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COLLEGE

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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1. SPIN REPRESENTATIONS.
2. REPRESENTATIONS OF SU(2).

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SPIN REPRESENTATIONS

1. Clifford algebras.

Let V be a finite dimensional complex vector space with a non-degenerate inner product $< , >$. The Clifford algebra $C(V)$ together with the inclusion of V into $C(V)$ are determined by the following universal property.

Definition Let A be an algebra and let $\phi: V \longrightarrow A$ be an inclusion of vector spaces such that

$$\{\phi(v_1), \phi(v_2)\} = -2 < v_1, v_2 > \qquad \text{for all} \quad v_1, v_2 \in V$$

where $\{a, b\}$ is the anti-commutator defined by $\{a, b\} = ab + ba$. Then there is a unique algebra homomorphism $C(V) \longrightarrow A$ which extends the inclusion of V.

All non-degenerate inner products on V are equivalent so the algebra $C(V)$ depends only on the dimension of V.

This construction still gives an algebra when the inner product is degenerate. In particular, applying this construction when the inner product of v_1 and v_2 is 0 for all $v_1, v_2 \in V$ gives the exterior algebra on V. Now replace the inner product $< , >$ by $h < , >$ where $0 \leq h \leq 1$ and apply the construction. For $h = 1$ this gives the Clifford algebra and for $h = 0$ this gives the exterior algebra. This gives a vector space isomorphism between the two algebras. This shows that if the dimension of V is n then the dimension of the Clifford algebra is 2^n. The exterior algebra also has a natural Z_2-grading. This map also shows that the Clifford algebra has a natural Z_2-grading.

2. Representations of algebras.

Recall that a representation of an algebra A is a vector space M and a linear map $\theta: A \otimes M \longrightarrow M$ such that

$$\theta(a \otimes \theta(b \otimes m)) = \theta(ab \otimes v)$$

where $a, b \in A$ and $m \in M$.

Definition A representation M is reducible if there is a proper subspace N of M such that if $m \in N$ and $a \in A$ then $\theta(a \otimes m) \in N$.

A representation that is not reducible is called irreducible. The algebras in these notes will all have the property that every representation can be written as a direct sum of irreducible representations. However this needs to be proved using some special features of these algebras since not all algebras have this property. An example of an algebra which does not have this property is the algebra of 2×2 matrices whose bottom left hand entry is 0. This algebra has an obvious two dimensional representation. This representation is reducible because the vectors whose lower component is 0 form a proper invariant subspace. However this representation cannot be written as a direct sum of two one-dimensional representations.

A construction that will be used later is that if $\phi: A \longrightarrow B$ is an algebra homomorphism and $\theta: B \otimes M \longrightarrow M$ is a representation of B then there is a representation $\theta^*: B \otimes M \longrightarrow M$ of A defined by

$$\theta^*(a \otimes m) = \theta(\phi(a) \otimes m)$$

3.Representations of finite groups.

Let Γ be a finite group and denote the group algebra over the complex field by $C\Gamma$. Any representation of the group extends uniquely to a representation of the group algebra.

Theorem Every representation of Γ is a direct sum of irreducible representations.

Proof It is sufficient to show that if W is a representation and U is an invariant subspace then there is an invariant subspace V such that the representation W is the direct sum of the representations U and V. This follows from an inductive argument on the dimension of W.

An inner product on W is called Γ-invariant if

$$< v_1, v_2 >=< \gamma v_1, \gamma v_2 >$$

for all $v_1, v_2 \in W$ and for all $\gamma \in \Gamma$.

Let W be a representation with a proper invariant subspace U. Assume that W has a Γ-invariant non-degenerate inner product. Then the representation W is the direct sum of U and the orthogonal complement of U in W.

However any W admits a Γ-invariant non-degenerate inner product. Let $<\ ,\ >$ be any non-degenerate inner product and define an inner product $<<\ ,\ >>$ by

$$<< v_1, v_2 >>= \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} < \gamma v_1, \gamma v_2 >$$

Then $<<\ ,\ >>$ is a Γ-invariant non-degenerate inner product.

This theorem implies that the group algebra is a direct sum of matrix algebras. Hence the group algebra is determined up to isomorphism by the dimensions of the inequivalent irreducible representations. Note also that the sum of the squares of the dimensions of the inequivalent representations of Γ is the order of Γ.

Definition Two group elements γ_1 and γ_2 are conjugate if there exists a group element γ such that

$$\gamma\gamma_1\gamma^{-1} = \gamma_2$$

This is an equivalence relation on the group and an equivalence class is called a conjugacy class. For instance the conjugacy class of the identity contains only the identity.

Definition A character of Γ is a function from Γ to the complex numbers which is constant on each conjugacy class.

The set of characters is a vector space whose dimension is the number of conjugacy classes.

Let $\rho: \Gamma \longrightarrow End(W)$ be a representation of Γ then the character is the function defined by

$$\gamma \mapsto tr(\rho(\gamma))$$

where $\gamma \in \Gamma$ and tr denotes the trace of a matrix.

This construction shows that the dimension of the space of characters also has a basis indexed by the irreducible representations of Γ. These two ways of counting the dimension of the space of characters give the following result.

Theorem The number of irreducible representations is equal to the number of conjugacy classes.

Every conjugacy class in an abelian group only has one element. Hence the number of irreducible representations of an abelian group is the order of the group. Also every irreducible representation of an abelian group is one dimensional.

Definition The centre of Γ denoted $Z(\Gamma)$ is

$$Z(\Gamma) = \{\gamma \in \Gamma: \gamma\alpha = \alpha\gamma \text{ for all } \alpha \in \Gamma\}$$

This is a normal subgroup.

4. The extra special 2-groups.

Choose an orthonormal basis of V say $e_1, \ldots, e_n$. Then these elements in the Clifford algebra satisfy the relations

$$e_ie_j + e_je_i = \begin{cases} -2 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Hence these elements in the Clifford algebra are units and generate a subgroup of the group of units in the Clifford algebra whose order is 2^{n+1}. This finite group is denoted by Γ_n and is known as an extra special 2-group.

There is clearly a surjective algebra homomorphism from the group algebra of Γ_n to the Clifford algebra. Hence every representation of the Clifford algebra gives a representation of Γ_n. However not every representation of Γ_n arises in this way.

The representation theory of Γ_n depends on whether n is even or odd.

Assume n is even.

In this case the centre of Γ_n is $\{+1, -1\}$ and the conjugacy class of each of these elements contains only that element. If γ is any other element of Γ then the conjugacy class of γ is $\{\gamma, -\gamma\}$. Hence there are $(2^{n+1} - 2)/2$ conjugacy classes containing two elements and two conjugacy classes containing one element. This gives a total of $2^n + 1$ conjugacy classes.

In the quotient group $\Gamma_n/Z(\Gamma_n)$ the generators $e_1, \ldots, e_n$ all commute so the quotient Γ/Z_2 is the abelian group Z_2^n. In other words there is a short exact sequence

$$0 \longrightarrow Z_2 \longrightarrow \Gamma_n \longrightarrow Z_2^n \longrightarrow 0$$

This shows that every representation of Z_2^n gives a representation of Γ_n. Since Z_2^n has 2^n inequivalent one dimensional irreducible representations so also does Γ_n.

Now Γ_n has $2^n + 1$ inequivalent irreducible representations and has 2^n inequivalent one-dimensional irreducible representations. Hence there is one irreducible representation that has not been accounted for.

The dimension of this representation say d can be calculated using the fact that the sum of the squares of the dimensions of the inequivalent irreducible representations is the order of the group. The sum of the squares of the dimensions is $2^n + d^2$ and the order of the group is 2^{n+1}. Hence $d = 2^{n/2}$. This representation is the spin representation of the Clifford algebra.

Assume n is odd.

In this case the centre of Γ_n is

$$\{+1, -1, e_1e_2 \ldots e_n, -e_1e_2 \ldots e_n\}$$

and the conjugacy class of each of these elements contains only that element. If γ is any other element of Γ then the conjugacy class of γ is

$$\{\gamma, -\gamma, e_1e_2 \ldots e_n\gamma, -e_1e_2 \ldots e_n\gamma\}$$

. Hence there are $(2^{n+1} - 4)/4$ conjugacy classes containing four elements and four conjugacy classes containing one element. This gives a total of $2^{n-1} + 3$ conjugacy classes.

In the quotient group $\Gamma_n/Z(\Gamma_n)$ the generators $e_1, \ldots, e_n$ all commute so the quotient $\Gamma/Z(\Gamma_n)$ is the abelian group Z_2^{n-1}. In other words there is a short exact sequence

$$0 \longrightarrow Z(\Gamma_n) \longrightarrow \Gamma_n \longrightarrow Z_2^{n-1} \longrightarrow 0$$

This shows that every representation of Z_2^{n-1} gives a representation of Γ_n. Since Z_2^{n-1} has 2^{n-1} inequivalent one dimensional irreducible representations so also does Γ_n.

Now Γ_n has $2^{n-1} + 3$ inequivalent irreducible representations and has 2^{n-1} inequivalent one-dimensional irreducible representations. Hence there are three irreducible representations that have not been accounted for. The sum of the squares of the dimensions of these three inequivalent representations is $2^{n+1} - 2^{n-1} = 3.2^{n-1}$. These three representations are indexed by the three non-trivial irreducible representations of the centre. Each of these representations when restricted to Γ_{n-1} is the irreducible representation of dimension $2^{(n-1)/2}$.

The restriction of these representations to C_{n-1} is a direct sum of two representations given by the basis of the last copy of M. Each of these representations is irreducible because it is irreducible when regarded as a representation of C_{n-2} They are inequivalent representations of C_{n-1} since by Schurs lemma the map which gives the equivalence as representations of C_{n-2} is unique up to scalar multiplication and these maps do not give an equivalence of representations of C_{n-1}.

Finally there is the problem of which of the irreducible representations of Γ_n come from irreducible representations of the Clifford algebra. It is easy to see that the Clifford algebra has no one dimensional representations. This shows that for n even the only possible representation of the Clifford algebra has dimension $2^{n/2}$. Since the dimension of the Clifford algebra is 2^n this shows that the Clifford algebra is a matrix algebra. However a more restrictive condition is that a representation of Γ_n in which the element -1 acts trivially cannot come from a representation of the Clifford algebra. This again excludes the one dimensional representations. For n odd this condition also excludes one of the irreducible representations of dimension $2^{(n-1)/2}$ in the case when n is odd. The remaining two irreducible representations each come from inequivalent irreducible representations of the Clifford algebra. In one of these representations the central element $e_1 \dots e_n$ acts by 1 and the other by -1.

Theorem For n even the complex Clifford algebra is isomorphic to the algebra of $2^{n/2} \times 2^{n/2}$ matrices. For n odd the complex Clifford algebra is isomorphic to the direct sum of two copies of the algebra of $2^{(n-1)/2} \times 2^{(n-1)/2}$ matrices.

5.Construction of the spin representations.

The Clifford algebras are Z_2-graded algebras and for any vector spaces V and W there is an algebra isomorphism

$$C(V \oplus W) \cong C(V) \otimes C(W)$$

where the tensor product is the Z_2-graded tensor product. If A and B are Z_2 graded algebras then the Z_2-graded tensor product is the algebra whose vector space is $A \otimes B$ and whose multiplication is defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = \begin{cases} a_1 a_2 \otimes b_1 b_2 & \text{if } b_1 \text{ and } a_2 \text{ are both even or both odd} \\ -a_1 a_2 \otimes b_1 b_2 & \text{if } b_1 \text{ and } a_2 \text{ are even and odd} \end{cases}$$

In particular if V is even dimensional say $dim(V) = 2m$ then there is an algebra isomorphism

$$C(V) \cong \otimes^m C(C^2)$$

Now $C(C^2)$ is the quaternion algebra which is isomorphic to the algebra of 2×2 matrices. Let M be the two-dimensional irreducible representation of this algebra. Then the algebra isomorphism gives an action of $C(V)$ on $\otimes^m M$. The following carries out this construction and this is used to show explicitly that for n even the Clifford algebra is a matrix algebra.

The construction is by induction on n for n even. The start of the induction is the case $n = 2$. Define two matrices by

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then these matrices satisfy the Clifford relations and generate the whole matrix algebra. The Clifford algebra in this case is the algebra of quaternions. For the inductive step assume that $2^m \times 2^m$ matrices $e_1, \dots, e_{2m}$ have been defined and satisfy the Clifford relations. Then define $2^{m+1} \times 2^{m+1}$ matrices $E_1, \dots, E_{2m+2}$ by

$$E_i = \begin{pmatrix} e_i & 0 \\ 0 & e_i \end{pmatrix} \quad \text{for} \quad 1 \le i \le 2m$$
$$E_{2m+1} = \begin{pmatrix} ie_1 \dots e_{2m} & 0 \\ 0 & -ie_1 \dots e_{2m} \end{pmatrix}$$
$$E_{2m+2} = \begin{pmatrix} 0 & ie_1 \dots e_{2m} \\ ie_1 \dots e_{2m} & 0 \end{pmatrix}$$

Then the proof that these matrices satisfy the Clifford relations uses the following identities

$$(e_1 \dots e_{2m})^2 = 1$$
$$e_i(e_1 \dots e_{2m}) = -(e_1 \dots e_{2m})e_i$$

These identities are consequences of the Clifford relations.

Next we show that this algebra homomorphism from the Clifford algebra to the matrix algebra is surjective. Since both algebras have the same dimension it follows that this homomorphism is an isomorphism. Let $E_{i,j}$ denote the elementary matrix with a 1 in the (i,j) position and let F denote the matrix all of whose entries are 1. Then it is sufficient to show that the matrices $E_{i,i}$ and F are in the image because

$$E_{i,j} = E_{i,i} F E_{j,j}$$

6. The Dirac equation.

As an application we describe the Dirac's relativistic equation for the electron. The Clifford algebra of Minkowski space has dimension 16 and the spin representation has dimension 4. The spin representation is given explicitly by the following matrices

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

These matrices satisfy the relations

$$\gamma_{1,2,3}^2 = -1 \quad \gamma_4^2 = 1 \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0 \quad \text{for } \mu \neq \nu$$

The states of the electron are vector valued functions on Minkowski space and they take values in the spin representation. Since we have chosen a basis of the spin representation we can write a function Φ from Minkowski space to the spin representation as

$$\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)$$

The Dirac equation for a free electron is then

$$ih\gamma_4\partial_4\Phi - ihc\sum_{\mu=1}^{3}\gamma_\mu\partial_\mu\Phi = mc^2\Phi$$

This equation is usually written with $c = 1$ and $h = 1$ in the more concise form

$$\gamma_\mu\partial_\mu\Phi = -im\Phi$$

This represents a system of four simultaneous first-order linear partial differential equations. Replacing 1,2,3,4 by x, y, z, t respectively these equations

are :

$$\frac{\partial\Phi_x}{\partial t} - c\frac{\partial\Phi_z}{\partial t} - c\frac{\partial\Phi_t}{\partial x} + i\frac{\partial\Phi_t}{\partial y} = \frac{-imc^2}{h}\Phi_x$$
$$\frac{\partial\Phi_y}{\partial t} - c\frac{\partial\Phi_z}{\partial x} - ic\frac{\partial\Phi_x}{\partial y} - c\frac{\partial\Phi_t}{\partial z} = \frac{-imc^2}{h}\Phi_y$$
$$-c\frac{\partial\Phi_x}{\partial z} + c\frac{\partial\Phi_y}{\partial x} - ic\frac{\partial\Phi_y}{\partial y} - \frac{\partial\Phi_z}{\partial t} = \frac{-imc^2}{h}\Phi_z$$
$$c\frac{\partial\Phi_x}{\partial x} + ic\frac{\partial\Phi_z}{\partial y} - c\frac{\partial\Phi_y}{\partial z} - \frac{\partial\Phi_t}{\partial t} = \frac{-ihc^2}{h}\Phi_t$$

Suppose S is a transformation of the spin representation. Then S transforms the wave function by $\Phi' = S\Phi$ and the γ-matrices by $\gamma'_\mu = S\gamma_\mu S^{-1}$. Then it is an exercise to check that the Dirac equation is not changed.

The left hand side of these equations is the Dirac operator. Another excercise is to check that the square of the Dirac operator is a diagonal matrix with the Laplacian in each diagonal entry.

This equation is for Minkowski space. For a general space-time we need some additional geometric data. A space-time is an open 4-manifold with a Lorentz metric. Each tangent space is then a copy of Minkowski space. Since the construction of the Clifford algebra of a vector space is natural we can construct the Clifford algebra bundle. This is a vector bundle whose fibre at any point is the Clifford algebra of the tangent space at that point. A spin structure on the manifold is then a vector bundle such that the fibre at each point is a spin representation of the fibre of the Clifford algebra bundle. A general space-time may not admit any spin structure or there may be several inequivalent spin-structures. The number of inequivalent spin-structures on a space-time M is the order of the cohomology group $H^1(M : Z_2)$. In fact this group acts freely and transitively on the set of inequivalent spin-structures. The reason for this is that the cohomology group $H^1(M; Z_2)$ can be regarded as the group of equivalence classes of real line bundles over M where the group operation is tensor product. Now given a real vector bundle whose fibre at each point is a spin representation of the fibre of the real Clifford algebra bundle and a real line bundle then the tensor product is also a spin-structure.

Given this data then the Dirac operator acts on the global sections of the bundle of spin representations. The Dirac operator is a first order linear diferential operator and is defined by simply replacing ∂/∂_μ by ∇_μ

the covariant derivative in the direction μ.

REPRESENTATIONS OF $SU(2)$

The unit vectors in four dimensional Euclidean space form the compact three dimensional manifold S^3. This manifold is also a Lie group. There are three ways to define the group structure :

1 as $Sp(1)$ i.e. as the unit quaternions

2 as $SU(2)$ i.e. as the unitary 2×2 complex matrices with determinant 1

3 as $Spin(3)$ i.e. as the double cover of the identity component of the group of isometries of 3 dimensional Euclidean space

In this lecture we take the second definition.

Let $C[x, y]$ be the commutative algebra of polynomials over the complex field in two indeterminates x and y. The group $SU(2)$ acts on this algebra by algebra automorphisms. This action is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\sum_{i,j} p_{i,j} x^i y^j \right) = \sum_{i,j} p_{i,j} (dx - by)^i (-cx + ay)^j$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ and $\sum_{i,j} p_{i,j} x^i y^j \in C[x, y]$.

Let V^λ be the space of homogenous polynomials of degree λ for $\lambda = 0, 1, 2, \ldots$. Note that the dimension of V^λ is $\lambda + 1$. Then clearly V^λ is an invariant subspace and there is a direct sum decomposition

$$C[x, y] \cong \bigoplus_{\lambda \geq 0} V^\lambda$$

Theorem For every $\lambda \geq 0$ the space V^λ is an irreducible representation of $SU(2)$. Furthermore every irreducible representation of $SU(2)$ is equivalent to V^λ for some λ.

The Lie algebra of $SU(2)$ is a real 3 dimensional vector space and there is a definition of the Lie bracket for each of the three definitions of the group structure on the manifold S^3.

1 in terms of the unit quaternions the the Lie algebra is the vector space of imaginary quaternions. A vector space basis is i,j,k and the Lie bracket is given by

$$[i, j] = 2k \quad [j, k] = 2i \quad [k, i] = 2j$$

2 in terms of 2×2 complex matrices the Lie algebra is the vector space of 2×2 skew-Hermitian matrices whose trace is 0. The Lie bracket is the commutator. A vector space basis is

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

These satisfy the same commutation relations as i,j,k.

3 in terms of $SO(3)$ the Lie algebra is three dimensional space. The Lie bracket is the vector cross product. If e_1, e_2, e_3 is an orthonormal basis then this Lie bracket is given by

$$[e_1, e_2] = e_3 \quad [e_2, e_3] = e_1 \quad [e_3, e_1] = e_2$$

An isomorphism with the first definition is

$$e_1 \mapsto i/2 \quad e_2 \mapsto j/2 \quad e_3 \mapsto k$$

The Lie algebra of any compact Lie group has the property that any representation can be written as the direct sum of irreducible representations. This is equivalent to the property that if W is a representation and V is a proper invariant subspace then there is an invariant subspace V such that $W \cong U \oplus V$. Instead of giving a proof we describe a naturally ocurring Lie algebra which does not have this property.

Definition The Heisenberg Lie algebra is a 3 dimensional Lie algebra of differential operators. A vector space basis is $1,x$ and d/dx. Calculating the commutator $[d/dx, x]$ gives

$$\begin{aligned}[d/dx, x](f) &= \frac{d}{dx}(xf) - x\frac{df}{dx} \\ &= f\end{aligned}$$

In other words this Lie algebra satisfies the relations

$$[d/dx, x] = 1 \quad [x, 1] = 0 \quad [d/dx, 1] = 0$$

This Lie algebra has a three dimensional representation defined by

$$1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$x = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$d/dx = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The vector space spanned by the third co-ordinate vector is a proper invariant subspace. However there is no invariant complement.

Exponentiating each of these matrices shows that the Heisenberg Lie algebra is the Lie algebra of the Lie group of matrices are of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}$$

This Lie group is known as the Heisenberg group. As a manifold the Heisenberg group is a real 3 dimensional vector space.

In these notes we only consider complex representations both of the group and of the Lie algebra. The complex Lie algebra is defined by taking any of the three constructions and regarding the matrices as a basis for a complex vector space. This gives the Lie algebra of $SL(2,C)$. There are other Lie groups whose complex Lie algebra are isomorphic to this complex Lie algebra but none of them are compact. Examples of such Lie groups are $SL(2,R)$,$SO(2,1)$,$U(1,1)$ and their covering groups. The following is the Cartan presentation of the Lie algebra of $SL(2,C)$. This is a complex Lie group and is the complexification of $SU(2)$. Regarded as a presentation of a real Lie algebra this definition gives the Lie algebra of $SL(2,R)$.

Definition A vector space basis for $sl(2)$ is the matrices

$$e_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad e_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices satisfy the relations

$$[h, e_\alpha] = 2e_\alpha \quad [h, e_{-\alpha}] = -2e_{-\alpha} \quad [e_\alpha, e_{-\alpha}] = h$$

Definition Let $\phi: SU(2) \longrightarrow GL(V)$ be a representation of $SU(2)$ on the vector space V. Then V is also a representation for $su(2)$ the Lie algebra of $SU(2)$. The homomorphism of Lie algebras $\phi_*: su(2) \longrightarrow gl(V)$ is defined by

$$\phi_*(X) = \frac{d}{dt}\phi(\exp tX)\,|_{t=0}$$

where $X \in su(2)$ and $t \mapsto \exp tX$ is a particular function which represents X.

Apply this definition to the action of $SU(2)$ on $C[x,y]$. The exponentials of e_α and $e_{-\alpha}$ are

$$\exp(te_\alpha) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad \exp(te_{-\alpha}) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

Now calculating the derivative gives

$$\begin{aligned} \frac{d}{dt}\phi(\exp t\alpha)(p) &= \frac{d}{dt}p(x - ty, y) \\ &= -y\frac{\partial}{\partial x}p(x - ty, y) \end{aligned}$$

Evaluating at $t = 0$ gives

$$\phi_*(e_\alpha) = -y\frac{\partial}{\partial x}$$

A similar calculation shows that

$$\phi_*(e_{-\alpha}) = -x\frac{\partial}{\partial y}$$

Now another calculation or the relation $[e_\alpha, e_{-\alpha}] = h$ shows that

$$\phi_*(h) = y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}$$

This defines an action of $su(2)$ on the vector space $C[x,y]$. Again it is clear that for each λ the subspace V^λ is invariant and there is a direct sum decomposition

$$C[x,y] = \bigoplus_\lambda V^\lambda$$

Next we prove the following theorem.

Theorem For every $\lambda \geq 0$ the space V^λ is an irreducible representation of $sl(2)$. Furthermore every irreducible representation of $sl(2)$ is equivalent to V^λ for some λ.

Assume that V is an irreducible representation of the Lie algebra. Define subspaces V_μ by

$$V_\mu = \{v \in V : hv = \mu v\}$$

If $V_\mu \neq 0$ then μ is called a weight of V. Now if $v \in V_\mu$ then $e_\alpha v \in V_{\mu+2}$ and $e_{-\alpha}v \in V_{\mu-2}$. For this reason e_α is called a raising operator and $e_{-\alpha}$ is called a lowering operator. The proof is a short calculation using the relations.

$$he_\alpha v = [h, e_\alpha]v + e_\alpha hv = 2e_\alpha v + \mu e_\alpha v = (\mu + 2)e_\alpha v$$

and similarly for $e_{-\alpha}v$.

Now because V is finite dimensional there is a largest weight say λ. Choose $v_0 \in V_\lambda$ such that $v_0 \neq 0$. Then for $1 \leq \mu$ define $v_\mu \in V_{\lambda - 2\mu}$ by

$$v_\mu = \frac{1}{\mu!}e_{-\alpha}^\mu v_0$$

Then let m be the integer such that $v_m \neq 0$ and $v_{m+1} = 0$. Then we have

$$0 = e_\alpha . v_{m+1} = (\lambda - m)v_m$$

and since $v_m \neq 0$ this shows that $m = \lambda$. Then the vector subspace of V spanned by $v_0, \ldots, v_\lambda$ is an invariant subspace of V. Since V is irreducible this must be all of V. Also these are linearly independent because they are eigenvectors of h with distinct eigenvalues. This shows that they are a basis for V.

There is an isomorphism between these irreducible representations and the representations on homogenous polynomials. To describe the isomorphism note that

$$hx^p y^q = (q-p)x^p y^q$$

This shows that the highest weight vector is y^n and that h is already diagonal.

There is another approach to the representation theory of compact Lie groups in terms of characters. A character is a smooth complex valued function on the group which is invariant under conjugation. Every element of the group lies in a maximal torus. Any two maximal tori are conjugate. For $U(n)$ this follows from the observation that any unitary matrix can be diagonalised. This implies that a character is determined by its restriction to a maximal torus. The maximal torus of $SU(2)$ is $U(1)$. The one parameter subgroup generated by h is a maximal torus. Since we have already determined the eigenvectors of h in each irreducible representation we have already determined the restriction of each irreducible character to $U(1)$. Write q for $e^{i\theta}$. Then the character of V^n is

$$q^{-n} + q^{-n+2} + \cdots + q^n = q^{-n}\left(\frac{1-q^{2n+2}}{1-q^2}\right)$$

The Clebsch-Gordan formula gives the decomposition of the tensor product of two irreducibles into a direct sum of two irreducibles. Assume $n \leq m$ then the formula is

$$V^n \otimes V^m = \bigoplus_{p=0}^{n} V^{m-n+2p}$$

This can be proved using characters. The character of a tensor product is the product of the characters. To prove the Clebsch-Gordan formula it is sufficient to show how to write the product of two irreducible characters as a sum of irreducible characters. For example

$$\begin{aligned}(q^{-3}+q^{-1}+q+q^3)(q^{-2}+1+q^2) \\ = \; q^{-5}+q^{-3}+q^{-1}+q \\ +q^{-3}+q^{-1}+q+q^3 \\ +q^{-1}+q+q^3+q^5\end{aligned}$$

Now taking the top edge and the right hand edge of this rectangle gives $q^{-5}+q^{-3}+\cdots q^5$. Removing these two edges and then repeating gives $q^{-3}+q^{-1}+q+q^3$. The remaining terms are $q^{-1}+q$. This shows that $V^3 \otimes V^2$ is $V^5 \oplus V^3 \oplus V^1$. This method also proves the general case.