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SYMPLECTIC MANIFOLDS AND GEOMETRIC QUANTIZATION.

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Symplectic Manifolds and Geometric Quantization

by

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A **symplectic manifold** (M,ω) is a smooth manifold M equipped with a closed 2-form ω which is non-degenerate as a bilinear form on each tangent space.

In the case M is finite-dimensional the second condition forces M to be even-dimensional. If the dimension is 2n, then each of the exterior powers $\omega, \omega^2, \dots, \omega^n$ is non-degenerate. In particular M is orientable and if it is compact the de Rham classes $[\omega], [\omega^2], \dots, [\omega^n]$ are non-zero. The 2n-form $(-1)^{[n/2]}\omega^n/n!$ is called the **Liouville volume.**

Example 1 The most basic example of a symplectic manifold is $\mathbb{R}^{2n}$ with the symplectic structure

$$\omega = \sum_i dp_i \wedge dq^i$$

where we have divided the usual 2n coordinates on $\mathbb{R}^{2n}$ into two groups of n: $p_1,\dots,p_n$ and $q^1,\dots,q^n$. More generally, given any symplectic vector space V, the skew-symmetric bilinear form translates around V to yield a symplectic form making V into a symplectic manifold. We refer to these examples as the **linear symplectic manifolds.**

Example 2 $M = T^*N$ for any smooth manifold N. If $\pi : M \to N$ is the cotangent projection then the tautological 1-form θ defined by $\theta_p = p \circ \pi_*$ can be written using coordinates $x^1,\dots,x^n$ on $U \subset N$ as follows: set $q^i = x^i \circ \pi$ and for $p \in \pi^{-1}(U)$ let

$$p = \sum_i p_i(p)\, dx^i(\pi(p))$$

which gives smooth functions p_i on $\pi^{-1}(U)$. Then it is easy to see that $\theta = \sum_i p_i dq^i$. Thus $\omega = d\theta = \sum_i dp_i \wedge dq^i$ is a symplectic structure on M. Note that in these special charts on M the symplectic form is the same as that in Example 1.

Example 3 A manifold M is a Kaehler manifold if it has a Riemannian metric g and an almost complex structure J compatible with g (that is $g(JX,JY) = g(X,Y)$) such that $\nabla J = 0$ for the Levi-Civita connection. J is necessarily integrable and $\omega(X,Y) = g(JX,Y)$ is skew-symmetric. Obviously $\nabla\omega = 0$ so $d\omega = 0$ and ω is non-degenerate. Hence (M,ω) is a symplectic manifold.

Example 4 Let G be any Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual vector space. G acts on $\mathfrak{g}$ by the adjoint representation and on $\mathfrak{g}$ by its contragredient - called the coadjoint representation. Let $M \subset \mathfrak{g}^*$ be an orbit and $f \in M$. If G_f is the stabilizer of f and $\mathfrak{g}_f$ its Lie algebra then

$$\mathfrak{g}_f = \{\xi \in \mathfrak{g} : \xi.f = 0\},$$
$$= \{\xi \in \mathfrak{g} : f \circ ad\xi = 0\}.$$

Consider the bilinear form on $\mathfrak{g}$ given by

$$B_f(\xi,\eta) = f([\eta,\xi])$$
$$= -f \circ ad\xi\,(\eta).$$

It follows that B_f has kernel $\mathfrak{g}_f$, so induces a non-degenerate skew-symmetric form on $\mathfrak{g}/\mathfrak{g}_f = T_f M$. This gives M a natural non-degenerate 2-form ω. The Jacobi identity for $\mathfrak{g}$ implies that $d\omega = 0$ so (M,ω) is a symplectic manifold. The 2-form ω is known as the Kirillov-Kostant-Souriau 2-form of the coadjoint orbit M.

Example 5 Let Σ be a Riemann surface and $P \to \Sigma$ a principal G-bundle. Let $\mathcal{A}$ be the space of all connections on P. $\mathcal{A}$ is an affine space whose underlying vector space is $\Omega^1(ad(P))$ - the $P\times_G\mathfrak{g}$-valued 1-forms where $\mathfrak{g}$ is the Lie algebra of G. Let $\mathfrak{g}$ have an invariant inner product $(.,.)$ and take $B_1, B_2 \in \Omega^1(ad(P))$ then we can take $(B_1 \wedge B_2)$

which is a 2-form on Σ, and set

$$\omega(B_1,B_2) = \int_\Sigma (B_1 \wedge B_2).$$

Such a non-degenerate skew-symmetric bilinear form on $\Omega^1(ad(P))$ translates around $\mathcal{A}$ as in example 1 to give it a symplectic structure.

The fact that in example 2 we could find coordinates on T^*N such that the symplectic form was the same as example 1 is not an accident. This is the content of Darboux's Theorem:

<u>Theorem</u> If (M,ω) is a symplectic manifold and x a point of M then there is a neighbourhood U of x in M and a diffeomorphism of U onto an open neighbourhood of the origin in TM_x which pulls back the linear symplectic form on TM_x to coincide with ω on U.
<u>Proof</u> See [Abraham & Marsden].

If M is finite dimensional then there is a basis for the tangent space at x so that ω_x has the same form as the linear symplectic structure on $\mathbb{R}^{2n}$. So any 2n-dimensional symplectic manifold has a coordinate system $(p_1,...,p_n,q^1,...,q^n)$ in a neighbourhood of any given point such that $\omega = \sum_i dp_i \wedge dq^i$. These coordinates we call Darboux coordinates on M.

It is not known in general what are the topological restrictions on a smooth manifold so that it admits a symplectic structure. Obviously it must be even-dimensional and orientable, and if it is compact all its even Betti numbers are non-zero. A futher restriction is that it must admit an almost complex structure. To see this, we pick any Riemannian metric g on the symplectic manifold (M,ω). Then we can represent ω with respect to g by a skew-symmetric endomorphism A of TM: $\omega(X,Y) = g(AX,Y)$. $-A^2$ is then positive definite, so has a positive square root B. Set $J = AB^{-1}$. Since A and B commute, $J^2 = -1$, whilst $\omega(JX,JY) = -g(BX,AB^{-1}Y) = -g(X,AY) = \omega(X,Y)$. On the other hand $\omega(X,JX) = g(AX,AB^{-1}X) = g(X,BX)$ which is strictly positive for X non-zero. Note that this says that the modified metric $g'(X,Y) = g(BX,Y)$ gives M the structure of an almost Kaehler manifold with Kaehler 2-form ω. An almost complex structure J on M such that $\omega(X,JY)$ is an almost Kaehler metric we call a **positive compatible almost complex structure.**

For a long time the only known examples of compact symplectic manifolds were all Kaehler manifolds (such as projective spaces, Grassmannians, flag manifolds). Then Thurston constructed an example of a torus bundle over a torus with an odd first Betti number, but whose total space was symplectic. McDuff used Gromov's symplectic blowing up technique and Thurston's example to construct simply-connected non-Kaehlerian compact symplectic manifolds.

If (M,ω) is a symplectic manifold then a **symplectic diffeomorphism** of M (or a **canonical transformation**) is a diffeomorphism $\sigma : M \to M$ which preserves ω:

$$\sigma^*\omega = \omega.$$

Example 1 The group of translations of $\mathbb{R}^{2n}$ acts symplectically whilst the linear symplectic diffeomorphisms of $\mathbb{R}^{2n}$ form the simple Lie group $Sp(2n,\mathbb{R})$.

Example 2 If σ is any diffeomorphism of N then $p \to p\circ\sigma^{-1}{}_*$ gives a symplectic diffeomorphism of T^*N since in fact it already preserves the 1-form θ.

Example 3 The group of holomorphic isometries of a Kaehler manifold M will consist of canonical transformations of (M,ω).

Example 4 The Lie group G obviously acts on its coadjoint orbits, and by the natural way that the Kirillov-Kostant-Souriau form was defined, it is invariant under the action of G.

Example 5 The group of gauge transformations acts on the space of connections $\mathcal{A}$ so as to preserve the symplectic structure.

Let σ_t be a 1-parameter group of symplectic diffeomorphisms of (M,ω) with generating vector field X then

$$\sigma_t^*\omega = \omega,$$
$$\Rightarrow \quad \mathcal{L}_X\omega = 0,$$
$$\Rightarrow \quad d(i(X)\omega) + i(X)(d\omega) = 0.$$

Hence $i(X)\omega$ is a closed 1-form. Vector fields with this last property are called **locally Hamiltonian**. If the stronger property that $i(X)\omega$ is exact holds then X is called **Hamiltonian.** We denote by $Ham(M,\omega)$ the space of Hamiltonian vector fields on M. If $f \in C^\infty(M)$ then X_f denotes the Hamiltonian vector field with

$$i(X_f)\omega = df.$$

We thus have a map

$$C^\infty(M) \to Ham(M,\omega),$$
$$f \to X_f.$$

If X and Y are locally Hamiltonian then for any other vector field Z,

$$0 = (\mathcal{L}_X\omega)(Z,Y) = X\omega(Z,Y) - \omega([X,Z],Y) - \omega(Z,[X,Y])$$
$$0 = (\mathcal{L}_Y\omega)(X,Z) = Y\omega(X,Z) - \omega([Y,X],Z) - \omega(X,[Y,Z])$$
$$0 = (d\omega)(X,Y,Z) = X\omega(Y,Z) - Y\omega(X,Z) + Z\omega(X,Y)$$
$$- \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X).$$

Adding the three equations yields

$$0 = Z\omega(X,Y) - \omega([Y,X],Z).$$

Thus

$$[X,Y] = -X_{\omega(X,Y)}.$$

Hence the bracket of two locally Hamiltonian vector fields is always Hamiltonian. In particular the Hamiltonian vector fields form a Lie algebra.

Now consider the operation $X_f(g)$ of a Hamiltonian vector field X_f on a function g. Set

$$\{f,g\} = X_f(g)$$

then

$$\{f,g\} = dg(X_f) = \omega(X_g,X_f).$$

Thus $\{f,g\} = -\{g,f\}$. One can further show that the Jacobi identity holds for this bracket operation (essentially due to ω being closed) and so $C^\infty(M)$ with the operation $\{.,.\}$ is a Lie algebra. $\{.,.\}$ is called the **Poisson bracket** on $C^\infty(M)$. Note that

$$[X_f,X_g] = -X_{\omega(X_f,X_g)} = X_{\{f,g\}}$$

which shows that the map $f \to X_f : C^\infty(M) \to \mathrm{Ham}(M,\omega)$ is a homomorphism of Lie algebras.

In terms of Darboux coordinates

$$X_f = \sum_i \partial f/\partial q^i\, \partial/\partial p_i - \partial f/\partial p_i\, \partial/\partial q^i$$

and

$$\{f,g\} = \sum_i \partial f/\partial q^i\, \partial g/\partial p_i - \partial f/\partial p_i\, \partial g/\partial q^i.$$

It follows from these formulas that the integral curves for the vector field $-X_f$ satisfy the differential equations:

$$dq^i/dt = \partial f/\partial p_i,\ dp_i/dt = -\partial f/\partial q^i.$$

These are, of course, **Hamilton's equations**.

Now assume we have a symplectic manifold (M,ω) and a Lie group G acting smoothly on M by symplectic diffeomorphisms. We call this a **symplectic action of G**. Then each 1-parameter subgroup $t \to \exp t\xi$ of G gives rise to a 1-parameter group of diffeomorphisms of M whose generator we denote by $-X(\xi)$. This sign is chosen so that $\xi \to X(\xi)$ is a homomorphism of Lie algebras. In general $X(\xi)$ is a locally Hamiltonian vector field. If $X(\xi)$ is in fact in $\mathrm{Ham}(M,\omega)$ for all ξ then we say the action is **strongly symplectic**. If M is simply-connected, or if $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$ then a symplectic action will always be strongly symplectic. All the examples except possibly example 3 are of strongly symplectic group actions.

Given a strongly symplectic action of G on (M,ω) we can choose a linear map $\lambda : \mathfrak{g} \to C^\infty(M)$ such that $X(\xi) = X_{\lambda(\xi)}$ for each $\xi \in \mathfrak{g}$. We say the action is **Hamiltonian** if we can choose λ to be a homomorphism of Lie algebras and call such a λ a **Hamiltonian** for the action. This is not always possible, and the obstruction can be studied using Lie algebra cohomology. For a strongly symplectic action of G there is always a central extension of G which does admit a Hamiltonian.

An example of a non-Hamiltonian strongly symplectic action is given by the group of translations of $\mathbb{R}^{2n}$ with its linear symplectic structure (exercise). The central extension of the group of translations needed in this case is the Heisenberg group.

It is instructive to write down the Hamiltonians for the examples with strongly symplectic actions.

Example 1 Denote the linear symplectic form on $\mathbb{R}^{2n}$ by Ω. The Lie algebra of the group $Sp(2n,\mathbb{R})$ consists of all linear endomorphisms ξ with the form $\Omega(\xi v,w)$ symmetric. The function $\lambda(\xi) = \frac{1}{2}\Omega(\xi v,v)$ is the required Hamiltonian.

Example 2 The Lie algebra of the diffeomorphism group of N is the space of all vector fields $\mathcal{X}(N)$. Any vector field X on N defines a function $\lambda(X)$ on T^*N in the obvious way

$$\lambda(X)(p) = -p(X(\pi(p)))$$

and this is the required Hamiltonian.

Example 4 In this case the Hamiltonian function is given by duality:

$$\lambda(\xi)(f) = \langle f,\xi\rangle.$$

Example 5 The Lie algebra of the group of gauge transformations is $\Omega^0(\mathrm{ad}(P))$ so its dual will be $\Omega^2(\mathrm{ad}(P))$ with the duality given by first pairing in the Lie algebra and then integrating the resulting 2-form over Σ. If $\xi \in \Omega^0(\mathrm{ad}(P))$ then an easy calculation shows that $X(\xi)(A) = D^A\xi$ where $A \in \mathcal{A}$ and $D^A\xi$ denotes the covariant differential of ξ. Then

$$(i(X(\xi))\omega)_A(B) = \omega_A(D^A\xi,B) = \int_\Sigma (D^A\xi \wedge B).$$

But

$$(D^A\xi \wedge B) = d(\xi,B) - (\xi,D^AB)$$

so integration gives

$$(i(X(\xi))\omega)_A(B) = -\int_\Sigma (\xi,D^AB)$$

which is the derivative of the function

$$A \to -\int_\Sigma (\xi,F^A)$$

where F^A denotes the curvature of the connection A. Thus a Hamiltonian is given by

$$\lambda(\xi)(A) = -\int_\Sigma (\xi,F^A).$$

Returning to the general setting, let G be a Lie group acting strongly symplectically on (M,ω) with Hamiltonian function λ. We can dualise this situation by defining a map

$$\mu : M \to \mathfrak{g}^*$$

by

$$\mu(x)(\xi) = \lambda(\xi)(x).$$

We call the map μ obtained this way the **momentum map** for the Hamiltonian action of G. It generalizes to arbitrary groups of symmetries the linear momentum which corresponds with translations of $\mathbb{R}^n \subset \mathbb{R}^{2n}$, and angular momentum which corresponds with rotational symmetries.

Early versions of momentum maps were introduced by Mackey and Smale for cotangent bundles, but Kostant and Souriau developed the general theory.

Perhaps the most interesting momentum map of the above examples is the last one where we have

$$\mu(A) = -F^A.$$

That is: the curvature is the momentum map for the natural action on the space of connections of the group of gauge transformations of a principal bundle over a closed Riemann surface.

Let (M,ω) have a Hamiltonian G-action with momentum map μ. The image of μ is a union of coadjoint orbits. If M is homogeneous the image is a single orbit and it is a theorem of Kostant and Souriau that μ is then a covering map. In the non-homogeneous case there is a far-reaching construction due to Marsden and Weinstein called **reduction** which generalises to symplectic manifolds with Hamiltonian group actions the process which in classical mechanics is known as eliminating cyclic variables.

There are several versions of reduction of varying generality. Let us take the following case: Suppose $f \in \mathfrak{g}^*$ is in the image of the momentum map $\mu : M \to \mathfrak{g}^*$ and consider the inverse image $\mu^{-1}(f)$ which we suppose is a submanifold of M. It is no longer a G-space, but the stabilizer G_f of f still acts. The symplectic form ω will be degenerate when restricted to $\mu^{-1}(f)$, but it is not hard to see the characteristic directions are tangent to the orbits of G_f. If the quotient $\mu^{-1}(f)/G_f$ is a manifold it will inherit a symplectic structure. This new symplectic manifold we call the **(Marsden-Weinstein) reduced phase space.**

Take the momentum map for example 5. If we take the origin as the element of the dual of the Lie algebra at which to perform the reduction, then the stabilizer is the whole group of gauge transformations and the set $\mu^{-1}(0)$ is the set of flat connections. The reduced phase space is the set of equivalence classes of flat connections, and the above argument shows that at least where it is a smooth manifold, it has a symplectic structure.

Let us now turn to the question of quantization of (M,ω). The object is to try to find a Hilbert space H and and associate to each $f \in C^{\infty}(M)$ an operator $Q(f)$ on H so that

$$Q(f)Q(g)-Q(g)Q(f) = i\hbar Q(\{f,g\}).$$

Where $\hbar$ is a positive constant h divided by 2π. Apart from the factor of $i\hbar$ this means that $f \to Q(f)$ is a representation of the Lie algebra $C^{\infty}(M)$. The most naive solution is to take $Q(f) = i\hbar X_f$, but this has the consequence that $Q(1)$ is zero, so that if we apply the procedure to $\mathbb{R}^{2n}$, it would imply that the operators corresponding with position and momentum commute and that disagrees with the standard quantization of a particle with one degree of freedom.

Ideally we want $Q(1)$ to be the identity operator which means we could try $Q(f) = i\hbar X_f + f$, but a calculation shows that this is no longer a homomorphism. We can add another term depending linearly on X_f so that it does not spoil the normalization for the identity operator. In other words we can take a 1-form α on M and set

$$Q(f) = i\hbar (X_f + \alpha(X_f)) + f$$

then the commutator can be computed to give

$$[Q(f),Q(g)] = i\hbar Q(\{f,g\}) + i\hbar (i\hbar\, d\alpha(X_f,X_g) - \omega(X_f,X_g))$$

so we have a possible solution if α can be chosen so that $\omega = i\hbar d\alpha$.

This is not possible in general, since ω is not always exact. The way around that is to observe that since ω is closed, it is at least locally exact by the Poincare Lemma, so we should only interpret the formula as being true on some open set. We recognize the term $X + \alpha(X)$ as a covariant differentiation in the direction X, so the correct global formulation involves a complex line bundle L over M with a connection D. If s is a

section of L then we define

$$Q(f)\, s = i\hbar D_{X_f} s + f\, s$$

and and observe that this reduces to the previous operator when the bundle is trivialised so that α is the connection 1-form. Then the condition $\omega = i\hbar\, d\alpha$ says that the connection must be chosen so that it has curvature $\omega/i\hbar$. There is, of course, a topological restriction resulting from this, since the Chern-Weil theorem says that $i/2\pi$ times the curvature gives the real Chern class of L. Hence we have to assume $\omega/2\pi\hbar = \omega/h$ is an integral 2-form and then choose a line bundle L with this as its real Chern class. Such a bundle always has a connection with curvature $\omega/i\hbar$ and this we use to define the quantization as above. This condition on ω we call the **quantization condition.**

In order to form a Hilbert space from the sections of L we need it to have a Hermitian structure, then the Hilbert space consists of all sections which are square-integrable with respect the Liouville volume. If we choose a metric connection for D then the operators $Q(f)$ will be formally self-adjoint for real functions f. Such a connection has a pure-imaginary 2-form for its curvature - which is consistent with the quantization condition. We call a Hermitian line bundle L with a metric connection D having curvature $\omega/i\hbar$ a **prequantization** of (M,ω).

The above procedure appears to be a solution. When the manifolds in question are compact we get an integrality condition on the symplectic form which leads to discreteness of the parameters describing the quantization. Where it fails is in giving the 'right' answers for even the simplest of models. For example we take $\mathbb{R}^2$ with coordinates p and q, and the form $\omega = dp\wedge dq$. We interpret p and q as momentum and position respectively, then

$$X_p = -\partial/\partial q, \quad X_q = \partial/\partial p,$$

so that if we take the connection form $\alpha = p dq/i\hbar$, then

$$Q(p) = -i\hbar\, \partial/\partial q, \quad Q(q) = i\hbar\, \partial/\partial p + q.$$

The first operator is the standard one for momentum in the Schroedinger picture, but the second operator is only correct if it acts on functions of q alone rather than on the functions of p and q which we have ended up with. There is an equally bad result if we apply the formula to the harmonic oscillator $\frac{1}{2}(p^2 + q^2)$. This gives

$$Q(\tfrac{1}{2}(p^2 + q^2)) = i\hbar(q\,\partial/\partial p - p\,\partial/\partial q) + \tfrac{1}{2}(q^2 - p^2),$$

which is an operator with spectrum $(-\infty,+\infty)$. A partial solution can be found by taking the space of functions depending on q alone. Then the formulas for position and momentum are correct. The harmonic oscillator is not even defined in this picture as the operator above does not preserve the space of functions of q.

An alternative which does give an operator corresponding with the harmonic oscillator is to use a complex coordinate $z = p + iq$. Then we take the form $\alpha = -\bar{z}dz/2\hbar$ and obtain

$$Q(\tfrac{1}{2}z\bar{z}) = \hbar z\,\partial/\partial z - \hbar\bar{z}\,\partial/\partial\bar{z}.$$

This can be made to have positive spectrum by restricting it to the holomorphic functions of z. The norm in this case has to take account of the complex connection form α. The formula is:

$$\|f\|^2 = \int_{\mathbb{C}} |f|^2 e^{-|z|^2/2\hbar}\, dz d\bar{z}$$

Which means that this time the quantization is on the Segal-Bargmann space of holomorphic functions which are square-integrable with respect to a gaussian measure.

The only problem with the above operator is that it will give a purely integer spectrum to the harmonic oscillator, rather than the correct half-integer spectrum. This can be cured by introducing a symplectic analogue of spinors called half-forms which give an intrinsic way to form Hilbert spaces on symplectic manifolds.

The above examples show that we should try to cut out half the variables in order to get

the correct spectral and irreducibility properties. The geometrical way to do this is to introduce the notion of a **polarization**. We form the complexified tangent bundle $TM^{\mathbb{C}}$ and consider a subbundle F of half the fibre dimension. Extending the symplectic form ω to be bilinear on $TM^{\mathbb{C}}$ we say F is a polarization if it is closed under Lie brackets and isotropic with respect to ω.

Given a polarization F and a prequantization (L,D) of the symplectic manifold (M,ω) we form the space $\Gamma_F(L)$ of **polarized sections** of L, namely

$$\Gamma_F(L) = \{ s \in \Gamma(L) : D_X s = 0,\ \forall X \in \Gamma(F) \}.$$

This subspace of the space of sections is stable under Q(f) if $[X_f, \Gamma(F)] \subset \Gamma(F)$. This last space of functions we call $C^1{}_F$. It is easy to check that $C^1{}_F$ is a Lie subalgebra of $C^\infty(M)$ and we call it the space of **quantizable functions** for the polarization F.

Examples:

A) A polarization of $\mathbb{R}^2$ is given by taking F to be $\mathbb{C}\partial/\partial p$. When α is $pdq/i\hbar$ the polarized sections are then given by functions of q. A generalization of this example can be obtained by taking M to be a cotangent bundle T^*N. The complexification of the vertical distribution gives a polarization of M. Since the symplectic structure is exact we can take L to be trivial and $\alpha = \theta/i\hbar$ as the connection form. The polarized sections are then the functions on the base manifold N and $C^1{}_F$ consists of the functions on M which are polynomials of degree 1 on each cotangent space (i.e. linear in the momentum variables). In this cotangent case the symplectic spinors alluded to above would correspond with the bundle of half-densities on N. These form a natural Hilbert space $L^2(N)$ on which elements of $C^1{}_F$ act by formally self-adjoint first order differential operators.

B) A second example is provided by the identification $\mathbb{R}^2 = \mathbb{C}$. Then $F = \mathbb{C}\partial/\partial\bar{z}$ is a polarization and the polarized sections are the functions independent of $\bar{z}$ – i.e. the holomorphic functions. More generally, the space $T''M$ of antiholomorphic tangent vectors is a polarization for the Kaehler 2-form of a Kaehler manifold. If L is a Hermitian line bundle with connection D such that D has curvature $\omega/i\hbar$ then D has curvature of type (1,1) so L has a unique holomorphic structure such that a local section s of L is holomorphic if and only if Ds vanishes on antiholomorphic tangents. But that is just the condition for s to be a polarized section of L. Hence $\Gamma_F(L)$ is the space of global holomorphic sections of L, whilst it is easy to see that a function f is in $C^1{}_F$ if and only if X_f is the real part of a holomorphic vector field on M. The symplectic spinors in this case correspond with a square root $K^{\frac{1}{2}}$ of the canonical bundle of M and the quantization of $C^1{}_F$ is constructed on holomorphic sections of $L\otimes K^{\frac{1}{2}}$.

A polarization F of (M,ω) is said to be **real** if $F = \bar{F}$. It is easy to see that a real polarization is the tangent space to a foliation by real Lagrangian submanifolds and that locally such a foliation is symplectically equivalent to the vertical distribution of a cotangent bundle.

A polarization F of (M,ω) is said to be (pseudo-) **Kaehler** if $F\cap\bar{F} = 0$. For dimensional reasons we then have $TM^{\mathbb{C}} = F\oplus\bar{F}$ and so there is a real endomorphism J of TM with -i eigenspace F. J is an almost complex structure on M and the fact that we require F to be closed under Lie brackets says that J is integrable by the Nirenberg-Newlander theorem. The fact that F is isotropic translates to the compatibility of J with ω, but J may not be positive. This leads to the following notion:

Let (M,ω) be a symplectic manifold and F a polarization of M then F is said to be **positive** if $i\omega(X,\bar{X}) \geq 0$ for all $X \in F$. A positive Kaehler polarization is then the antiholomorphic tangent bundle of some Kaehler manifold M. Notice that real polarizations are also positive. The general positive polarization is a mixture of the two cases, but there are technical problems even at the formal level with integrability conditions so we assume the intersection $F\cap\bar{F}$ has constant dimension and $F + \bar{F}$ is integrable.

For another approach to the quantization of Kaehler manifolds which is closely related to geometric quantization see [Berezin].

There are various deficiencies in the over-simple approach above which rules out many

interesting cases by requiring L and $K^{\frac{1}{2}}$ to exist separately when all that is needed is the tensor product $L\otimes K^{\frac{1}{2}}$. A way around this is described in [Robinson & Rawnsley]. The fact that in the cotangent case we only handle Hamiltonians depending linearly on the momentum misses the standard quadratic kinetic energy terms. These can be handled by a generalization described in [Kostant]. In some situations one is forced to use non-positive polarizations or for reasons of holonomy there are no polarised sections; in this case there is a notion of cohomology which may still provide the Hilbert space see [Rawnsley], [Rawnsley, Schmid and Wolf].

Let us look briefly at the case of coadjoint orbits. It is a theorem of Kostant that if a momentum map exists then the Hamiltonian action of G on M lifts into the line bundle L as automorphisms of the connection D, so there is an induced action on the space of sections of L. Further the infinitesimalization of this action is just the quantization of the function $\lambda(\xi)$ for each $\xi \in \mathfrak{g}$. If the polarization is in addition G-invariant we get a representation of G on $\Gamma_F(L)$. There are two basic cases where invariant polarizations always exist for coadjoint orbits. If G is nilpotent then Kirillov showed that all the unitary representations of G can be obtained by geometric quantization of the coadjoint orbits. For this case all the orbits are simply-connected and have real polarizations giving them the structure of cotangent bundles of homogeneous spaces of G.

If G is compact semi-simple, $\mathfrak{g}$ and $\mathfrak{g}^*$ can be identified using the Killing form, so the coadjoint orbits as homogeneous spaces are the same as G modulo the centralizer of a torus. Such spaces are simply-connected. If M is the orbit of ξ, then M has a complex structure defined as follows: the stabilizer of ξ is its centralizer, which has Lie algebra the kernel of adξ. The tangent space to the orbit of ξ can then be identified with the sum of the non-zero eigenspaces of adξ. There is thus a natural splitting of the tangent space at ξ into two halves according as the eigenvalue is above or below the real axis. It is not hard to see that this gives the orbit of ξ a positive Kaehler polarization, which is G-invariant by construction. Hence each coadjoint orbit of a compact semi-simple Lie group has a unique positive invariant polarization. If the orbit is integral there is a unique Hermitian holomorphic line bundle L with the holomorphic connection having curvature $\omega/i\hbar$, and the group G has a representation on the space of holomorphic sections of L. That these representations are irreducible and exhaust the representations of G is the content of the classical Borel-Weil theorem.

For more general Lie groups there are some difficulties finding all the data needed for geometric quantization of all the integral coadjoint orbits, but for example for type I solvable Lie groups or for real semisimple Lie groups, enough orbits can be quantized to give the unitary representations occuring in the Plancherel theorem.

In example 5 the linear symplectic structure is exact so we can use the trivial line bundle to quantize the space of connections. A polarization may be found by observing that the choice of a complex structure on the Riemann surface Σ induces a splitting of the complex 1-forms into (1,0) and (0,1) forms, and this splitting of $\Omega^1(\mathrm{ad}(P))^{\mathbb{C}}$ is isotropic for the symplectic structure. So the space of connections on $P \to \Sigma$ has a natural Kaehler manifold structure.

Let us look at the compatibility of reduction and geometric quantization. Suppose we have a Hamiltonian G-manifold (M,ω) with momentum map μ, and that (M,ω) has a quantization (L,D) and invariant polarization F. If we choose f in the image of μ then because the orbits of G_f are isotropic for ω they are isotropic for the curvature of D. Hence the pull-back of L to a G_f-orbit is flat. If L can be parallelized on each G_f-orbit there will be an induced line bundle on the reduced space $\mu^{-1}(f)/G_f$ and a connection in this bundle whose curvature is the induced symplectic structure. One can also see that there is an induced polarization, at least generically, so that the whole of the quantization can be reduced. For example if this is applied to example 5, it induces a holomorphic line bundle on the moduli space of flat bundles. In the case $G = S^1$, the moduli space is the Jacobian of Σ and the line bundle is the universal bundle on the Jacobian.

I hope that in these brief notes I have conveyed some of the fascination of symplectic geometry and its relations with the of questions of global analysis which are the subject of this College.

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