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COLLEGE

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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HODGE THEORY.

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REVIEW OF MULTILINEAR ALGEBRA

Let V be a finite dimensional real vector space and V^* it's dual.

Definition: A tensor of type (p,q) is a multilinear map

$$t: \underbrace{V^* \times \cdots \times V^*}_{p \text{ times}} \times \underbrace{V \times \cdots \times V}_{q\text{- times}} \longrightarrow \mathbb{R}$$

The set of (p,q)-tensor has a natural structure of vector space that we will denote by $\mathcal{T}^{(p,q)}(V)$.

Examples $\mathcal{T}^{(0,1)}(V) = V^*$, $\mathcal{T}^{(1,0)}(V) = (V^*)^* \cong V$ and we will set $\mathcal{T}^{(0,0)} = \mathbb{R}$.

Given $t \in \mathcal{T}^{(p,q)}(V)$ and $s \in \mathcal{T}^{(p',q')}(V)$ the tensor product $t \otimes s$ is defined as the $(p+p', q+q')$-tensor

$$t\otimes s(\tilde{X}_1, \dots, \tilde{X}_{p+p'}, X_1, \dots, X_{q+q'}) =$$

$$= t(\tilde{X}_1, \dots, \tilde{X}_p, X_1, \dots, X_q)\cdot s(\tilde{X}_{p+1}, \dots, \tilde{X}_{p+p'}, X_{q+1}, \dots, X_{q+q'})$$

where $\tilde{X}_i \in V^*$ and $X_j \in V$.

It is easily seen that the tensor product is associative and distributive with respect to ~~the~~ tensor sums ~~[illegible]~~ and therefore induces a structure of (graded) algebra in the graded vector space

$$\mathcal{T}(V) = \bigoplus_{p,q \geq 0} \mathcal{T}^{(p,q)}(V)$$

We will be interested in special tensors of type $(0,q)$. Let Σ_q be the group of permutations of the set $\{1, \dots, q\} \subseteq \mathbb{N}$. For $\sigma \in \Sigma_q$ we will denote by $|\sigma|$ its parity, i.e. $|\sigma| = 0$ (resp. $|\sigma| = 1$) if σ is the product of an even (resp. odd) number of transpositions.

A tensor $t \in \mathcal{T}^{(0,q)}(V)$ is said to be symmetric if $\forall \sigma \in \Sigma_q$, $\{X_1, \dots, X_q\} \subseteq V$ we have

$$t(X_1, \dots, X_q) = t(X_{\sigma(1)}, \dots, X_{\sigma(q)})$$

and a tensor $\omega \in \mathcal{T}^{(0,q)}(V)$ is said to be skew symmetric if

$$\omega(X_1, \dots, X_q) = (-1)^{|\sigma|}\, \omega(X_{\sigma(1)}, \dots, X_{\sigma(q)})$$

A skew symmetric tensor of type $(0,q)$ will be called an exterior form of degree q or simply a q-form. We will denote by $A^q(V)$ the space of q-forms.

Examples: (a) Let V be an inner product space. The map $t(X,Y) = \langle X, Y\rangle$ is a symmetric 2-tensor.

(b) Let V be a Lie algebra with an inner product. Fix $Z \in V$. The map $\omega(X,Y) = \langle [X,Y]$ is a 2-form. ~~[illegible]~~

(c) Consider $V = \mathbb{R}^n$ and let $E_1, \dots, E_n$ be the standard basis. The map

$$\mathrm{Det}(X_1, \dots, X_n) = \det(\langle X_i, E_j \rangle)$$

is an n-form. More generally, fix $1 \le j_1 < \dots < j_q \le n$ and consider

$$D_{j_1 \dots j_q}(X_1, \dots, X_q) = \det(\langle X_i, E_{j_k} \rangle)_{1 \le i, k \le p}$$

The maps above are p-forms and form a basis for the space of p-forms in $\mathbb{R}^n$.

We have a natural projection $A : \mathcal{T}^{(0,q)}(V) \to A^q(V)$ given by

$$(At)(X_1, \dots, X_q) = \frac{1}{q!} \sum_{\sigma \in \Sigma_q} (-1)^{\sigma} t(X_{\sigma(1)}, \dots, X_{\sigma(q)})$$

The tensor product of two forms is not in general a form, but we can use A to obtain a form.

<u>Definition</u>: The <u>exterior</u> or <u>wedge product</u> of two forms $\omega \in A^p(V)$ and $\tau \in A^q(V)$ is defined as:

$$\omega \wedge \tau = A(\omega \otimes \tau)$$

The following properties of the wedge product are easly checked:

(a) $\omega \wedge (\phi \wedge \tau) = (\omega \wedge \phi) \wedge \tau$, $\omega \wedge (\phi + \tau) = \omega \wedge \phi + \omega \wedge \tau$

(b) if $\omega \in A^p(V)$, $\tau \in A^q(V)$ then $\omega \wedge \tau = (-1)^{pq} \tau \wedge \omega$

(c) If $\tilde{X}_1, \dots, \tilde{X}_n$ is a basis for $V^* = A^1(V)$ then the set $\{ \tilde{X}_{i_1} \wedge \dots \wedge \tilde{X}_{i_p} : 1 \le i_1 < \dots < i_p \le n \}$ is a basis for $A^p(V)$.

<u>Remark</u>: The properties above imply that $A^*(V) = \bigoplus_{p \ge 0} A^p(V)$ is an associative, graded commutative algebra (extending the wedge product by linearity) of dimension 2^n (note that $A^p(V) = 0$ for $p > n$!).

Conversely suppose G is a 2^n-dimensional algebra over the reals such that:

1) $1 \in G$, $V^* \subseteq G$ and 1 and V^* generates G

2) $\varphi \wedge \varphi = 0 \quad \forall \varphi \in G$

then G is isomorphic to $A^*(V)$.

Let V, W be vector spaces and $F : V \to W$ a liner map. Then F gives a map

$$F^* : A^*(W) \longrightarrow A^*(V)$$

extending by linearity the map

$$\left[(F^*)^p \omega \right] (X_1, \dots, X_p) = \omega(FX_1, \dots, FX_p).$$

It is easily seen that F^* is an algebra morphism, i.e. $F^*(\omega \wedge \tau) = F^*\omega \wedge F^*\tau$ and verifies the relations:

a) $V = W$ $F = Id_V \Rightarrow F^* = 1_{A^*(V)}$

b) Given $F: V \to W$, $G: W \to U$, $(G\circ F)^* = F^* \circ G^*$ so that we have a contravariant functor from the category of finite dimensional vector spaces and linear map to the category of graded algebras.

Now, let ~~now~~ V be a finite dimensional vector space with inner product $\langle\,,\rangle$. The inner product induces an isomorphism:

$$\sim : V \to V^*\ ,\quad X \rightsquigarrow \tilde{X}\ ,\quad \tilde{X}(Y) = \langle X, Y\rangle$$

which gives an isomorphism $\sim : A^p(V^*) \to A^p(V)$

$$X_1 \wedge \cdots \wedge X_p \longrightarrow \widetilde{X_1 \wedge \cdots \wedge X_p} = \tilde{X}_1 \wedge \cdots \wedge \tilde{X}_p$$

We ~~have also~~ also have an induced inner product in $A^p(V)$ extending the ~~bilinear~~ bilinear map

$$\langle \tilde{X}_1 \wedge \cdots \wedge \tilde{X}_p , \tilde{Y}_1 \wedge \cdots \wedge \tilde{Y}_p \rangle = \det(\langle X_i, Y_j\rangle)$$

~~We~~ If $X_1, \dots, X_n$ is an orthonormal base for V the above inner product is characterized by the fact that $\{\tilde{X}_{i_1} \wedge \cdots \wedge \tilde{X}_{i_p} : 1 \le i_1 < \cdots < i_p \le n\}$ is an orthonormal basis for $A^p(V)$.

A basis $\{X_1, \dots, X_n\}$ of V determines a generator $\tilde{X}_1 \wedge \cdots \wedge \tilde{X}_n$ of $\Lambda^n(V) \cong \mathbb{R}$ and two such generators are related by the determinant of the change of basis. In particular two basis determine the same orientation if and only if the associate n-forms are in the same

component of $(\Lambda^n(V) - \{0\})$.

Let us now fix an orientation on the inner product space V. For every $p = 0, \dots, n$ define the operator

$$* : A^p(V) \longrightarrow A^{n-p}(V)$$

extending by linearity the map

$$*(\tilde{X}_{i_1} \wedge \cdots \wedge \tilde{X}_{i_p}) = \tilde{X}_{j_1} \wedge \cdots \wedge \tilde{X}_{j_{n-p}}$$

where $\{X_1, \dots, X_n\}$ is an orthonormal base in V, $\{\tilde{X}_1, \dots, \tilde{X}_n\}$ its dual basis and $j_1, \dots, j_{n-p}$ are such that $\{i_1, \dots, i_p, j_1, \dots, j_{n-p}\}$ is an <u>even</u> permutation of $\{1, \dots, n\}$.

Alternatively $*$ is characterized by the condition

$$(\tilde{X}_{i_1} \wedge \cdots \wedge \tilde{X}_{i_p}) \wedge *(\tilde{X}_{i_1} \wedge \cdots \wedge \tilde{X}_{i_p}) = \tilde{X}_1 \wedge \cdots \wedge \tilde{X}_n\ .$$

The operator $*$ is called the <u>Hodge operator</u> and it is "almost an involution" in the sense that

$$* \circ * = (-1)^{p(n-p)} (\mathrm{Id})$$

DIFFERENTIAL FORMS AND de RHAM COHOMOLOGY

Let M^n be an n-dimensional smooth (i.e. C^∞) differentiable manifold. We will denote by T_xM the tangent space of M at $x \in M$ and by $TM = \bigcup_{x\in M} T_xM \xrightarrow[\pi]{} M$ the tangent bundle.

We can apply the costructions of the previous paragraph and obtain:

$\mathcal{T}^{p,q}_x(M) = \mathcal{T}^{p,q}(T_xM)$ = space of tensors of type (p,q) at x

$A^q_x(M) = A^q(T_xM)$ = space of q-forms at x

$\mathcal{T}^{p,q}(M) = \bigcup_{x\in M} \mathcal{T}^{p,q}_x(M)$ = bundle of tensors of type (p,q)

$A^q(M) = \bigcup_{x\in M} A^q_x(M)$ = bundle of q-forms.

As in the case of TM we can give to $\mathcal{T}^{(p,q)}(M)$ and $A^q(M)$ a structure of smooth manifolds such that the canonical projections onto M are smooth vector bundles.

If $U \subseteq M$ we will denote by $\mathcal{F}(U)$ the space of smooth real valued function on U and by $\mathcal{H}(U)$ the space of smooth vector fields, i.e. smooth sections of $TM|_U \to U$.

(If U is not open, a function or vector field on U is smooth if it admits a smooth extention to some open neighborhood of U).

Definition: A differential p-form on U is a smooth section of the bundle $A^p(M)|_U \to U$. Similarly a smooth tensor field of type (p,q) on U is a smooth section of $\mathcal{T}^{(p,q)}(M)|_U \to U$. We will denote by $\Lambda^p(M)$ the space of differential p-forms and set $\Lambda^*(M) = \oplus \Lambda^p(M)$.

Remark: Given $\omega: U \to A^p(M)|_U$, $\omega(x) \in A^p_x(M)$, ω is smooth if and only if given $X_1, \dots, X_p \in \mathcal{H}(U)$ the function $x \mapsto \omega(x)(X_1(x), \dots, X_p(x))$ is ~~smo~~ smooth.

Remark: (tensoriality criterium). Given a (p,q)-tensor field $t: U \to \mathcal{T}^{(p,q)}(M)|_U$, t induces an $\mathcal{F}(U)$-multilinear map

$$\tilde{t}: \Lambda^1(U)\times\dots\times\Lambda^1(U)\times\mathcal{H}(U)\times\dots\times\mathcal{H}(U) \to \mathcal{F}(U)$$

Conversely given an $\mathbb{R}$-multilinear map

$$\tilde{\ell}: \Lambda^1(U)\times\dots\times\Lambda^1(U)\times\mathcal{H}(U)\times\dots\times\mathcal{H}(U) \to \mathcal{F}(M)$$

the $\tilde{\ell}$ is induced by a tensor if and only if it is $\mathcal{F}(M)$-multilinear.

Examples (a) Let M be a riemannian manifold with Levi-Civita connection ∇. The map

$$\tilde{\nabla}: \mathcal{H}(M)\times\mathcal{H}(M)\times\mathcal{H}(M) \to \mathcal{F}(M)$$

$$\tilde{\nabla}(X,Y,Z) = \langle \nabla_X Y, Z \rangle$$

is <u>not</u> induced by a tensor field. However as a function of the 1st and 3rd variable (i.e. fixing Y) it is a tensor field.

(b) In the same situation the map $\tilde{R}: \mathcal{X}(M)\times\mathcal{X}(M)\times\mathcal{X}(M)\times\mathcal{X}(M)\to\mathcal{F}(M)$

$$\tilde{R}(X,Y,Z,W) = \langle \bar\nabla_X\bar\nabla_Y Z - \bar\nabla_Y\bar\nabla_X Z - \bar\nabla_{[X,Y]} Z, W\rangle$$ is induced by a tensor, the Riemann curvature tensor.

Let M, N be smooth manifolds and $\varphi: M \to N$ a smooth map. The differential of φ at $x \in M$ is a linear map $(d\varphi)_x : T_x M \to T_{\varphi(x)} N$ which induces a map at the level of differential forms

$$\varphi^* : \Lambda^p(N) \to \Lambda^p(M); \quad \varphi^*\omega(X_1,\dots,X_p) = \omega(d\varphi X_1,\dots,d\varphi X_p)$$

The form $\varphi^*\omega$ is called the pull back of ω.

Let now $M = \Omega \subseteq \mathbb{R}^n$ be an open set and $\{x_1,\dots,x_n\}$ coordinates in $\mathbb{R}^n$. Then $\mathcal{X}(\Omega)$ is spanned over $\mathcal{F}(\Omega)$ by the coordinate vector fields $\{\partial/\partial x_i\}$ and $\Lambda^1(\Omega)$ by the dual 1-forms dx_i, $dx_i(\partial/\partial x_j) = \delta_{ij}$.

If $\omega \in \Lambda^p(M)$ then there exist functions $\omega_{i_1\dots i_p}$ such that

$$\omega = \sum_{i_1<\dots<i_p} \omega_{i_1\dots i_p}\, dx_{i_1}\wedge\dots\wedge dx_{i_p}$$

since for $x \in \Omega$ $\{(dx_{i_1})_x\wedge\dots\wedge(dx_{i_p})_x : i_1<\dots<i_p\}$ is a basis for ~~[illegible]~~ $\Lambda^p_x(\Omega)$

The smoothness of ω is equivalent to the smoothness of the $\omega_{i_1\dots i_p}$.

If $f \in \mathcal{F}(\Omega) = \Lambda^0(M)$, $df \in \Lambda^1(\Omega)$, $(df)(X) = Xf =$ directional derivative of f in the X direction.

The operator $d: \Lambda^0(\Omega) \to \Lambda^1(\Omega)$ may be extended to an operator

$$d : \Lambda^*(\Omega) \to \Lambda^*(\Omega)$$

defining, for $\omega = \sum_{i_1<\dots<i_p} \omega_{i_1\dots i_p}\, dx_{i_1}\wedge\dots\wedge dx_{i_p} \in \Lambda^p(\Omega)$, the $(p+1)$-form

$$d\omega = \sum_{i_1<\dots<i_p} d(\omega_{i_1\dots i_p})\wedge dx_{i_1}\wedge\dots\wedge dx_{i_p} = \sum_{k,\, i_1<\dots<i_p} \partial/\partial x_k(\omega_{i_1\dots i_p})\, dx_k\wedge dx_{i_1}\wedge\dots\wedge dx_{i_p}.$$

It is easy to see that the operator defined above is the unique $\mathbb{R}$-linear operator such that

1) $d(\Lambda^p(\Omega)) \subseteq \Lambda^{p+1}(\Omega)$ and $d|\Lambda^0(\Omega)$ is the usual differential
2) if $\omega \in \Lambda^p(\Omega)$, $\tau \in \Lambda^q(\Omega)$, then $d(\omega\wedge\tau) = d\omega\wedge\tau + (-1)^p\omega\wedge d\tau$
3) $d \circ d = 0$

Such an operator can be defined on arbitrary manifolds:

<u>Proposition</u>: Let M be a differentiable manifold. Then there exists a unique $\mathbb{R}$-linear operator

$$d : \Lambda^*(M) \to \Lambda^*(M)$$

which ~~[illegible]~~ satisfies the conditions 1), 2), 3) above.

Proof: We will first prove that if such an operator exists it is unique. Suppose given such an operator:

Claim: Let $x \in M$, $\omega \in \Lambda^p(M)$ and $\omega \equiv 0$ in a neighborhood of x. Then $(d\omega)(x) = 0$.

Let V be such a neighborhood, $U \subseteq M$ an open set such that $x \in U \subseteq \bar{U} \subseteq V$ and $f \in \mathcal{F}(M)$ such that $f \equiv 0$ on $\bar{U}$ and $f \equiv 1$ outside V. Then $\omega = f \cdot \omega$ and

$$(d\omega)(x) = d(f\cdot\omega)(x) = (df)(x) \wedge \omega(x) + f(x) \wedge (d\omega)(x) = 0 .$$

In particular if two forms ω, τ coincide on an open set V then $d\omega = d\tau$ in V.

By the claim is enough to prove that given $\omega \in \Lambda^p(M)$, $x \in M$ and V a coordinate neighborhood of x with coordinates $x_1, \dots, x_n$, then $d(\omega|_V)$ is uniquely determined.

But
$$\omega|_V = \sum_{i_1 < \dots < i_p} \omega_{i_1 \dots i_p}\, dx_{i_1} \wedge \dots \wedge dx_{i_p}$$
(with the usual abuse of notations!). As we have observed before the condition on d force:

$$(*) \quad d(\omega|_V) = \sum_{i_1 < \dots < i_p} d(\omega_{i_1 \dots i_p}) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

and the above expression is uniquely determined since d is uniquely determined on functions (the $\omega_{i_1 \dots i_p}$'s and the x_j's).

Now $(*)$ suggests also how to prove existence of such an operator. In fact we observe that the right hand side of $(*)$ defines an operator d_V on $\Lambda^*(V)$ for which 1), 2), 3) hold.

By uniqueness such an operator does not depends on the choice of coordinates. Moreover if V, W are coordinate neighborhoods with non void intersection d_V and d_W act on $\Lambda^p(V \cap W)$ according to 1), 2), 3) so, again by uniqueness, $d_V(\omega|_{V\cap W}) = d_W(\omega|_{V \cap W})$.

So we can glue togheter the local formulas to get an operator on $\Lambda^*(M)$. ■

It is often usefull to have alternative descriptions of d. Let $\omega \in \Lambda^p(M)$, $X_1, \dots, X_{p+1} \in \mathfrak{X}(M)$.

Claim:
$$(d\omega)(X_1, \dots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} X_i \cdot \omega(X_1, \dots, \hat{X}_i, \dots, X_{p+1}) + \sum_{j<k} (-1)^{j+k} \omega([X_j, X_k], X_1, \dots, \hat{X}_j, \dots, \hat{X}_k, \dots, X_{p+1}) .$$

where $\hat{X}_t$ means the exclusion of X_t, $[\cdot, \cdot]$ is the usual Lie brakets of vector fields and $X \cdot$ is the directional derivative of a function.

To see this we can express the right hand side in terms of local coordinates and recover (*) or show directly that the right hand side defines a form (i.e. is a skew symmetric $\mathcal{F}(M)$ multilinear map) and that the operator so defined has the properties 1), 2), 3). ■

Now, let M be a riemannian manifold with Levi-Civita connection ∇. Then ∇ acts on $\Lambda^*(M)$ as:

$$(\nabla_X\omega)(X_1,\dots,X_p) = X\cdot\omega(X_1,\dots,X_p) - \sum_{i=1}^{p}\omega(X_1,\dots,\nabla_X X_i,\dots,X_p)$$

where $\omega\in\Lambda^p(M)$, $X, X_1,\dots,X_p\in\mathcal{H}(M)$. It is easy to see that $\nabla_X\omega$ is in fact a form, i.e. is skew symmetric and $\mathcal{F}(M)$ linear.

<u>Claim</u>: $(d\omega)(X_1,\dots,X_p) = \sum_{1}^{p+1}(-1)^{i+1}(\nabla_{X_i}\omega)(X_1,\dots,\hat{X}_i,\dots,X_{p+1})$ where $X_1,\dots,X_{p+1}$ are orthonormal vectors.

To prove the claim we observe that the right hand side is a form, so given $x\in M$ and $X_1,\dots,X_{p+1}\in T_xM$ we can choose arbitrary extentions of the X_i's in a neighborhood of x to do the calculations at x. It is convinient to extend the X_i's to normal coordinate vector fields that, in particular, have the properties $[X_i,X_j]=0$ and $(\nabla_{X_i}X_j)(x)=0$.

With this choice the above formula is just the one given in the previous claim. ■

The operator d has a good behavior with respect to maps:

<u>Proposition</u>: Let $\varphi: M\to N$ be a smooth map. Then

$$d\varphi^*\omega = \varphi^*d\omega$$

<u>Proof</u>: If $f\in\Lambda^0(N)$, $\varphi^*f = f\circ\varphi$. Therefore

$$d(\varphi^*f) = df\circ d\varphi = \varphi^*df.$$

Suppose the relation valid for q-forms, $q<p$ and let $\omega\in\Lambda^p(N)$. We can suppose ω is supported in a coordinate neighborhood. In this case ω is sum of forms of the type $\tau\wedge dx_i$, $\tau\in\Lambda^{p-1}$. Then

$d(\varphi^*(\tau\wedge dx_i)) = d(\varphi^*\tau\wedge\varphi^*dx_i) =$

$= d\varphi^*\tau\wedge\varphi^*dx_i + (-1)^{p-1}\varphi^*\tau\wedge d\varphi^*dx_i$

$= d\varphi^*\tau\wedge\varphi^*dx_i = \varphi^*d(\tau\wedge dx_i)$ ■

We have defined a sequence of vector spaces and $\mathbb{R}$-linear maps

$$0 \longrightarrow \Lambda^0(M) \xrightarrow{d} \Lambda^1(M) \longrightarrow \cdots \longrightarrow \Lambda^n(M) \longrightarrow 0$$

We set

$Z^p(M) = \operatorname{Ker}\{d : \Lambda^p(M) \to \Lambda^{p+1}(M)\}$ = space of <u>closed</u> forms

$B^p(M) = \operatorname{Im}\{d : \Lambda^{p-1}(M) \to \Lambda^p(M)\}$ = space of <u>exact</u> forms

the condition $d \circ d = 0$ is equivalent to $B^p(M) \subseteq Z^p(M)$ and therefore we can consider the quotient space

$$H^p_{dR}(M) = Z^p(M) / B^q(M)$$

which is called the <u>p^{th} deRham cohomology group of M</u>.

The following properties are easy to check:

a) If $\omega_i \in Z^{p_i}(M)$ $i = 1,2$, $\phi \in B^q(M)$ then

$$\omega_1 \wedge \omega_2 \in Z^{p_1+p_2}(M), \quad \omega_i \wedge \phi \in B^{p_i+q}(M)$$

In particular the wedge product induces a structure of graded algebra in the vector space

$$H^*_{dR}(M) = \bigoplus_{p=0}^{n} H^p_{dR}(M) \; ; \; [\alpha]\cdot[\beta] = [\alpha \wedge \beta]$$

b) If $\varphi : M \to N$ is a smooth map then

$$\varphi^*(Z^p(N)) \subseteq Z^p(M) ; \quad \varphi^*(B^q(M)) \subseteq B^q(M)$$

and therefore φ^* induces a graded algebra homomorphism

$$\varphi^* : H^*_{dR}(N) \longrightarrow H^*_{dR}(M)$$

Moreover if $\varphi = 1 : M \to M$ is the identity map then $\varphi^* = 1 : H^*_{dR}(M) \to H^*_{dR}(M)$ is the identity map and if $\varphi : M \to N$, $\psi : N \to P$ are smooth maps then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$. In other words we have constructed a functor, the de-Rham cohomology, from the category of smooth manifolds and smooth maps to the category of graded algebras and graded algebras morphisms.

<u>Examples</u>: (1) $H^0_{dR}(M)$ is the space of real valued function with zero differential. So if M is connected $H^0_{dR}(M) \cong \mathbb{R}$. In general $H^p_{dR}(M)$ is the direct product of the p^{th} cohomology of the connected components of M.

(2) Let $\Omega \subseteq \mathbb{R}^n$ be an open connected set, $\omega = \sum \omega_i \, dx_i \in \Lambda^1(\Omega)$. If $\gamma : [a,b] \subseteq \mathbb{R} \to \Omega$ is a smooth path, ~~then~~ $\gamma(t) = (x_1(t), \ldots, x_n(t))$ then

$$\int_\gamma \omega = \int_a^b \sum \omega_i (x_1(t) \ldots x_n(t)) \, \dot{x}_i(t) \, dt.$$

the following facts are well known from basic calculus:

(a) If $\gamma_1, \gamma_2 : S^1 \to \Omega$ are two closed curves homotopic through closed curves and ω is a closed form, then

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$$

(b) a closed form ω is exact, i.e. $\omega = df$ for some function f, if and only if for every closed curve $\gamma : S^1 \to \Omega$

$$\int_{\gamma} \omega = 0$$

In particular if the fundamental group of Ω is finite $H^1_{dR}(\Omega) = 0$! In fact if γ is a closed curve and $\pi_1(\Omega)$ is finite there exist a natural number k such that $\gamma^k = \gamma * \cdots * \gamma$ is homotopic to a constant. Therefore for all closed 1-form ω

$$\int_{\gamma} \omega = k^{-1} \int_{\gamma^k} \omega = 0 .$$

INTEGRATION AND de RHAM THEOREM.

We have looked at the problem of differentiation of a differential form. We now want to give ~~now~~ a quick look at the problem of integration and the relation between the two.

If $p \geq 1$ is an integer, we define the standard p-simplex

$$\Delta^p = \{ (x_1, \dots, x_p) \in \mathbb{R}^p : \sum x_i \leq 1 \quad x_i \geq 0 \}$$

For $p = 0$ we set $\Delta^0 = \{0\}$.

<u>Definition</u>: A <u>singular</u> (smooth) <u>p-simplex</u> on a differentiable manifold M is a smooth map

$$\sigma : \Delta^p \to M$$

Let $\sigma : \Delta^p \to M$ be a singular p-simplex and $\tilde{\sigma} : U \supseteq \Delta^p \to M$ a smooth extention of σ to an open neighborhood U of Δ^p in $\mathbb{R}^p$. Let $\omega \in \Lambda^p(M)$. The pull-back $\tilde{\sigma}^* \omega$ is a p-form in an open set $U \subseteq \mathbb{R}^p$ and therefore ~~then~~ there exits a function $f : U \to \mathbb{R}$ such that

$$\tilde{\sigma}^* \omega = f \, dx_1 \wedge \dots \wedge dx_p$$

We define

$$\int_{\sigma} \omega = \int_{\Delta^p} \tilde{\sigma}^* \omega = \int_{\Delta^p} f \, dx_1 \dots dx_p$$

A basic theorem in integration is Stokes theorem. To state it in our context we have to define integration along the boundary so we will start defining the boundary

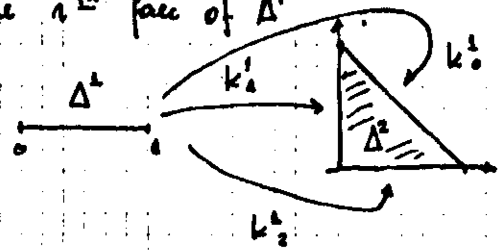
If $p \geq 0$ we define maps $K_i^p : \Delta^p \to \Delta^{p+1}$ $0 \le i \le p+1$

$K_0^0(0) = 1$ $\quad K_1^0(0) = 0$

$K_0^p(x_1 \dots x_p) = (1 - \sum_1^p x_i, x_1, \dots, x_p)$

$K_i^p(x_1 \dots x_p) = (x_1, \dots, x_{i-1}, 0, x_i, \dots, x_p)$ $\quad i \geq 1$

K_i^p is then the natural identification of Δ^p with the "i^{th} face of Δ^{p+1}"



So we define the i^{th} face of singular simplex $\sigma : \Delta^{p+1} \to M$ as the singular simplex $\sigma^{(i)} : \Delta^p \to M$

$$\sigma^{(i)} = \sigma \circ K_i^p$$

Really $\sigma^{(i)}$ is the restriction of σ to the i^{th} face of Δ^{p+1}.

The boundary of a simplex is a "sum of faces". To make this precise we introduce the concept of chain:

<u>Definition</u>: Let $S_p^\infty(M)$ denote the set of singular p-simplices. The group of p-chain is the real vector space with basis $S_p^\infty(M)$ and will be

denoted by $C_p^\infty(M)$.

A p-chain is therefore a finite (formal) linear combination of singular p-simplices with real coefficients.

We define the boundary operator

$$\partial_p : C_p^\infty(M) \to C_{p-1}^\infty(M)$$

extending by linearity the map

$$\partial_p(\sigma_p) = \sum_{i=0}^{p} (-1)^i \sigma^{(i)}$$

So we can extend the integral defined at the beginning to a bilinear map

$$\int : \Lambda^p(M) \times C_p^\infty(M) \to \mathbb{R}.$$

And Stokes theorem can be written as:

<u>THEOREM</u>: $\displaystyle\int_\gamma d\omega = \int_{\partial\gamma} \omega$

the proof reduces easily to the standard Green-Stokes theorems in $\mathbb{R}^p$.

We have an analogous situation as we had for the de Rham cohomology: We have a sequence of vector spaces and linear maps

$$\cdots \to C_{p+1}^\infty(M) \xrightarrow{\partial_{p+1}} C_p^\infty(M) \xrightarrow{\partial_p} \cdots \to C_0^\infty(M) \to 0$$

and (with a little patience we see that) $\partial_p \circ \partial_{p+1} = 0$ which is equivalent to

$B_p^\infty(M) = \operatorname{Im} \partial_{p+1}$ = space of p-boundaries $\subseteq$

$\subseteq Z_p^\infty(M) = \ker \partial_p$ = space of p-cycles.

So we can define the quotient space

$$H_p^\infty(M) = Z_p^\infty(M) / B_p^\infty(M)$$

which is called the p^{th} homology group of M with real coefficients.

Remarks (a) the homology groups of M are usually defined using the above construction but starting with simplices $\sigma : \Delta^p \to M$ which are just continuous. The resulting groups, usually denoted by $H_p(M; \mathbb{R})$ are isomorphic to the ones we have defined.

(b) Instead of taking chains as linear combination of singular simplices with real coefficients we could use any (fixed) commutative ring with unity A. The above construction yields, in this case, the homology with coefficients in A, usually denoted by $H_p(M; A)$. The most interesting case is when $A = \mathbb{Z}$, the

ring of integers.

(c) If M is a compact manifold standard algebraic topology garantees that $H_p(M; \mathbb{Z})$ is a finitely generated abelian group. From the structure of such groups we get

$$H_p(M; \mathbb{Z}) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r\text{-times}} \oplus \mathbb{Z}_{p_1^{s_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{s_t}} .$$

The universal coefficients theorem garantee that $H_p(M, \mathbb{R}) \cong \mathbb{R}^r$.

Going back to our integration, a basic fact is the following:

Proposition : Integration induces a bilinear map

$$I : H_{dR}^p(M) \times H_p^\infty(M) \to \mathbb{R}$$

$$I([\omega], [c]) = \int_c \omega$$

Proof : Clearly I is well defined on $Z^p(M) \times Z_p(M)$. Now if $\omega_1, \omega_2 \in Z^p(M), \omega_1 - \omega_2 \in B^p(M)$, $c \in Z_p(M)$

$$\int_c \omega_1 - \omega_2 = \int_c d\rho = \int_{\partial c} \rho = 0$$

and if $\omega \in Z^p(M)$, $c_1, c_2 \in Z_p(M)$ with $c_1 - c_2 \in B_p(M)$

$$\int_{c_1} \omega - \int_{c_2} \omega = \int_{\partial b} \omega = \int_b d\omega = 0$$

∎

We now state the main result of this section

THEOREM (de Rham): The bilinear map I is non degenerate and ~~therefore~~ induces an isomorphism

$$\tilde{I} : H^p_{dR}(M) \longrightarrow (H_p^n(M))^* .$$

We notice explicitly that the right hand side is a topological invariant (see preceding remarks) while H^p_{dR} depends, a priori, on the differentiable structure of M.

Beside integration on chains we want to integrate ~~fo~~ functions on a ~~Rhm~~ manifold. We start by integrating n-forms. We recall that an orientation on M is a choice of a smooth atlas $\{\varphi_\alpha : \Omega_\alpha \to M\}$ such that the change of coordinates ~~compositions~~ have positive Jacobian. It is known that this is equivalent to the choice of a nowhere zero n-form or also to the choice of an orientation on each tangent space which can be locally given by a smooth frame field.

Let M be an oriented manifold $\omega \in \Lambda^n(M)$ and we suppose that $\text{supp}\,\omega = \overline{\{x \in M : \omega(x) \neq 0\}}$ is compact and contained in a coordinate neighbourhood $\varphi_\alpha(\Omega_\alpha)$.

If $x_1, \dots, x_n$ are local coordinates induced by φ_α and $\omega = f(x_1 \dots x_n)\, dx_1 \wedge \dots \wedge dx_n$, we define

$$\int_M \omega = \int_{\Omega_\alpha} f(x_1 \dots x_n)\, dx_1 \dots dx_n$$

Let $\varphi_\beta : \Omega_\beta \to M$ be another chart of the given atlas with the same property, and let $y_1, \dots, y_n$ be the induced coordinates. Then if $\omega = g(y_1, \dots, y_n)\, dy_1 \wedge \dots \wedge dy_n$, we have

$$(*) \quad f(x) = g(y(x)) \det\left(\frac{\partial y_i}{\partial x_j}\right)$$

and therefore by the usual change of coordinates in multiple integrals we have:

$$\int_{\Omega_\alpha} f(x_1 \dots x_n)\, dx_1 \dots dx_n = \int_{\Omega_\beta} g(y_1 \dots y_n)\, dy_1 \dots dy_n$$

To define the integral of a form $\omega \in \Lambda^n(M)$ we take a partition of unity subordinated to the given atlas, $\{h_i\}$ and define

$$\int_M \omega = \sum_i \int_M h_i \cdot \omega$$

if the series converges.

We observe that we need orientability to guarantee

$$\left|\det\left(\frac{\partial y_i}{\partial x_i}\right)\right| = \det\left(\frac{\partial y_i}{\partial x_i}\right)$$

in the change of variable step.

We can really think of an n-form as a collection of functions given in coordinate neighbourhoods and related by (*) in the intersections. Instead of this we could consider a collection of functions which change in the intersection of coordinate neighbourhood with the absolute value of the Jacobian. Such objects are called <u>densities</u> and can be integrated even in the non orientable case.

For example if M is a riemannian manifold we can define the volume density, as given, in local coordinates $\{x_1, \dots, x_n\}$, by

$$V_g(x) = (\det g_{ij}(x))^{1/2} \quad \text{where } g_{ij} = \langle \partial/\partial x_i, \partial/\partial x_j \rangle$$

So we can integrate functions on a riemannian manifold which is not necessarily orientable, by setting:

$$\int_M f = \int_M f \cdot V_g$$

<u>Remark</u> To be more precise a density can be defined as a section of $\Lambda^n(M) \otimes L$ where L is the orientation line bundle on M.

It is a natural question to ask what is the relation between integration of n-forms and integration on chains.

At least in the compact orientable case it can be seen that (actually) integration on manifold is integration on the fundamental homology class.

We can do exactly the same costruction in the case of a manifold with boundary. If M is an oriented manifold with boundary ∂M, the orientation on M induces an orientation on ∂M and Stokes theorem can be stated in this situation as:

<u>THEOREM</u>: if $\omega \in \Lambda^{n-1}(M)$

$$\int_M d\omega = \int_{\partial M} i^*\omega$$

where $i : \partial M \to M$ is the inclusion. In particular if $\partial M = \emptyset$, $\int_M d\omega = 0$.

We will now give an easy application. We recall that a symplectic form on a $2n$-dimensional manifold M is a 2-form ω which is closed and non degenerate i.e. ω^n is nowhere zero. A symplectic manifold is a manifold with a simplectic form. Such a manifold is naturally oriented by ω^n. So if M is compact

$$\int_M \omega^n \neq 0$$

Now observe that $d\omega = 0$ implies $d\omega^k = 0$ and therefore ω^k are closed forms for $k = 0, \dots, n$. On the other hand they can not be ~~exact~~ exact since if $\omega^k = d\beta$, then $\omega^n = \omega^{n-k} \wedge d\beta = d(\omega^{n-k} \wedge \beta)$ and therefore we would have

$$0 \neq \int_M \omega^n = \int_M d(\omega^{n-k} \wedge \beta) = 0$$

So we have proved:

THEOREM: A compact symplectic manifold has non-vanishing even dimensional cohomology.

HARMONIC FORMS AND THE HODGE THEOREM.

Let M be a riemannian manifold and we will denote, as usual by $\langle\, , \rangle$ the scalar product and by ∇ the Levi-Civita connection.

If $f \in \Lambda^0(M)$ is a smooth function, the gradient of f is the vector field dual to the differential $df \in \Lambda^1(M)$, i.e.

$$\langle \operatorname{grad} f, X \rangle = (df)(X) \quad \forall\, X \in \mathcal{X}(M)$$

Let $x \in M$ and $X \in \mathcal{X}(M)$. Consider the linear transformation $L : T_xM \to T_xM$ given by $L(Y) = \nabla_Y X$. The divergence of X is the trace of L. If $X_1, \dots, X_n$ is an orthonormal basis, then

$$\operatorname{div} X = \sum_{i=1}^{n} \langle \nabla_{X_i} X, X_i \rangle$$

Example: Let $M = \mathbb{R}^n$ with coordinates $(x_1, \dots, x_n)$, and $f : \mathbb{R}^n \to \mathbb{R}$ a smooth function and $X = \sum \varphi_i\, \partial/\partial x_i$ a smooth vector field. Then:

$$\operatorname{grad} f = \sum \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

$$\operatorname{div} X = \sum \frac{\partial \varphi_i}{\partial x_i}$$

so that the operator $\operatorname{div} \circ \operatorname{grad} : \Lambda^0(M) \to \Lambda^0(M)$ takes the well known aspect

$$\operatorname{div} \operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

For general riemannian manifolds we copy the above costruction defining the Laplace Beltrami operator on zero-forms

$$\Delta : \Lambda^0(M) \to \Lambda^0(M) \ , \ \Delta f = -\operatorname{div} \operatorname{grad} f$$

To extend Δ to an operator on p-forms we will extend the operators grad and div. For the first we have the de Rham differential and for the second we proceed as follows:

Let $X_1, \dots X_n$ be an orthonormal basis for $T_x M$ and $\omega \in \Lambda^p(M)$. Define a $(p-1)$-form by:

$$(\operatorname{div} \omega)(X_{i_1}, \dots, X_{i_{p-1}}) = \sum_1^n (\nabla_{X_k} \omega)(X_k, X_{i_1}, \dots, X_{i_{p-1}})$$

It is not difficult to see that the above definition does not depend on the choice of the frame and defines a $(p-1)$-form. The divergence of a 1-form coincides with the divergence of the dual vector field, as defined at the beginning.

Notation : we denote by $\delta : \Lambda^p(M) \to \Lambda^{p-1}(M)$ the opposite of the divergence operator and call it the codifferential.

The Laplace-Beltrami operator is then defined as the operator:

$$\Delta : \Lambda^p(M) \to \Lambda^p(M) \ ; \ \Delta\omega = (d\delta + \delta d)\omega$$

Notice that $\delta^2 = 0$ and therefor, thinking of d and δ as operators on $\Lambda^{\mp}(M)$, $\Delta = (d+\delta)^2$.

Suppose now M is oriented and $* : \Lambda^p(M) \to \Lambda^{n-p}(M)$ is the Hodge star operator.

Proposition : $\operatorname{div} \omega = (-1)^{n(p+1)} * d * \omega \quad \forall \omega \in \Lambda^p(M)$.

Proof : Let $\{X_1, \dots, X_n\}$ be a positive orthonormal basis for $T_x M$ and extend them to local field such that $(\nabla_{X_i} X_j)(x) = 0$. Then we have:

$$(* \operatorname{div} \omega)(X_1, \dots, X_{n-p+1}) = \sum_{k=1}^n (\nabla_{X_k} \omega)(X_k, X_{n-p+2}, \dots, X_n)$$

$$= \sum_{k=1}^{n-p+1} (\nabla_{X_k} \omega)(X_k, X_{n-p+2}, \dots, X_n)$$

On the other hand, up to terms which vanish in x, we have

$$(d*\omega)(X_1, \dots, X_{n-p+1}) = \sum_{k=1}^{n-p+1} (-1)^{k+1} \left[\nabla_{X_k}(*\omega)\right](X_1, \dots, \hat{X}_k, \dots, X_{n-p+1})$$

$$= \sum_{k=1}^{n-p+1} (-1)^{k+1} X_k \cdot \left[*\omega(X_1, \dots, \hat{X}_k, \dots, X_{n-p+1}) \right] =$$

$$= \sum_{k=1}^{n-p+1} (-1)^{k+1+n-p+1-k} X_k \cdot \omega(X_k, X_{n-p+2}, \dots, X_n)$$

So we have

$$* \operatorname{div} \omega = (-1)^{n-p} d*\omega$$

and applying $*$ to both sides we get the conclusion ∎

Let M be a compact riemannian manifold.

In $\Lambda^p(M)$ we define the L^2 inner product integrating the pointwise inner product with respect to the riemannian volume density:

$$(\omega_1, \omega_2) = \int_M \langle \omega_1(x), \omega_2(x) \rangle \, dV_g$$

If M is oriented we can write:

$$(\omega_1, \omega_2) = \int_M \omega_1 \wedge * \omega_2$$

It is easy to see that the above bilinear form is positive definite and therefore induces a structure of pre-Hilbert space in $\Lambda^p(M)$.

We will assume from now on that M is a compact oriented riemannian manifold.

Proposition $\forall \omega \in \Lambda^{p-1}(M)$, $\tau \in \Lambda^p(M)$ we have

$$(d\omega, \tau) = (\omega, \delta\tau)$$

and therefore $(\Delta \omega_1, \omega_2) = (\omega_1, \Delta \omega_2) \quad \forall\, \omega_1, \omega_2 \in \Lambda^p(M)$

Proof:

$$d(\omega \wedge * \tau) = d\omega \wedge * \tau + (-1)^{p-1} \omega \wedge d * \tau =$$
$$= d\omega \wedge * \tau + (-1)^{(p-1)+(p-1)(n-p+1)} \omega \wedge ** d * \tau =$$
$$= d\omega \wedge * \tau - \omega \wedge * \delta \tau \,,$$ By Stokes theorem:

$$0 = \int_M d(\omega \wedge * \tau) = (d\omega, \tau) - (\omega, \delta\tau)$$ ■

Corollary (Hopf lemma) if $f \in \Lambda^0(M)$ then $\int_M (\Delta f) * 1 = 0$. In particular if $\Delta f \geq 0$ then $\Delta f = 0$.

Proof:

$$\int_M \Delta f * 1 = (\Delta f, 1) = (f, \Delta 1) = 0$$ ■

Proposition: $\Delta \omega = 0 \iff d\omega = 0$ and $\delta\omega = 0$

Proof: Clearly $d\omega = 0, \delta\omega = 0 \Rightarrow \Delta\omega = 0$. Conversely if $\Delta\omega = 0$ we get

$$0 = (\Delta\omega, \omega) = (d\delta\omega, \omega) + (\delta d\omega, \omega) = \|\delta\omega\|^2 + \|d\omega\|^2$$ ■

Definition: A p-form $\omega \in \Lambda^p(M)$ is harmonic if $\Delta\omega = 0$.

We will denote by $\mathcal{H}^p$ the space of harmonic forms.

Remark: The above proposition tells us that on a compact manifold a form is harmonic if and only if it is closed and coclosed (i.e. $\delta\omega = 0$). Without the compactness assumption it is obviously still true that a closed and coclosed form is harmonic but not the converse.

If M is compact, as we have assumed, an harmonic form is closed and so it determines a de Rham cohomology class. The following is a nice ~~characterization~~ characterization of harmonic forms in terms of the geometry of the pre-Hilbert space $\Lambda^p(M)$:

Proposition: $\omega \in \Lambda^p(M)$ is harmonic if and only if it is closed and ω has the minimum L^2 norm in its cohomology class.

Proof: let $\omega \in \mathcal{H}^p$. Then

$$\|\omega + d\beta\|^2 = (\omega + d\beta, \omega + d\beta) = \|\omega\|^2 + \|d\beta\|^2 + 2(\omega, d\beta)$$
$$= \|\omega\|^2 + \|d\beta\|^2 + 2(\delta\omega, \beta) = \|\omega\|^2 + \|d\beta\|^2$$

so ω has the smallest L^2 norm in its cohomology class.
Conversely suppose $\|\omega\|^2 = \inf\{\|\omega + d\beta\|^2 : \beta \in \Lambda^{p-1}(M)\}$ and $d\omega = 0$. Set $g_\beta(t) = \|\omega + t\,d\beta\|^2$. Then $g'_\beta(0) = 0 \;\; \forall \beta \in \Lambda^{p-1}(M)$.
But $g'_\beta(0) = 2(\omega, d\beta) = 2(\delta\omega, \beta) = 0 \;\; \forall \beta \in \Lambda^{p-1}(M)$
which ~~implies~~ implies that ω is coclosed and hence harmonic.

Let us observe that harmonic forms are orthogonal to exact forms. In particular if two harmonic forms differ by an exact one they coincide. An other way of ~~seeing this~~ proving uniqueness of harmonic representatives in a given cohomology class is observing that in a convex set in a pre Hilbert space there is at most one element of smallest norm and apply the above proposition to the affine subspace $[\omega] = \{\omega + d\beta\,;\, \beta \in \Lambda^{p-1}(M)\} \subseteq \Lambda^p(M)$.

By the above, we have a linear injection

$$\mathcal{H}^p \longrightarrow H^p_{dR}(M)$$

The main result of Hodge theory is that the above map is actually surjective and therefore an isomorphism. In other words in every cohomology class there exist a (unique) element of smallest L^2 norm.

We will give now an idea of the proof of the following:

THEOREM: (Hodge decomposition theorem). For each $p = 0, \dots, n$ we have the following orthogonal direct sum decomposition:

$$\Lambda^p(M) = \Delta(\Lambda^p M) \oplus \mathcal{H}^p$$

Remark: Since M is compact, by general arguments of algebraic topology the cohomology of M is finite dimensional. By de Rham theorem the de Rham cohomology is also finite dimensional and therefore $\mathcal{H}^p$ which injects in $H^p_{dR}(M)$ is finite dimensional. Without using this algebraic topological fact the finite dimensionality of $\mathcal{H}^p$ can be deduced from one of the facts we will discuss below.

Idea of the proof:

Since $\mathcal{H}^p$ is finite dimensional we have a projection $\Lambda^p(M) \to \mathcal{H}^p$, $\omega \longmapsto \omega_H$ and an orthogonal direct sum decomposition

$$\Lambda^p(M) = \mathcal{H}^p \oplus (\mathcal{H}^p)^\perp$$

So to prove the theorem we will prove $(\mathcal{H}^p)^\perp = \Delta(\Lambda^p(M))$.
If $\omega \in \Delta(\Lambda^p(M))$ and $\varphi \in \mathcal{H}^p$ then, setting $\omega = \Delta\beta$

$$(\omega, \varphi) = (\Delta\beta, \varphi) = (\beta, \Delta\varphi) = 0$$

and therefore $\Delta(\Lambda^p(M)) \subseteq (\mathcal{H}^p)^\perp$.

The main point is then to prove $(\mathcal{H}^p)^\perp \subseteq \Delta(\Lambda^p(M))$ i.e. given $\alpha \in (\mathcal{H}^p)^\perp$ then exist a solution of the equation $\Delta\varphi = \alpha$.

We describe a general strategy for the problem:

Suppose ω is a solution of $\Delta\omega = \alpha$. Then $\forall \varphi \in \Lambda^p(M)$

$$(\Delta\omega, \varphi) = (\omega, \Delta\varphi) = (\alpha, \varphi)$$

So ω determines a bounded linear operator $\ell : \Lambda^p(M) \to \mathbb{R}$ $\ell(\beta) = \langle \omega, \beta \rangle$ which verifies

$$(*) \quad \ell(\Delta\varphi) = (\alpha, \varphi) \qquad \forall \varphi \in \Lambda^p(M).$$

It turns out that in general it is easier to find such a linear functional and so it deserves a special name: A functional verifying $(*)$ is called a <u>weak solution</u> of the equation $\Delta\omega = \alpha$. So every (ordinary or classical) solution determines a weak solution.

Conversely suppose ℓ is a weak solution, then ℓ is represented in the completement $\hat{\Lambda}^p(M)$ of $\Lambda^p(M)$ (with respect to the L^2 product) by an element $\hat{\omega} \in \hat{\Lambda}^p(M)$, i.e.

$$\ell(\beta) = (\hat{\omega}, \beta)$$

Suppose now $\hat{\omega} \in \Lambda^p(M) \subseteq \hat{\Lambda}^p(M)$. Then

$$(\Delta\hat{\omega}, \beta) = (\hat{\omega}, \Delta\beta) = \ell(\Delta\beta) = (\alpha, \beta) \quad \forall \beta \in \Lambda^p(M)$$

Therefore $\hat{\omega}$ is an (ordinary or classic) solution of $\Delta\omega = \alpha$.

The two main facts about the operator Δ that we will use are the following:

(a) <u>Regularity theorem</u>: Any weak solution of $\Delta\omega = \alpha$ is represented by a smooth form and therefore this yields an ordinary solution.

(b) <u>Compactness theorem</u>: Given a sequence $\{\alpha_n\} \subseteq \Lambda^p(M)$ such that $\|\alpha_n\| \le c$, $\|\Delta\alpha_n\| \le c$, for some constant c, then $\{\alpha_n\}$ has a Cauchy subsequence in $\Lambda^p(M)$.

<u>Remark</u>: the above properties are showed by a great class of differential operator, the elliptic differential operator which are the main argument of the parallel course of Prof. Gilkey and we refer to his notes for further informations.

<u>Remark</u>: Fact (b) ~~easily~~ easily implies the finite dimensionality of $\mathcal{H}^p$. In fact if $\dim \mathcal{H}^p = \infty$ we could find an orthonormal sequence of harmonic forms which clearly will not have any Cauchy subsequence.

Facts (a),(b) imply that Δ is invertible in $(\mathcal{H}^p)^\perp$. More precisely

<u>Claim</u>: there exist a constant $c>0$ such that

$$\|\beta\| \le c\,\|\Delta\beta\| \quad \forall\, \beta\in(\mathcal{H}^p)^\perp$$

<u>Proof</u>: Suppose not. Then there exists a sequence of forms β_i such that $\|\beta_i\|=1$, $\|\Delta\beta_i\|\to 0$, $\beta_i\in(\mathcal{H}^p)^\perp$. From (b) we can assume that β_i is a Cauchy sequence and we can define a linear functional $\ell:\Lambda^p(M)\to\mathbb{R}$,

$$\ell(\varphi)=\lim_{i\to\infty}(\beta_i,\varphi)$$

then ℓ is well defined and bounded. Moreover

$$\ell(\Delta\varphi)=\lim_{i\to\infty}(\beta_i,\Delta\varphi)=\lim_{i\to\infty}(\Delta\beta_i,\varphi)=0$$

and therefore ℓ is a weak solution of $\Delta\omega=0$. By (a) there exists a classical solution ω, $\ell(\varphi)=(\omega,\varphi)$ $\forall\,\varphi\in\Lambda^p(M)$. So β_i is a Cauchy sequence which weakly converg to ω and therefore converges to ω. So $\|\omega\|=1$ and $\omega\in(\mathcal{H}^p)^\perp$ contradicting $\Delta\omega=0$.

Let $\alpha\in(\mathcal{H}^p)^\perp$. Define a linear functional $\ell:\Delta(\Lambda^p(M))\to\mathbb{R}$ by

$$\ell(\Delta\varphi)=(\alpha,\varphi)$$

Now

$$|\ell(\Delta\varphi)|=|\ell\,\Delta(\varphi-\varphi_H)|=|(\alpha,\varphi-\varphi_H)|\le\|\alpha\|\,\|\varphi-\varphi_H\|$$
$$\le c\,\|\alpha\|\,\|\Delta(\varphi-\varphi_H)\|=c\,\|\alpha\|\,\|\Delta\varphi\|$$

therefore, by the Hahn Banach theorem, ℓ extends to a linear bounded functional on all of $\Lambda^p(M)$ which is clearly a weak solution of $\Delta\omega=\alpha$. By Fact (a) again there exists a classical solution of $\Delta\omega=\alpha$. ∎

As a consequence of the above theorem we get:

<u>Theorem</u> (Hodge): The linear injection $\mathcal{H}^p\to H^p_{dR}(M)$ is onto

<u>Proof</u>: Let $\alpha\in\Lambda^p(M)$ be a closed form. From the decomposition theorem we have:

$$\alpha=\alpha_H+\Delta\beta=\alpha_H+d\beta_1+\delta\beta_2$$

but $d\alpha=0\implies(\alpha,\delta\beta_2)=0$ and therefore

$$\alpha-\alpha_H=d\beta_1$$

As a first application of the Hodge theorem we get a quick proof of Poincaré Duality:

<u>Theorem</u>: If M is a compact <u>oriented</u> n-dimensional riemannian manifold, then the map:

$$P: H^p_{dR}(M)\times H^{n-p}_{dR}(M)\longrightarrow\mathbb{R}$$
$$P([\alpha],[\beta])=\int_M\alpha\wedge\beta$$

is a well defined bilinear non-singular map and therefore determines an isomorphism

$$\tilde{P}: H^p_{dR}(M) \longrightarrow \left(H^{n-p}_{dR}(M)\right)^*$$

Proof: that P is well defined follows from Stokes theorem. In fact

$$\int_M (\alpha + d\beta)\wedge\beta = \int_M \alpha\wedge\beta + \int_M d(\gamma\wedge\beta) = \int_M \alpha\wedge\beta$$

Clearly P is bilinear. Moreover if $\alpha \in \mathcal{H}^p$, $*\alpha \in \mathcal{H}^{n-p}$ since $*$ commutes with Δ. Therefore if $\alpha \neq 0$

$$P([\alpha],[*\alpha]) = \|\alpha\|^2 \neq 0 \qquad \blacksquare$$

Remarks: (a) Poincaré's duality depends essentially on the orientability of M although the Hodge theorem does not.

(b) In the ~~[illegible]~~ orientable case, $*$ induces an isomorphism

$$*: \mathcal{H}^p(M) \to \mathcal{H}^{n-p}(M)$$

(c) The Laplace-Beltrami operator depends on the metric on M. Hodge's theorem garantees that the dimension of the space of harmonic form depends only on the homotopy type of M (in the category of compact manifolds).

(d) As observed the Laplace-Beltrami operator and therefore the harmonic form depends on the metric on M. Suppose now $\dim M = 2k$ and g_1, g_2 two conformally equivalent metrics, i.e. $g_2 = f^2 g_1$ for some nowhere zero function $f: M \to \mathbb{R}$. Then if we denote by $*_1, *_2$ the relative Hodge operator we see easily that $*_1 = *_2$ on $\Lambda^k(M)$. So, in this case the harmonic forms in the middle dimension are invariant under a conformal change of metric.

Now we will give another application of the theorem of Hodge. We have seen that is a compact manifold admits a symplectic two-form, then the even dimensional cohomology is non zero.

THEOREM: Let M be a ~~[illegible]~~ compact $2n$-dimensional riemannian manifold. If M admits a parallel symplectic 2-form then

$$b_p(M) \leq b_{p+2}(M) \qquad 0 \leq i < n$$

where $b_p(M) = \dim H^p(M;\mathbb{R})$ is the p^{th} Betti number of M.

Proof: Let Ω be the parallel symplectic 2-form. Define a map

$$\Phi: \Lambda^p(M) \to \Lambda^{p+2}(M) \qquad \phi(\omega) = \Omega\wedge\omega.$$

Claim: Φ is injective if $p < n$

In fact a theorem of Darboux guarantees that for each $x \in M$ there are local coordinates around x, $(x_1, \dots, x_{2n})$, such that

$$\omega = \sum_{i=1}^{n} dx_i \wedge dx_{i+n}$$

Now writing $\mu \in \Lambda^p(M)$ in those coordinates we have $\mu = \sum \mu_{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$ and if $\mu \neq 0$, $p < n$ it is easily seen that $\Omega \wedge \mu \neq 0$.

Claim if $\Delta\mu = 0$ then $\Delta(\Omega \wedge \mu) = 0$

Clearly $\Omega \wedge \mu$ is a closed form. So if we show that $\Omega \wedge \mu$ is coclosed we prove the claim. Before proving this let us recall a few facts.

For a given $X \in T_x M$ we have the interior multiplication $i_X : A^p_x(M) \to A^{p-1}_x(M)$ given by $(i_X \omega)(Y_1, \dots, Y_{p-1}) = \omega(X, Y_1, \dots, Y_{p-1})$. i_X is the adjoint of exterior multiplication by $\tilde{X} = \langle X, \cdot \rangle$ and acts as an antiderivation on $A^*_x(M)$, i.e. $i_X(\omega \wedge \tau) = (i_X \omega) \wedge \tau + (-1)^p \omega \wedge i_X \tau$ if $\omega \in A^p_x(M)$. At level of forms i_X is related to the exterior derivative by

$$(i_X \circ d + d \circ i_X)\, \omega\, (X_1, \dots, X_p) =$$

$$= X \cdot \omega(X_1, \dots, X_p) - \sum_{i=1}^{p} \omega(X_1, \dots, X_{i-1}, [X, X_i], X_{i+1}, \dots, X_p)$$

$$= (\text{def})\, (\mathcal{L}_X \omega)(X_1, \dots, X_p).$$

$\mathcal{L}_X \omega$ is the Lie derivative of ω in the direction X.

Let us return to the proof of the claim. Let $x \in M$ and $X_1, \dots, X_{2n}$ a frame field in a neighbourhood of x such that $(\nabla_{X_i} X_j)(x) = 0$. Then is easily seen that, at x, $\mathcal{L}_{X_i} \mu = \nabla_{X_i} \mu$ and $\mathcal{L}_{X_i} \Omega = \nabla_{X_i} \Omega = 0$.

Therefore we get:

$$\delta(\Omega \wedge \mu) = -\sum_j i_{X_j} \nabla_{X_j} \Omega \wedge \mu =$$

$$= -\sum_j i_{X_j} \left[(\nabla_{X_j} \Omega) \wedge \mu + \Omega \wedge \nabla_{X_j} \mu \right] = -\sum_j i_{X_j} [\Omega \wedge \nabla_{X_j} \mu]$$

$$= \left(\sum_j - \left[(i_{X_j} \Omega) \wedge \nabla_{X_j} \mu \right] \right) - \left(\Omega \wedge \sum_j i_{X_j} \nabla_{X_j} \mu \right) =$$

$$= -\sum_j i_{X_j} \Omega \wedge \nabla_{X_j} \mu .$$

Let us consider the form $\beta = \sum_{i=1}^{2n} i_{X_i} \Omega \wedge i_{X_i} \mu$. Then:

$$d\beta = \sum_i d(i_{X_i} \Omega) \wedge i_{X_i} \mu - i_{X_i} \Omega \wedge d(i_{X_i} \mu) =$$

$$= \sum_i \left[-i_{X_i} d\Omega + \mathcal{L}_{X_i} \Omega \right] \wedge i_{X_i} \mu - i_{X_i} \Omega \wedge \left[-i_{X_i} d\mu + \mathcal{L}_{X_i} \mu \right]$$

$$= -\sum_i i_{X_i} \Omega \wedge \nabla_{X_i} \mu = \delta(\Omega \wedge \mu)$$

But the only exact and coexact form is zero and therefore $\delta(\Omega \wedge \mu) = 0$ and therefore $\Omega \wedge \mu$ is harmonic.

The above claims prove that Φ induces an injection $\mathcal{H}^p \to \mathcal{H}^{p+2}$ and the conclusion follows. $\blacksquare$

THE WEITZENBÖCK FORMULA

We want to establish a formula for the Laplace-Beltrami operator, involving explicitly one of the most important geometric invariants of a riemannian manifold, the curvature tensor.

We recall that the curvature tensor of a riemannian manifold is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Since the Levi Civita connection acts on forms we can extend $R(X,Y)$ to an operator on $\Lambda^p(M)$

$$R(X,Y)\,\omega = \nabla_X \nabla_Y \omega - \nabla_Y \nabla_X \omega - \nabla_{[X,Y]}\,\omega$$

It is not difficult to see that $R(X,Y)$ is antisymmetric and, if $\{X_1, \dots, X_n\}$ is an orthonormal basis with dual basis $\{\tilde{X}_1, \dots, \tilde{X}_n\}$, we have

$$R(X,Y)\,(\tilde{X}_{i_1} \wedge \dots \wedge \tilde{X}_{i_p}) = \sum_{k=1}^{p} \tilde{X}_{i_1} \wedge \dots \wedge \tilde{X}_{i_{k-1}} \wedge R(X,Y)\tilde{X}_{i_k} \wedge \dots \wedge \tilde{X}_{i_p}$$

If $\{X_1, \dots, X_n\}$ is an orthonormal basis, the <u>Ricci tensor</u> $Q : T_x M \to T_x M$ is defined by

$$Q(X) = \sum_{k=1}^{n} R(X, X_k) X_k$$

And the <u>Ricci curvature</u> at $X \in TM$ is given by

$$\mathrm{Ricc}\,(X) = \langle Q(X), X \rangle$$

It is easy to see that the above definitions do not depend on the choice of the basis.

The Ricci tensor extends to an operator on $(0,p)$-tensors:

$$(Q_p\omega)(X_{i_1},\dots,X_{i_p}) = \sum_{k=1}^{n}\sum_{j=1}^{p}\left[R(X_k,X_{i_j})\omega\right](X_{i_1},\dots,\hat{X}_{i_j},X_k,\dots,X_{i_p})$$

The above definition does not depend on the choice of the basis and for 1-forms coincides with the Ricci tensor modulo $TM \cong T^*M$.

The formula we are looking for is the following:

Weitzenböck formula: If $X_1,\dots,X_n$ is an orthonormal basis and $\omega \in \Lambda^p(M)$, then

$$\langle \Delta\omega,\omega\rangle = \frac{1}{2}\Delta(\|\omega\|^2) + \sum \|\nabla_{X_i}\omega\|^2 + \langle Q_p\omega,\omega\rangle$$

Proof: We extend $X_1,\dots,X_n$ to a local orthonormal frame around $x\in M$ such that $(\nabla_{X_i}X_j)(x)=0$.

Forgetting terms which vanish at x we have:

$$(d\delta\omega)(X_{i_1},\dots,X_{i_p}) = \sum_{k=1}^{p}(-1)^{k+1}X_{i_k}\cdot\left[\delta\omega(X_{i_1},\dots,\hat{X}_{i_k},\dots,X_{i_p})\right] =$$

$$= \sum_{k=1}^{p}(-1)^{k+1}X_{i_k}\cdot\left(-\sum_{s=1}^{n}\nabla_{X_s}\omega(X_s,X_{i_1},\dots,\hat{X}_{i_k},\dots,X_{i_p})\right)$$

$$= \sum_{k=1}^{p}\sum_{s=1}^{n}(-1)^{k}(\nabla_{X_{i_k}}\nabla_{X_s}\omega)(X_s,X_{i_1},\dots,\hat{X}_{i_k},\dots,X_{i_p})$$

Similarly

$$(\delta d\omega)(X_{i_1},\dots,X_{i_p}) = -\sum_{s=1}^{n}X_s\cdot\left[d\omega(X_s,X_{i_1},\dots,X_{i_p})\right] =$$

$$= -\sum_{s=1}^{n}X_s\cdot\left[\nabla_{X_s}\omega(X_{i_1},\dots,X_{i_p}) + (-1)^k\sum_{k=1}^{p}(\nabla_{X_{i_k}}\omega)(X_s,\dots,\hat{X}_{i_k},\dots,X_{i_p})\right]$$

$$= -\sum_{s=1}^{n}\left[(\nabla_{X_s}\nabla_{X_s}\omega)(X_{i_1},\dots,X_{i_p}) + \sum_{k=1}^{p}(-1)^k(\nabla_{X_s}\nabla_{X_{i_k}}\omega)(X_s,\dots,\hat{X}_{i_k},\dots,X_{i_p})\right]$$

Summing the two expressions we get at x

$$(*)\quad (\Delta\omega)(x) = -\sum_{s=1}^{n}(\nabla_{X_s}\nabla_{X_s}\omega)(x) + Q_p(\omega).$$

Finally

$$-\sum_{s=1}^{n}\langle\nabla_{X_s}\nabla_{X_s}\omega,\omega\rangle = -\sum_{s=1}^{n}\left[X_s\cdot\langle\nabla_{X_s}\omega,\omega\rangle - \langle\nabla_{X_s}\omega,\nabla_{X_s}\omega\rangle\right]$$

$$= -\frac{1}{2}\sum_{s=1}^{n}X_sX_s\langle\omega,\omega\rangle + \sum_{s=1}^{n}\|\nabla_{X_s}\omega\|^2 =$$

$$= \frac{1}{2}\Delta(\|\omega\|^2) + \sum_{s=1}^{n}\|\nabla_{X_s}\omega\|^2$$

Then the conclusion follows immediately from $(*)$.

Remark: The connection ∇ induces an operator

$$\nabla : \mathcal{T}^{(0,q)}(M) \longrightarrow \mathcal{T}^{(0,q+1)}(M)$$

$$(\nabla\omega)(X,Y_1,\dots,Y_q) = (\nabla_X\omega)(Y_1,\dots,Y_q)$$

This operator has a formal adjoint $\nabla^*: \mathcal{T}^{(0,q+1)}(M) \to \mathcal{T}^{(0,q)}(M)$.

The operator $\nabla^*\nabla$ is called the rough Laplacian and if $\{X_1, \dots, X_n\}$ is an orthonormal frame field we have:

$$\nabla^*\nabla = -\sum_{i=1}^{n} \left(\nabla_{X_i}\nabla_{X_i} - \nabla_{\nabla_{X_i}X_i} \right)$$

So formula ⊛ above takes the form

$$\Delta\omega = \nabla^*\nabla\omega + Q_p(\omega)$$

and the rough Laplacian differs from the Laplace-Beltrami operator by a 0-order differential operator, i.e. a tensor.

If M is a compact riemannian manifold we can integrate the Weitzenböck formula on M obtaining:

$$(\Delta\omega, \omega) = \|\nabla\omega\|^2 + \tilde{Q}_p(\omega)$$

where $(\ ,\ )$ is the L^2-scalar product, $\|\cdot\|$ the relative norm and $\tilde{Q}_p(\omega) = \int_M \langle Q_p(\omega), \omega \rangle$

A good strategy to prove vanishing theorems for cohomology or to get information on harmonic forms, is to study the sign of $\tilde{Q}$. For example:

(A) If $\tilde{Q}_p$ is positive definite then $H^p(M,\mathbb{R}) = \{0\}$

(B) If $\tilde{Q}_p$ is positive semidefinite then every harmonic form is parallel.

The following classical result illustrates (A):

THEOREM: If M is a compact riemannian manifold whose Ricci curvature is non-negative and positive at some point then $H^1(M,\mathbb{R}) = 0$.

Remark: With the above hypothesis it follows that $H_1(M,\mathbb{Z})$ is finite (universal coefficients theorem) and so if the fundamental group $\pi_1(M)$ is abelian, it is also finite (Hurewitz theorem).

Really the condition that $\pi_1(M)$ is abelian is not necessary for proving the finiteness of $\pi_1(M)$. This can be deduced, using the above arguments, from the following result:

THEOREM (Cheeger-Gromoll): If M is a compact riemannian manifold with non-negative Ricci curvature, then the fundamental group $\pi = \pi_1(M)$ contains a finite normal subgroup Φ such that $\pi^* := \pi/\Phi$ contains a free abelian normal subgroup $\Gamma^r \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (r times) of finite index.

We want to apply some of the above results to study the topology of compact Lie groups. We recall that compact Lie groups admits bi-invariant metrics and for such a metric and left invariant fields X, Y, Z and W we have:

i) $\langle [X,Y], Z \rangle = \langle X, [Y,Z] \rangle$

ii) $R(X,Y)Z = -\frac{1}{4}\,[[X,Y], Z]$

Proposition: If G is a compact Lie group whose Lie algebra $\hat{G}$ has trivial center, then $H^1(G,\mathbb{R}) = 0$ and (consequently) $\pi_1(G)$ is finite.

Proof: We will show that the Ricci curvature of a bi-invariant metric is positive. In fact, if $X \in \hat{G}$ and $\{E_1, \dots, E_n\}$ is an orthonormal basis for $\hat{G}$, we have:

$$\mathrm{Ricc}(X) = \sum_k \langle R(X,E_k)E_k, X\rangle = \frac{1}{4}\sum \| [X,E_k] \|^2$$

which is positive since otherwise X would belong to the center of $\hat{G}$.

Remark: In general a simply connected Lie group with bi-invariant metric splits as $G \times \mathbb{R}^k$ where G is compact and $\hat{G}$ has trivial center.

If G is compact and $\hat{G}$ has trivial center, there is a special bi-invariant metric given by

$$\langle X, Y\rangle = -\,\mathrm{Trace}\ ad_X \circ ad_Y$$

where $ad_X : \hat{G} \to \hat{G}$ is given by $ad_X(Y) = [X,Y]$. It is easy to see that such a metric turn G into an Einstein manifold with constant $1/4$, i.e. the Ricci tensor is given by

$$\langle \mathrm{Ricc}\, X, Y\rangle = 1/4 \langle X, Y\rangle$$

Using this fact we will prove the following result:

Proposition. If G is a compact Lie group and $\hat{G}$ has trivial center, then $H^2(G,\mathbb{R}) = 0$.

Proof: Let $\mu \in \Lambda^2(G)$ and $\{E_1, \dots, E_n\}$ be an orthonormal basis for $\hat{G}$ with dual basis $\{\tilde{E}_1, \dots, \tilde{E}_n\}$. Then:

$$Q_2(\mu)(E_i,E_j) = \sum_{k=1}^{n} [R(E_k,E_i)\mu](E_k, E_j) + [R(E_k,E_j)\mu](E_i,E_k)$$

$$= \sum_{k=1}^{n} \langle R(E_k,E_i)\mu, \tilde{E}_k \wedge \hat{E}_j\rangle + \langle R(E_k,E_j)\mu, \tilde{E}_i \wedge \tilde{E}_k\rangle$$

$$= \sum_{k=1}^{n} \langle \mu, R(E_i,E_k)(\tilde{E}_k \wedge \tilde{E}_j)\rangle + \langle \mu, R(E_j,E_k)(\tilde{E}_i \wedge \tilde{E}_k)\rangle$$

$$= \sum_{k=1}^{n} \langle \mu, (R(E_i,E_k)\tilde{E}_k) \wedge \tilde{E}_j + \tilde{E}_k \wedge (R(E_i,E_k)\tilde{E}_j)\rangle$$

$$+ \langle \mu, (R(E_j,E_k)\tilde{E}_i) \wedge \tilde{E}_k + E_i \wedge (R(E_j,E_k)\tilde{E}_k)\rangle$$

$$= \frac{1}{4}\langle \mu, \tilde{E}_i \wedge \tilde{E}_j\rangle + \frac{1}{4}\langle \mu, \tilde{E}_i \wedge \tilde{E}_j\rangle$$

$$+ \sum_{k=1}^{n} \langle \mu, (R(E_j,E_k)\tilde{E}_i + R(E_k,E_i)\tilde{E}_j) \wedge \tilde{E}_k\rangle$$

$$= \frac{1}{2}\mu(E_i,E_j) - \langle \mu, (R(E_i,E_j)\tilde{E}_k) \wedge \tilde{E}_k\rangle$$

Define an operator $\rho : \Lambda^2(G) \to \Lambda^2(G)$ extending linearly the map

$$\langle \rho(\tilde{X} \wedge \tilde{Y}), \tilde{Z} \wedge \tilde{W}\rangle = \langle R(X,Y)W, Z\rangle$$

ρ is a symmetric operator called the curvature operator.
we will prove ~~[illegible]~~ later on the following fact:

FACT In our hypothesis the eigenvalues of ρ are between 0 and $1/8$.

Going back to our calculation we get:

$$Q_2(\mu)(E_i, E_j) = \tfrac{1}{2}\mu(E_i, E_j) - 2\langle \rho(\mu), \tilde{E}_i \wedge \tilde{E}_j \rangle$$

Therefore:

$$\langle Q_2(\mu), \mu \rangle = \frac{\|\mu\|^2}{2} - 2\sum_{i<j} \langle \rho(\mu), \tilde{E}_i \wedge \tilde{E}_j \rangle \mu(E_i, E_j)$$

$$= \frac{\|\mu\|^2}{2} - 2\langle \rho(\mu), \mu \rangle \geq \frac{\|\mu\|^2}{2} - \frac{\|\mu\|^2}{4} > 0.$$

So the conclusion follows from the integrated ~~[illegible]~~ Weitzenböck formula ∎

We can go a little further:

Proposition: If G is as above and $(\dim G) \geq 3$ then $H^3(G, \mathbb{R}) \neq 0$

Proof: Let us define a 3-form in G by

$$\mu(U, V, W) = \langle [U, V], W \rangle$$

So $\mu \neq 0$ since G is not abelian. Since $(\nabla_U V) = \frac{1}{2}[U, V]$ for left invariant fields U, V, we have:

$$(\nabla_X \mu)(U, V, W) = \tfrac{1}{2}\langle [[X, U], V] + [[U, V], X] + [[V, X], U], W \rangle$$

which is zero by the Jacobi identity. Therefore μ is parallel and hence harmonic, and non-zero, and therefore determines a non-zero 3-dimensional cohomology class ∎

MANIFOLDS WITH NON NEGATIVE CURVATURE OPERATOR

One of the basic problems in riemannian geometry is the study of the topology of manifolds with positive (or non-negative) curvature. There are many "curvatures" obtained from the Riemann curvature ~~tensor~~ tensor and one of the most interesting is the sectional curvature. Although many results are known in this direction, the problem is far from being well understood. We will make a stronger positivity assumption (stronger than the positivity of the sectional ~~p~~ curvature) and discuss a classification theorem for such manifolds.

The Riemann curvature tensor induces a symmetric endomorphism at level of 2-forms called the curvature operator obtained extending linearly the map ρ:

$$\langle \rho(X\wedge Y), Z\wedge W\rangle = \langle R(X,Y)W, Z\rangle$$

The fact that ρ is well defined and symmetric is an immediate consequence of the symmetries of R.

The sectional curvatures of the plane spanned by X and Y is given by

$$k(X,Y) = \frac{\langle R(X,Y)Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2} = \frac{\langle \rho(X\wedge Y), X\wedge Y\rangle}{\|X\wedge Y\|^2}$$

The positivity of ρ implies ~~that~~ the positivity of the sectional curvatures. The converse is not true in general; the reason being that eigenvectors of ρ may not be decomposable (i.e. of the form $X\wedge Y$).

An important class of manifolds with non-negative curvature operator is the class of compact symmetric spaces. In this class there is, for example, the complex projective space $\mathbb{CP}^n$ whose sectional curvatures take values in an interval of the type $[A^2, 4A^2]$, where $A\neq 0$ depends on the normalization of the metric. However, as we will see, the curvature operator of $\mathbb{CP}^n$ has non-trivial kernel.

As an example we prove the following fact (that we used in the proof of an earlier theorem):

<u>Proposition</u>: If G is a compact Lie group and $\hat{G}$ has trivial center then the curvature operator has eigenvalues in the interval $[0, 1/8]$ (with respect to the Killing metric of constant Ricci curvature $1/4$)

<u>Proof</u>: Let $M: \Lambda^2(\hat{G}) \to G$ be the linear extension of $M(X,Y) = [X,Y]$ and $M^*: \hat{G} \to \Lambda^2(\hat{G})$ its adjoint. Then

$$\langle \rho(X\wedge Y), Z\wedge W\rangle = -\frac{1}{4}\langle [[X,Y],W], Z\rangle = \frac{1}{4}\langle M(X\wedge Y), M(Z\wedge W)\rangle$$

So $\rho = \frac{1}{4}M^*M$

Now let $X, Y \in \hat{G}$ and $\{E_1, \dots, E_n\}$ be an orthonormal basis for $\hat{G}$ with dual basis $\{\tilde{E}_1, \dots, \tilde{E}_n\}$. Then

$$M^* X = \sum_{i<j} \langle M^* X, \tilde{E}_i \wedge \tilde{E}_j \rangle \tilde{E}_i \wedge \tilde{E}_j = \sum_{i<j} \langle X, [E_i, E_j] \rangle \tilde{E}_i \wedge \tilde{E}_j$$

therefore

$$2 \langle X, M M^* Y \rangle = 2 \langle M^* X, M^* Y \rangle = 2 \sum_{i<j} \langle X, [E_i, E_j] \rangle \langle Y, [E_i, E_j] \rangle$$

$$= \sum_i \langle [X, E_i], [Y, E_i] \rangle = \langle X, Y \rangle$$

so $M M^* Y = \frac{1}{2} Y$.

Now $\langle \rho(\omega), \omega \rangle = \frac{1}{4} \| M \omega \|^2$.

If $\omega = M^* X$ for some $X \in \hat{G}$ we have

$$\| M \omega \|^2 = \langle M M^* X, M M^* X \rangle = \langle M^* X, M^* M M^* X \rangle =$$

$$= \frac{1}{2} \| \omega \|^2$$

If $\omega \in (\operatorname{Im} M^*)^\perp$ then $M \omega = 0$. So

$$0 \le \langle \rho(\omega), \omega \rangle \le \| \omega \|^2 / 8$$

$\blacksquare$

Now we will give an expression of the generalized Ricci tensor Q_p in terms of eigenvalues and eigenvectors of ρ.

Given a 2-form $\omega \in A^2_x(M)$ we define an operator $\theta_\omega : A^p_x(M) \to A^p_x(M)$ extending linearly

$$\theta_\omega (\tilde{X}_{i_1} \wedge \dots \wedge \tilde{X}_{i_p}) = \sum_{k=1}^{p} (i_{X_{i_k}} \omega) \wedge \tilde{X}_{i_1} \wedge \dots \wedge \widehat{\tilde{X}_{i_k}} \wedge \dots \wedge \tilde{X}_{i_p}$$

where, as usual, $\{X_1, \dots, X_n\}$ is an orthonormal basis for $T_x M$, $\{\tilde{X}_1, \dots, \tilde{X}_n\}$ its dual basis and i_X is interior multiplication.

It is not difficult to see that θ_ω is a well defined antisymmetric operator.

The curvature operator is symmetric hence diagonalizable and we will denote by $\{\omega_s\}$ an orthonormal basis of eigenvectors of ρ and by λ_s the relative eigenvalues.

<u>THEOREM</u> : $\langle Q_p \omega, \tau \rangle = \sum_{s=1}^{\binom{n}{2}} \lambda_s \langle \theta_{\omega_s} \omega, \theta_{\omega_s} \tau \rangle$.

<u>Proof</u> : A direct calculation gives:

(a) $\widetilde{R(X,Y)} Z = - \sum_{s=1}^{\binom{n}{2}} \lambda_s \langle \omega_s, \tilde{X} \wedge \tilde{Y} \rangle i_Z \omega_s$

(b) $\theta_{\omega_s} (V_1 \wedge \dots \wedge V_p) = \sum_{i=1}^{n} \sum_{k=1}^{p} \omega_s (V_k, X_i) \tilde{V}_1 \wedge \dots \wedge \widehat{\tilde{V}_k} \wedge \tilde{X}_k \wedge \dots \wedge \tilde{V}_p$

(c) $\sum_s \lambda_s \omega_s (V, X_i) \theta_s (\tilde{W}_1 \wedge \dots \wedge \tilde{W}_p) =$

$$= \sum_h \tilde{W}_1 \wedge \dots \wedge \widehat{\tilde{W}_h} \wedge R(V, X_i) \tilde{W}_h \wedge \dots \wedge \tilde{W}_p$$

where $\{X_1, \dots, X_n\}$ is an orthonormal basis with dual basis $\{\tilde{X}_1, \dots, \tilde{X}_n\}$

So we have:

$$\langle Q_p(\tilde{v}_1\wedge\dots\wedge\tilde{v}_p),\ \tilde{w}_1\wedge\dots\wedge\tilde{w}_p\rangle =$$

$$= \sum_{h,t}\langle R(X_h,V_t)(\tilde{v}_1\wedge\dots\wedge\hat{\tilde{v}}_t\wedge\tilde{X}_h\wedge\dots\wedge\tilde{v}_p),\ \tilde{w}_1\wedge\dots\wedge\tilde{w}_p\rangle =$$

$$= \sum_{h,t}\langle \tilde{v}_1\wedge\dots\wedge\hat{\tilde{v}}_t\wedge\tilde{X}_h\wedge\dots\wedge\tilde{v}_p,\ \sum_h \tilde{w}_1\wedge\dots\wedge R(V_t,X_h)\tilde{w}_h\wedge\dots\wedge\tilde{w}_p\rangle =$$

$$\overset{(c)}{=} \sum_{h,t,s}\langle \lambda_s\,\omega_s(V_t,X_h)\,\tilde{v}_1\wedge\dots\wedge\hat{\tilde{v}}_t\wedge\tilde{X}_h\wedge\dots\wedge\tilde{v}_p,\ \theta_{\omega_s}(\tilde{w}_1\wedge\dots\wedge\tilde{w}_p)\rangle$$

$$\overset{(b)}{=} \sum_s \lambda_s\langle\theta_{\omega_s}(\tilde{v}_1\wedge\dots\wedge\tilde{v}_p),\ \theta_{\omega_s}(\tilde{w}_1\wedge\dots\wedge\tilde{w}_p)\rangle$$

■

<u>Proposition</u>: If $\theta_{\omega_s}\omega = 0$ $\forall s = 1,\dots,\binom{n}{2}$, then $\omega = 0$ if $p\neq 0,n$

<u>Proof</u>: take ~~the new~~ an orthonormal basis $X_1,\dots,X_n$.

<u>claim</u> $A^p_x(M) = \operatorname{span}\{\theta_{\omega_s}\beta : \beta\in\Lambda^p_x(M)\}$.

In fact

$$\tilde{X}_{i_1}\wedge\dots\wedge\tilde{X}_{i_p} = \theta_\tau(\tilde{X}_j\wedge\tilde{X}_{i_2}\wedge\dots\wedge\tilde{X}_{i_p})$$

where $\tau = \tilde{X}_j\wedge\tilde{X}_{i_1}$ and $j\notin\{i_1,\dots,i_p\}$

Now suppose $\theta_{\omega_s}\omega = 0$ $\forall s$. Then $\forall\beta\in A^p_x(M)$

$$0 = \langle\theta_{\omega_s}\omega,\beta\rangle = -\langle\omega,\theta_s\beta\rangle$$

and the claim implies $\omega = 0$

■

As a corollary we get the following result

<u>THEOREM</u> (Meyer): Let M be a compact riemannian manifold whose curvature operator is non negative and positive at some point. Then $\forall p = 1,\dots,n-1$, $H^p(M,\mathbb{R}) = 0$.

<u>Proof</u>: By the previous theorem, the integrated Weitzenböck formula can be written as:

$$(\Delta\omega,\omega) = \|\nabla\omega\|^2 + \sum_s\int_M \lambda_s\|\theta_{\omega_s}\omega\|^2$$

By the last proposition in our hypothesis the last term is positive for all $\omega\neq 0$ and $p = 1,\dots,n-1$ so that the only harmonic form is the zero form.

■

Even in the case where the curvature operator is only non negative we can get strong information on the manifold since harmonic forms will necessarily be parallel and existence of parallel forms is rare. Let us discuss this point a little more.

We will assume for simplicity, from now on that the manifold M is compact and <u>simply connected</u>.

Let $x\in M$. For all piecewise smooth loops γ at x we have an orthogonal, orientation preserving transformation of T_xM obtained by parallel translation along γ. Such transformations form a subgroup of $SO(T_xM)$ called the holonomy group of M at x. It is easy to see that holonomy groups at two different points are isomorphic. Holonomy groups are "often trivial" i.e. often the ~~all~~ whole of $SO(n)$. When this is not so we get strong conclusions as, for example, the following result (and others that we will see later on):

<u>THEOREM</u> (de Rham decomposition) In our hypothesis if the holonomy group splits as a product $G_1 \times \dots \times G_k$ then M is the riemannian product of manifolds M_i with holonomy G_i.

We observe also that the de-Rham decomposition is unique up to order.

We will now give an idea of the proof of the following result:

<u>THEOREM</u> (Gallot-Meyer) Let M be a compact simply connected riemannian manifold with non negative curvature operator. Then M is a riemannian product of manifolds of the following types:

(A) n-dimensional manifolds with holonomy $SO(n)$ and real cohomology as the n-sphere.

(B) $2k$-dimensional manifolds with holonomy $U(k)$ and real cohomology as the complex projective space.

(C) Symmetric spaces.

<u>Idea of the Proof</u>: First we observe that each factor in the de-Rham decomposition of M has non negative curvature operator so we can work separately on each irreducible component.

We look at a point $x \in M$ and the action of the holonomy group on the unit sphere of T_xM. A theorem of Berger guarantees that if this action is not transitive then the manifold is a symmetric space. Subgroups of $SO(n)$ which act transitively on the unit sphere where classified by Berger. They are:

1. $SO(n)$, $U(d)$ $\quad n=2d$
2. $Spin(9)$ $\quad n=16$
3. $Sp(1)\cdot Sp(d)$ $\quad n=4d$
4. $SU(d)$ $(n=2d)$, $Sp(d)$ $(n=4d)$, $Spin(7)$ $(n=8)$, G_2 $(n=7)$

Brown and Gray proved that a compact simply connected manifold with holonomy $Spin(9)$ is isometric to the Cayley projective plane, hence is symmetric. If the holonomy is $Sp(1)\cdot Sp(d)$ then M is an Einstein manifold (Berger) and with our hypothesis, again a symmetric space (Tachibana). Finally manifolds with holonomy as in (4) are Ricci flat (Berger) and therefore, in our hypothesis are flat which is incompatible with ~~compactness and simply~~ the fact that M is compact and simply connected.

So the only possible groups in the non symmetric case are $SO(n)$ and $U(d)$.

From the integrated Weitzenböck formula it follows that harmonic forms are parallel if $\rho \geq 0$. But parallel forms are exactly the forms left invariant by the holonomy group and therefore those groups determine, in our case, the real cohomology, which is obviously the one of a sphere in the $SO(n)$ case and the one of $\mathbb{C}P^d$ in the $U(d)$ case.

$\blacksquare$

<u>Remark</u>: Recent results allow us to be more precise in the above theorem proving that in case (A) the manifold is actually homeomorphic to a sphere and in case (B) is bi-holomorphic to $\mathbb{C}P^n$.