathbb{R}^4 \to su(2) .$$

A ***gauge transformation*** is a smooth function $g : \mathbb{R}^4 \to su(2)$ and gauge transformations act on connections as follows:

$$g^*(A) = g^{-1}Ag + g^{-1}dg .$$

Here, to make sense of this formula, remember that we are dealing with matrices, so for example dg is just the differentiated matrix. Two connections A_1 and A_2 are ***gauge equivalent*** if there is a gauge transformation g such that $g^*A_1 = A_2$.

The ***curvature*** of A is the matrix of 2-forms

$$F_A = dA + A \wedge A$$

where $A \wedge A$ is defined using matrix product and exterior product of forms: in coordinates

$$F = \sum_{\mu < \nu} F_{\mu\nu} dx_\mu dx_\nu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

where if X, Y are matrices, then $[X,Y] = XY - YX$. In particular these formulas show that F is an su(2)-valued 2-form.

The next ingredient is the ***∗-operator***. This is the operator on 2-forms given by

$$*(dx_i dx_j) = \pm\, dx_k dx_\ell$$

where $\{i,j,k,\ell\} = \{1,2,3,4\}$ and the sign is + if $(1234) \to (ijk\ell)$ is an even permutation and − if it is odd.

The connection A defines the ***covariant derivative*** operator $D_A = d + A$; so for example if F is a su(2)-valued 2-form on $\mathbb{R}^4$ then

$$D_A F = dF + A \wedge F.$$

It is now possible to state the Yang-Mills equations.

THE YANG-MILLS EQUATIONS:

$$D_A F_A = 0$$
$$D_A(*F_A) = 0$$

These equations are a system of second order non-linear partial differential equations for the components of the connection A_μ. In fact the first of these equations is an identity, the ***Bianchi identity***, it is always valid if F_A is the curvature of the connection A. This observation leads to the self dual Yang-Mills equations.

SELF DUAL YANG-MILLS EQUATION:

$$*F_A = F_A$$

Of course one could equally well look at the anti-self dual equation $*F_A = -F_A$. In view of the Bianchi identity a solution of the self dual equations, or anti-self dual equations, is automatically a solution of the full Yang-Mills equations. The self dual equations are a system of first order non-linear equations for the A_μ.

A solution of the self dual equations will be called a ***self dual connection***. It is easy to check that if A is self dual then so is g^*A or more generally if A is a solution of the full Yang-Mills equations then so is g^*A. This leads to the introduction of the ***moduli space of self dual connections***

$$M = \frac{\text{self-dual connections}}{\text{gauge equivalence}}$$

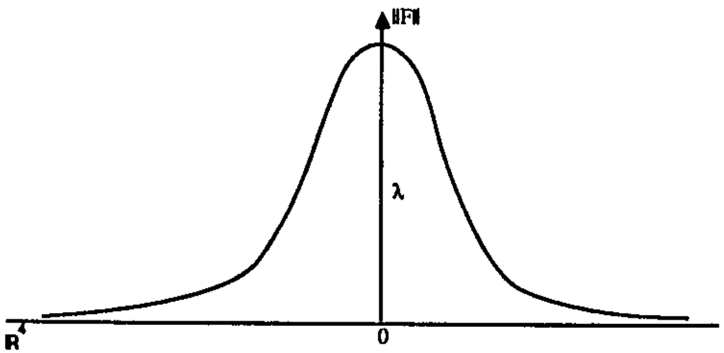
All this can be done globally on a smooth 4-manifold equipped with a Riemannian metric. I will not really try to say anything precise about this here but I will make one important point. If we are working on a compact 4-manifold X, then we must really work with a principal SU(2) bundle over X and there is one moduli space for each such bundle. Principal SU(2) bundles on X are classified by an integer k and I will denote the corresponding moduli space by M_k. This integer k is often referred to as the ***topological quantum number***.

There is a strong interaction between the structure of the moduli spaces M_k and the topology of X and this is the point which Donaldson exploits to prove the above theorems. Here is a simple example of this interaction. Suppose that X satisfies our initial assumptions (i.e. simply connected, closed, ...) then it follows that, for a generic metric, the moduli space M_k is a smooth "manifold with singularities" of dimension

$$8k - 3(1 + b^-(X))$$

so that the dimension of the moduli space is determined by a topological property of X. Here the phrase generic metric means that this result is true for an open dense set of metrics.

One of the important technical points is the existence of point-like solutions to the self dual Yang-Mills equations. These point-like solutions provide the method for using the moduli spaces to get information about the points of the manifold X. On $\mathbb{R}^4$ the norm of curvature, or gauge field $\|F\|$ of these special solutions looks like this:



These solutions A^λ are determined by a single parameter λ, and as $\lambda \to \infty$ the function $\|F^\lambda\|$ converges to a delta function at the origin.

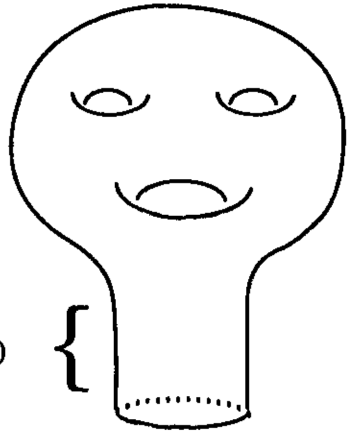
Now we try to take these solutions defined on $\mathbb{R}^4$ and "graft" them in to a general 4-manifold X by gluing them in to a neighbourhood of a chosen point x, to get a self dual connection $A^\lambda(x)$ on X, or more generally we try to superimpose k solutions $A^{\lambda_1}(x_1),\dots,A^{\lambda_k}(x_k)$. If successful this process will produce a self dual connection with topological quantum number k. However this is ***not always possible*** but there are extensive results due to Taubes on when this can be done and when it cannot. This process is very important; this is what gives a precise relation between the points of X and the solutions of the self dual equations.

Let me illustrate how to put all these ideas together by outlining, very briefly, the proof of the following special case of Theorem 1. Assume that X is spin, so that Q_X is even, and also that Q_X is definite. If we go back and look at Donaldson's Theorem 1 this means that $H(X) = 0$ and $Q_X = 0$ since Theorem 1 says that Q_X must be isomorphic (over the integers) to $x_1^2 + \dots + x_r^2$ where $r = \operatorname{rank} H(X)$. But if Q_X is even this is only possible if $r = 0$. I will outline how to prove that $H(X) = 0$.

Look at the k = 1 moduli space: Then for a generic metric it turns out that this is an open 5-dimensional manifold and outside a compact set C

$$M_1 \setminus C \cong X \times (1,\infty)$$

This copy of $X \times (1,\infty)$ is the space of solutions of the form $A^\lambda(x)$. The proof of these facts requires some hard analysis. In particular the proof that there is a compact set outside of which M_1 looks like $X\times(1,\infty)$ requires an existence theorem due to Cliff Taubes and a compactness theorem due to Karen Uhlenbeck. Here is a picture:



Given that this is the structure of the moduli space M_1, it is natural to compactify M_1 by adding a copy of X×1. This compactified moduli space $\overline{M}_1$ is a smooth 5-manifold with boundary X.

A simple general topological argument shows that if W is any 5-manifold with $\partial W = Y$, then

$$\text{signature } Q_Y = 0 .$$

In particular we conclude that the signature of Q_X is 0; but since Q_X is definite and unimodular this must mean that H(X) = 0.

There is another and equally interesting part of Donaldson's theory of 4-manifolds that there has not been time to touch on here. This is Donaldson's work on the Classification Problem: classify smooth 4-manifolds with a given intersection form. This leads to the theory of the Donaldson polynomials which I am sure we will hear much about later on during this meeting. However I hope I have said enough to convey one or two key points in the applications of Yang-Mills theory to 4-manifolds and enough to convince you that this is a very striking and deep subject.

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