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COLLEGE

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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YANG-MILLS FIELDS IN DIFFERENTIAL GEOMETRY.

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Complementary Lectures on Yang Mills fields in Differential Geometry (preliminary version)

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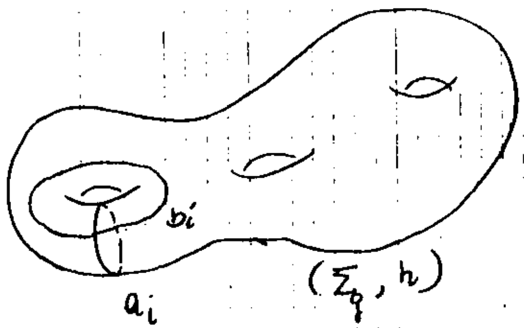
Abstract

These notes deal with the construction of "multiinstanton" solutions for $SU(2)$ on S^4 using the 'Levi Civita' trick, following the treatment of Atiyah [ ]; they are also aimed at giving an elementary introduction to YM theory on Riemann surfaces; the Jacobian variety pertaining to a R.S. is naturally interpreted as a symplectic quotient in the sense of Marsden-Weinstein; this is the most elementary example of the deep relationship existing among holomorphic geometry of vector bundles, YM equations, and symplectic geometry.

- **Maxwell theory on a Riemann surface**



Σ_g : compact Riemann surface of genus g (= algebraic curve) h metric

Kähler form attached to the metric $h = h_{z\bar z}\, dz\, d\bar z$

$$\omega = \frac{i}{2} h_{z\bar z}\, dz \wedge d\bar z$$

(z local coordinate)

$g = \frac{1}{2} b_1$ $\quad b_1$ = first Betti number $\equiv \dim H_1(\Sigma_g, \mathbb{R})$ $(= 2g)$

$\|$

handles $= \dim H^1(\Sigma_g, \mathbb{R})$ (De Rham) $=$

$= \dim \mathcal{H}^1(\Sigma_g)$ (Hodge)

$\mathcal{H}^1(\Sigma_g) = \{$ harmonic 1 forms $\}$

(g = dim of the (smooth) space of holomorphic 1 forms (abelian differentials))

Let $L \to \Sigma_g$ be a complex $U(1)$ line bundle on Σ_g i.e. L is hermitian line bundle $(L, \langle\ \rangle)$ ($\langle\ \rangle$ a metric on L)

$\Omega^0(L) \equiv$ sections of L

(a (finitely generated projective) module on $\Omega^0(\Sigma_g) \equiv$ functions on Σ_g

$s \in \Omega^0(L)$ $f \in \Omega^0(\Sigma_g)$ $\Rightarrow$ $fs \in \Omega^0(L)$ ...

the metric $<,>$ reads (locally)

$$<s' s>(x) = H(x)\, \bar{s'}(x)\, s(x) \qquad H > 0$$

(x a local coordinate)

Prehilbert space structure on $\Omega^0(L)$:

$$< s', s > := \int_{\Sigma_g} < s', s > (x) \, (*1)$$

volume form (= ω)

($*$ Hodge star)

$$\Omega^k(L) := \Omega^0 (L \otimes_{\Omega^0(\Sigma)} \Lambda^k T^* \Sigma_g)$$

(L valued forms)

Using h we get a metric on $\Lambda^k T^* \Sigma_g$ $\quad k = 0, 1, 2$

$$<\omega' \omega> = \int_{\Sigma_g} \omega' \wedge * \omega'$$

$\Rightarrow$ (by tensoring) get a metric on $\Omega^k(L)$

Connection:

$$\nabla : \Omega^0(L) \to \Omega^1(L) \qquad \text{(linear)}$$

$$\nabla(fs) = df\, s + f \nabla s \qquad \text{(Leibniz)}$$

compatibility with $<>$

$$d < s, s' > = < \nabla s, s' > + < s, \nabla s' >$$

The general given vector bundles E, F and connections ∇_E ∇_F, one can form

$$\nabla_{E \otimes F} = \nabla_E \otimes I_F + I_E \otimes \nabla_F$$

$\Rightarrow$ get a connection on $\Omega^k(L)$ (again denoted by ∇)

$\Rightarrow$ this boiling down to a generalized De Rham sequence

$$(*) \quad \Omega^0(L) \xrightarrow{\nabla} \Omega^1(L) \xrightarrow{\nabla} \Omega^2(L) \to \dots$$

in general $R_\nabla = \nabla^2$ does not vanish (so $(*)$ is not a complex)

R_∇ : the curvature operator

Basic property:

$$R_\nabla(fs) = f R_\nabla s$$

(module property, or linearity with r. to functions)

Given a connection ∇^0 (compatible with $<>$) any other compatible connection is given by

$$\nabla = \nabla^0 + a$$

$$a^* = -a \quad , a \in \Omega^1(\mathrm{End}\, L) = \Omega^1(L^* \otimes L)$$

$= \Omega^1(\Sigma_g)$ since $L^* \otimes L$ is a trivial line bundle. a is thus a imaginary valued 1 form

i.

$L^* \otimes L$ is trivial since its (first) Chern class

$$c_1(L^*\otimes L) = c_1(L^*) + c_1(L) = -c_1(L) + c_1(L) = 0$$

$\left(c_1(L) = \frac{i}{2\pi}\int_\Sigma R^\nabla \right)$ or equivalently if L corresponds to transition functions $\{g_{ij}\}$, L^* (as a hermitian bundle) corresponds to $\bar g_{ij} = g_{ij}^{-1}$.

(independent of ∇) (R^∇ is a closed form)

Let $\mathcal{Y}$ = {imaginary valued functions} $\equiv \Omega^0(\mathcal{Y})$

Then: the space of compatible connections $\mathcal{A}$ is an affine space modelled on $\Omega^1(\mathcal{Y})$

If $\nabla' = \nabla + a$ $\quad R^{\nabla'} = R^\nabla + da + a\wedge a = R^\nabla + da$

A connection induces a generalized De Rham sequence via the position:

$$\nabla(T) = [\nabla, T] \qquad T\in \operatorname{End} E \ (\text{in general } V.b)$$

$$\Omega^0(\operatorname{End}E) \xrightarrow{\nabla} \Omega^1(\operatorname{End}E) \xrightarrow{\nabla} \Omega^2(\operatorname{End}E) \to \dots$$

$$(\nabla(T) = [\nabla, (-1)^k T] \qquad , T\in\Omega^k(\operatorname{End}E))$$

$$\nabla^2(T) = [R^\nabla, T]$$

$$\nabla(R^\nabla) = 0 \qquad \text{(Bianchi identity)}$$

A symmetry group $\mathcal{G}$ (the gauge group) acts on $\mathcal{A}$

$$\mathcal{G} = \{ u \in \Omega^0(\Sigma_g), \quad u^* = u^{-1} \}$$

The "gauge algebra" is $\mathcal{Y}$

$$\mathcal{Y} = \{ a \in \Omega^0(\Sigma_g), \quad a^* = -a \}$$

The gauge group is not connected:

$$u : \Sigma_g \to S^1$$

$u \in \mathcal{G}_0$ (connected component of the identity) iff $u = e^a$ $\quad a\in\mathcal{Y}$

u belongs to the same component of v iff $uv^{-1}\in\mathcal{G}_0$ (homotopy equivalence)

or, equivalently, iff

$$\left[u^*\left(\frac{dz}{z}\right)\right] = \left[v^*\left(\frac{dz}{z}\right)\right] \quad \text{in } H^1(\Sigma_g, \mathbb{Z})$$

$*$ = pull back, $\frac{1}{2\pi}\frac{dz}{z} = d\varphi$ angular form on S^1

♦ $\mathcal{G}/\mathcal{G}_0 \simeq H^1(\Sigma_g, \mathbb{Z})$

$\mathcal{G}$ acts on $\mathcal{A}$: if $u\in\mathcal{G}$ then

$$\nabla^u := u\circ\nabla\circ u^{-1} = \nabla + u\nabla(u^{-1})$$

Locally $\nabla = d + a$; if $\xi\in\Omega^0(L)$

$$\nabla(u^{-1})\xi = [d+a, u^{-1}]\xi =$$

$$= (d+a)(u^{-1}\xi) - u^{-1}(d+a)\xi =$$

$$= d(u^{-1}\xi) + \cancel{a u^{-1}\xi} - u^{-1}d\xi - \cancel{u^{-1}a\xi}$$

$$= (du^{-1})\xi + \cancel{u^{-1}d\xi} - \cancel{u^{-1}d\xi}$$

whence

$$\nabla^u = \nabla + u\, d(u^{-1})$$

if $u = e^{i\varphi}$ $\quad \nabla^u = \nabla - i\, d\varphi$

(...)

$$R^{\nabla^u} = (\nabla^u)^2 = (\nabla + u\,d(u^{-1}))^2 =$$

$$= \nabla^2 + \nabla\cdot u\,du^{-1} + u\,du^{-1}\nabla + \underbrace{(u\,du^{-1})^2}_{=0}$$

$$= \nabla^2 + \underbrace{(d+a)\,u\,du^{-1} + u\,du^{-1}(d+a)}_{\Diamond}$$

$$(d+a)\,u\,du^{-1}\xi = d[u\,du^{-1}\xi] + a\,u\,du^{-1}\xi$$

$$(u\,du^{-1})(d+a)\xi = u\,du^{-1}\,d\xi + u\,du^{-1}a\,\xi$$

$$d[u\,du^{-1}\xi] = d(u\,du^{-1})\xi - u\,du^{-1}\,d\xi$$

$\Rightarrow \Diamond = du\cdot du^{-1}\cdot\xi$ $\quad$ but $d(uu^{-1}) = 0$
$\Rightarrow du\,u^{-1} = -u\,du^{-1}$

$\Rightarrow \Diamond = 0$

So $R^{\nabla^u} = R^\nabla$ (gauge invariance)

(in general $R^{\nabla^u} = u\,R^\nabla u^{-1}$: covariance)

The Yang Mills functional on it

$$ym(\nabla) = -\int_{\Sigma_g} R^\nabla \wedge * R^\nabla \geq 0 \equiv \langle R^\nabla R^\nabla \rangle$$

ym connections: critical pts of ym

upon variation we'll get:

∇ ym iff R^∇ is the unique (Hodge) harmonic representative of $c_1(L) \in H^2(\Sigma_g, \mathbb{Z})$ (which minimises ym)

In fact ∇ ym iff

$$\spadesuit \quad \frac{d}{dt}\, ym(\nabla + ta)\Big|_{t=0} = 0 \quad \forall a \in \Omega^1(\mathcal{Y})$$

$$\spadesuit = \frac{d}{dt}\langle R^\nabla + t\,da,\ R^\nabla + t\,da\rangle\Big|_{t=0}$$

$\Rightarrow \langle da,\ R^\nabla\rangle = 0 \quad \forall a \in \Omega^1(\mathcal{Y})$

$\Rightarrow \langle a, \delta R^\nabla\rangle = 0$ $\quad$ δ: Hodge adjoint

$$\Rightarrow \begin{cases} \delta R^\nabla = 0 \\ +\ dR^\nabla = 0 \end{cases} \Leftrightarrow \underset{d\delta + \delta d}{\Delta} R^\nabla = 0$$

Hessian at ∇ critical

$$H_\nabla(b,b) = \langle db\ db\rangle = \langle b, \delta db\rangle \geq 0$$

$\Rightarrow \nabla$ minimises ym $\quad$ ($H_\nabla = 0$ iff $db = 0$)

By gauge invariance ∇ critical $\Rightarrow \nabla^u$ critical

$\Rightarrow$ a solution is to be considered up to gauge equivalence

Let us examine the situation on Σ_g

If ∇ is critical $\nabla' = \nabla + a$ is critical

iff $R^{\nabla'} = R^{\nabla}$ i.e. $da = 0$

Hence $\{ym\}/\mathcal{G}_0 \simeq H^1(\Sigma_g, \mathbb{R})$

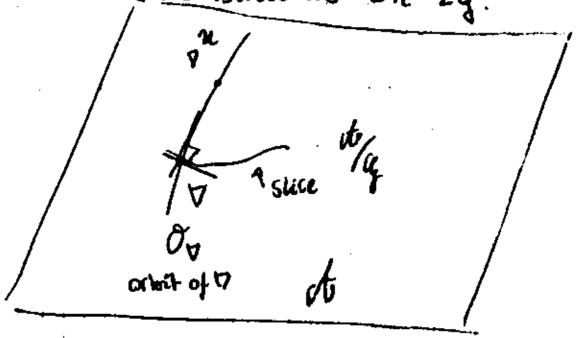
since if $u = e^{i\varphi}$ $\nabla^u = \nabla - i\,d\varphi$ (globally)

Thus $\{ym\}/\mathcal{G} \simeq H^1(\Sigma_g, \mathbb{R}) / H^1(\Sigma_g, \mathbb{Z})$ unitary ♦

$\mathcal{M} \cong$

moduli space

which by definition is the Jacobian $J(\Sigma_g)$ (a complex (algebraic) torus) attached to Σ_g.
It parametrises isomorphism classes of holomorphic line bundles on Σ_g.



Observe that infinitesimally

$(T\mathcal{O}_\nabla)^\perp = \{a / \delta a$

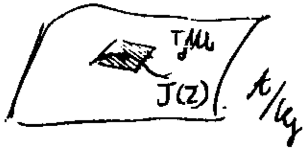
$\langle d\varphi, a\rangle = 0 \ \forall \varphi$ iff

$\delta a = 0$

Hence at least formally

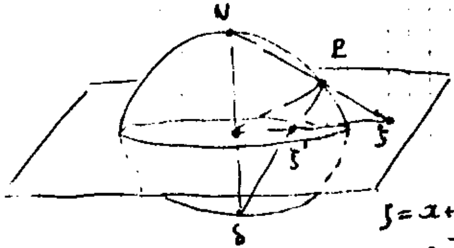
$T\mathcal{M} \simeq \mathcal{H}^1(\Sigma_g) = \{$ harmonic 1 forms on Σ_g

We shall reexamine this question later on



• ym on S^2 Here $J(S^2) \simeq \{pt\}$

via stereographic projection $S^2 \simeq \mathbb{C}^1$ as a R.S.



$P = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$= \left(\frac{2x}{1+r^2}, \frac{2y}{1+r^2}, \frac{r^2-1}{r^2+1}\right)$

$\zeta = \frac{z_1}{z_0}$

$\zeta' = \frac{z_0}{z_1} = \zeta^{-1}$

in $\mathcal{U}_0 \cap \mathcal{U}_1$

Levi Civita trick: $E \to X$ v. bundle rank $n < N$. Look at $X \times \mathbb{R}^N \to X$ trivial $\equiv I_N$

it as a subbundle of I_N

ex $\mathbb{R}^3$

$E_x = \mathrm{Im}\, u_x \equiv P_x \mathbb{R}^N$

$u_x : \mathbb{R}^n \to \mathbb{R}^N = \mathrm{Im}\, u_x \oplus \mathrm{Im}\, u_x^{\perp}$

$\mathrm{Im}\, u_x = P_x \mathbb{R}^N$ (a "gauge")

u "orthogonal":

$P = uu^*$ $u^* u = I_{\mathbb{R}^n}$ u: partial isometry

in block form $u = \begin{pmatrix} u_0 & 0 \\ 0 & 0 \end{pmatrix}$ $u_0 u_0^* = u_0^* u_0 = 1$

section of E: $f = ug$ ($g : X \to \mathbb{R}^n$) in the gauge u

$f \to df$ (in general) = flat connection on I_N

$\nabla f = P\, df$ (gives the Levi Civita)

$$\nabla f = P\,df = uu^* d(ug) = uu^* du\, g + u\cancel{u^* u}\, dg$$

$$= u(u^* du) g + u\, dg = u\left[(u^* du) g + dg\right]$$

in the gauge u: $\nabla = d + A$ $A = u^* du$

curvature: $dA + A \wedge A = du^* du + u^* du\, u^* du$

Apply:

$$u: \mathbb{C} \to \mathbb{C}^2$$

$$u = \begin{pmatrix} -1 \\ z \end{pmatrix} (1+|z|^2)^{-\frac{1}{2}}$$

z local coordinate for S^2

$$u^* u = (1+|z|^2)^{-1} (-1, \bar z) \begin{pmatrix} -1 \\ z \end{pmatrix} = I_{\mathbb{C}} = 1.$$

$$u u^* = P = (1+|z|^2)^{-1} \begin{pmatrix} -1 \\ z \end{pmatrix} (-1, \bar z) =$$

$$= (1+|z|^2)^{-1} \begin{pmatrix} 1 & -\bar z \\ -z & |z|^2 \end{pmatrix}$$

$$A = u^* du = (1+|z|^2)^{-\frac12} (-1, \bar z)\, d\, (1+|z|^2)^{-\frac12} \begin{pmatrix} -1 \\ z \end{pmatrix}$$

$$= (1+|z|^2)^{-1} (\bar z\, dz) + (1+|z|^2)^{\frac12} d\,(1+|z|^2)^{-\frac12} =$$

$$= (1+|z|^2)^{-1} \bar z\, dz + (1+|z|^2)^{\frac12} \left(-\frac{1}{2}\right) (1+|z|^2)^{-\frac32} d|z|^2$$

$$= (1+|z|^2)^{-1} \left[\bar z\, dz - \frac12 \bar z\, dz + \frac12 z\, d\bar z \right] :$$

$$= \frac12 (1+|z|^2)^{-1} (\bar z\, dz - z\, d\bar z)$$

$F = dA + A \wedge A \qquad A \wedge A = 0$

$$-|z|^2 d\bar z dz - |z|^2 d\bar z dz$$

$$dA = \frac12 \, \frac{2 d\bar z\, dz\, (1+|z|^2) + (\bar z\, dz - z\, d\bar z)(\bar z\, dz + z\, d\bar z)}{(1+|z|^2)^2}$$

$$= \frac{1}{\cancel{2}}\, \cancel{2} \left(\frac{d\bar z\, dz\, (1+\cancel{|z|^2} - \cancel{|z|^2})}{(1+|z|^2)^2} \right) = \frac{d\bar z\, dz}{(1+|z|^2)^2}$$

$$= 2i \, \frac{dx \wedge dy}{(1+x^2+y^2)^2}$$

$$\int_{S^2} F = 2i \iint \frac{dx\,dy}{1+x^2+y^2} :$$

$$= 2i \cdot 2\pi \int_0^\infty \frac{\rho\, d\rho}{(1+\rho^2)^2} = 2i\pi \int_0^\infty \frac{d\rho^2}{(1+\rho^2)^2}$$

$$= 2\pi i$$

"instanton" $(R = -c_1(L))$

$c_1(L) = -1$ (= tautological bundle)

$\Omega = -2\pi i\, \omega$

Solution:

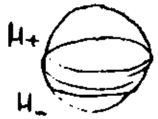
Holomorphic connection

$\nabla'' = \bar\partial$

$\nabla' = \partial \log h = h^{-1} \partial h$

$h = e^{-f} \qquad e^{f} \partial e^{-f} = \cancel{f}\cancel{f} - \partial f$

This is the monopole (E&H)



$U(1)$ principal bundle

$U_j \times G \xrightarrow{\Phi_j} \pi^{-1}(U_j)$

$(x, g) \to s_i(x) g$

$s_i = g_{ij} s_j$

$s_i : S^2 \to U(1)$

$H_\pm \times U(1)$

$s_\pm \ (\vartheta, \varphi) \to e^{i\psi_\pm} \qquad s_- = g_{-+} s_+$

$$g_{-+} = e^{i(\psi_- - \psi_+)} = e^{in\phi}$$

n integer!

connection $\omega = \begin{cases} A_+ + i\, d\psi_+ \\ A_- + d\psi_- \end{cases}$

$A_+ = A_- + i n \phi \qquad (\omega(x,g) = g^{-1} A(x) g + g^{-1} dg)$

$$A_\pm = \frac{n}{2} (\pm 1 - \cos\vartheta)\, d\varphi$$

$$F_\pm = dA_\pm = \frac{n}{2} \sin\vartheta\, d\vartheta \wedge d\varphi .$$

$\frac{1}{2\pi}\int_{S^2} F = +\frac{n}{2}\cdot 2 = n$ $c_1(L) = +n$ $c_1(E) = n$

$n=1$ Hopf bundle $S^3 \to S^2$ $\pi_3(S^2) \simeq \mathbb{Z}$

Compare the expressions

$\hat{A}_+ = \frac{1}{2}(1+\cos\vartheta)\,d\varphi = \left(1 + \frac{-1+r^2}{r^2+1}\right) d\varphi =$

$= \frac{+r^2 \cancel{-1} \cancel{+1} + r^2 = 2r^2}{r^2+1} \cdot \frac{-i}{2} \frac{\bar{z}\,dz - z\,d\bar{z}}{r^2}$

$= -i \frac{\bar{z}\,dz - z\,d\bar{z}}{|z|^2+1} = i A$ "$x^* dx$"

Using local charts technique one can describe the Riemannian connection on S^2

$R \in S^2$

$R = (\sin\vartheta\cos\varphi,\ \sin\vartheta\sin\varphi,\ \cos\vartheta)$

Let x_1, x_2 be two tangent v. fields

$x_1 = \partial_\vartheta R$
$x_2 = \partial_\varphi R$ $\to x = (x_1\ x_2) = \begin{pmatrix} \cos\vartheta\cos\varphi & -\sin\vartheta\sin\varphi \\ \cos\vartheta\sin\varphi & \sin\vartheta\cos\varphi \\ -\sin\vartheta & 0 \end{pmatrix}$

Consider the YM solution

$F = \frac{d\bar{z}\wedge dz}{(1+|z|^2)^2} = -2i \frac{dx\,dy}{(1+x^2+y^2)^2}$

$= 2i\ e_1 \wedge e_2$ $e_1 = \frac{1}{\sqrt{\pi}}\frac{dx}{(1+x^2+y^2)}$ $e_2 = \frac{1}{\sqrt{\pi}}\frac{dy}{(1+x^2+y^2)}$

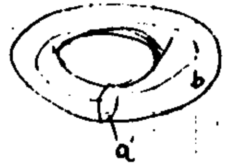
$\omega = \frac{1}{2\pi} F = \frac{+1}{\pi} \frac{dx\,dy}{(1+x^2+y^2)}$ $\omega = e_1 \wedge e_2 = e_1 \wedge * e_1$

$e_1 \perp e_2$ $*\omega = 1$

$\delta\omega = d\,1 = 0$

12 bis

- The Torus $(g=1)$



$M = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$

The Kähler form: $\omega = \frac{i}{2}(\mathrm{Im}\,\tau)^{-1} dz\wedge d\bar{z} = dq\wedge dp$

if I have local coordinates.

$z = q + \tau p$ inherited by

(q,p) coordinates on $\mathbb{R}^2 \simeq \mathbb{C}$

$\mathrm{Im}\,\tau > 0$

$dz = dq + \tau\,dp$ holomorphic 1-form

$(d\bar{z} = dq + \bar{\tau}\,dp$ antihol...$)$

$$\begin{pmatrix} dq = -\bar{\tau}(\tau-\bar{\tau})^{-1} dz + \tau(\tau-\bar{\tau})^{-1} d\bar{z} \\ dp = (\tau-\bar{\tau})^{-1} dz - (\tau-\bar{\tau})^{-1} d\bar{z} \end{pmatrix}$$

• S^4 (The basic instanton)

$\mathbb{C} \to \mathbb{H}$: $\left\{ \begin{array}{cccc} \begin{pmatrix}1&0\\0&1\end{pmatrix} & \begin{pmatrix}i&0\\0&-i\end{pmatrix} & \begin{pmatrix}0&1\\-1&0\end{pmatrix} & \begin{pmatrix}0&i\\i&0\end{pmatrix} \\ & i & j & k \end{array} \right\}$

$ij = k$ $\qquad Sp(1) \simeq S^3 \simeq SU(2)$

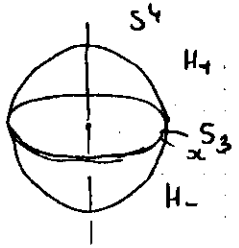
$S^4 \simeq P_1(\mathbb{H})$ $\qquad h_1' = h_1 \lambda$ $\qquad H^2 \simeq \mathbb{C}^4 \simeq \mathbb{R}^8$

$\qquad\qquad h_2' = h_1 \lambda$

$x = x_0 + x_1 i + x_2 j + x_3 k$

$\bar{x} = x_0 - x_1 i - x_2 j - x_3 k$

$x^{-1} = x \cdot |x|^{-2}$ $\qquad |x| = |x\bar{x}|^{\frac{1}{2}}$



$x_2 = |x_2| g_2 \in SU(2)$

$x_1 = |x_1| g_1 \sim S^3$

$|x| = x_2 x_1^{-1}$ if $|x| = 1$

$x = g_- g_+^{-1}$

$(e^{i\varphi} = e^{i\psi_-} e^{-i\psi_+}$

$x^k = g_- g_+^{-1}$

$$\omega = \frac{dx\, d\bar{x}}{(1+|x|^2)^2} \qquad \text{self dual}$$

$S^7 \xrightarrow{S^3} S^4$ $\qquad \pi_7(S^4) \simeq \mathbb{Z}$

• Symplectic geometry (basic notions)

(M, ω) smooth symplectic manifold

(ω closed non degenerate 2-form)

G Lie group

G acts symplectically on M $\qquad (x \to g \cdot x)$

$g^* \omega = \omega \quad \forall g \in G$

$\mathfrak{g}$ Lie algebra $\quad \mathfrak{g}^*$ dual of $\mathfrak{g}$

$\mu : M \to \mathfrak{g}^*$ equivariant moment map

iff $\quad \mu(gx) = Ad_g^* \mu(x)$

where Ad^*: coadjoint action of G on $\mathfrak{g}^*$
(dual to the adjoint action)

μ is symplectic since any coadjoint orbit in $\mathfrak{g}^*$ is a symplectic manifold
(Kirillov Kostant Souriau)

$f \in \mathcal{O}_{f_0}$ $\quad (f_0 \in \mathfrak{g}^*)$ orbit through f_0

Kirillov form:

$$B_f\left(ad^*(u) f,\ ad^*(v) f\right) := \langle f, [u,v] \rangle$$

$$(ad^* u\,(f))\,(v) \equiv -\langle f, \underset{(ad\,u.\,v)}{[u,v]} \rangle)$$

μ exists iff G acts on M in a hamiltonian way i.e if $\hat{x}$ is the vector field on M pertaining to x:

$$i_{\hat{x}}\,\omega = d\lambda_x \quad (\text{globally}) \; \lambda_x \text{ function on } M$$

Introducing Poisson brackets via:

$$\{\lambda_x, \lambda_v\} := \omega(\hat{x}, \hat{v})$$

$\Lambda = \{\lambda_x\}$ becomes a Lie algebra since $\{\lambda_x \lambda_v\} = \lambda_{[x,v]}$

For a coadjoint orbit $\mathcal{O}_{f_0}$

$$\lambda_x(f) = \langle f, x \rangle$$

Let $f \in \mathfrak{g}^*$ $\quad G_f \equiv \{ g \in G \,/\, Ad^*_g f = f \}$

isotropy group. Under suitable conditions

$$M_f := \frac{\mu^{-1}(f)}{G_f}$$

is a submanifold of M and inherits a symplectic structure ω_f ; (M_f, ω_f) is called the Marsden Weinstein reduced symplectic s. manifold corresponding to $f \in \mathfrak{g}^*$.

- Symplectic interpretation of ym

G (gauge group) suitably completed w.r.to a Sobolev norm is a Lie group $\mathcal{G}$ (suitably completed as well) is its Lie algebra.

Since $\mathcal{G} \simeq \Omega^2(\mathcal{G})$ and, via the map $\mathcal{G} \to \mathcal{G}^*$: $\varphi \to \langle \int_\Sigma * \varphi , \psi \rangle = \int_\Sigma * \varphi \cdot \psi \quad \varphi, \psi \in \mathcal{G}$

we can identify the coadjoint and the adjoint action

$\mathcal{A}$ is a symplectic manifold:

$$\omega_\nabla (a, b) := \int_\Sigma a \wedge b \qquad a, b \in \Omega^1(\mathcal{G})$$

is the symplectic form

$$\mu(\nabla) = R^\nabla \in \Omega^2(\mathcal{G}) \simeq \mathcal{G}$$

is the moment map : G acts symplectically on $\mathcal{A}$ and (in gen $R^{\nabla^x} = x R^\nabla x^{-1}$ = i.e. on the R^∇)

then $ym = \|\mu\|^2$

Let R^∇ be the curvature of a ym connection:

then

$$\mathcal{M}_b = \frac{\{ym\}}{G} = \frac{\mu^{-1}(R^\nabla)}{G_{R^\nabla} = G}$$

appears as a MW quotient

Let us describe the symplectic structure:

According to the previous discussion

$$T_0 \mathcal{A}b \simeq \mathcal{H}^1(\Sigma_g) \simeq \mathbb{R}^{2g}$$

$$\mathcal{A}b \simeq \mathbb{C}^g/\Lambda \quad \Lambda \text{ lattice of rank } 2g$$

We get a symplectic structure on $J(\Sigma_g)$:

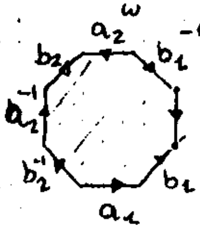
$$\omega(\vartheta, \eta) = \int_\Sigma \vartheta \wedge \eta \qquad \vartheta, \eta \in \mathcal{H}^1(\Sigma_g)$$

But this is the natural polarization of $J(\Sigma_g)$!

(If ϑ, η are any closed 1-forms

$$\omega(\vartheta, \eta) = \sum_{i=1}^g \int_{a_i} \vartheta \int_{b_i} \eta - \int_{a_i} \eta \int_{b_i} \vartheta)$$

One normalizes the invariant 1-forms as follows $\int_{a_i} \alpha_j = \delta_{ij}$ $\int_{b_j} \alpha_j = 0$



a_1^{-1}

$(g=2)$

$\{a_i, b_i\}_{i=1..g}$ are the homology 1 cycles considered at the beginning; If $\{\omega_1 \dots \omega_g\}$ is a basis of holomorphic 1 forms (abelian differentials) and we normalize as follows

$$\int_{a_i} w_j = \delta_{ij} \quad , \text{then} \quad \int_{b_i} w_j = z_{ij}$$

and Z is such that $Z = Z^t$ $\operatorname{Im} Z > 0$

The traditional map $\Sigma \to J(\Sigma)$ is given, fixing $P_0 \in \Sigma$ by

$$\Sigma \ni P \to \left(\int_{P_0}^P \omega_1 \dots \int_{P_0}^P \omega_g \right)$$

$$\Lambda = \mathbb{Z}^g + Z\,\mathbb{Z}^g \qquad J(\Sigma_g) \simeq \mathbb{C}^g/\Lambda$$

(i.e belongs to the so called Siegel domain H_g)

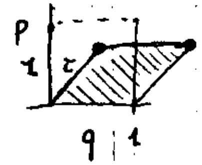
$J(\Sigma_g)$ is also Kähler ; in suitable coordinates ω becomes:

$$\circledast \quad \omega = \sum_{i=1}^g dq_i \wedge dp_i = \frac{i}{2} \sum_{\alpha\beta} W_{\alpha\beta}\, dz_\alpha \wedge d\bar{z}_\beta$$

$$W = (\operatorname{Im} Z)^{-1} \qquad (T^* J(\Sigma_g) \simeq T^{*(1,0)} J(\Sigma_g))$$

(If $g=1$ $\quad z = q + \tau p \quad \operatorname{Im} \tau > 0$)

$J(\Sigma_g) \simeq \Sigma_g$



$g=1$

$J(\Sigma_g)$ is a principally polarized abelian variety

We said that $J(\Sigma_g)$ also parametrizes isomorphism class of holomorphic line bundles (i.e having holomorphic transition functions)

This can be understood in the YM framework as follows:

A holomorphic structure on $L \to \Sigma_g$ consists in choosing $\langle\ \rangle$ on L and taking the antiholomorphic part of a compatible connection ; conversely

in a holomorphic line bundle equipped with a metric $\langle\ \rangle$, there is one and only one connection compatible with both the holomorphic structure and the metric:

this means, more precisely

if $\nabla = \nabla' + \nabla''$

extend ∇: $\nabla : \Omega^0(L) \to \Omega^0(L) \otimes \Omega^1_{\mathbb{C}}(\Sigma_g)$

$\nabla' : \Omega^0(L) \to \Omega^0(L) \otimes \Omega^{(1,0)}(\Sigma_g)$

$\nabla'' \; \Omega^0(L) \to \Omega^0(L) \otimes \Omega^{(0,1)}(\Sigma_g)$

$(\Omega^1_{\mathbb{C}} = \Omega^{(1,0)} \oplus \Omega^{(0,1)})$

and if locally we have

$\langle s\, s \rangle = h\, \bar{s} s$

then $\begin{cases} \nabla'' = \bar\partial & \text{(compatibility with the hol structure)} \\ \nabla' = \partial + \partial \log h \end{cases}$

$(\Rightarrow \nabla$ is compatible with $\langle \; \rangle)$

∇ is called the Chern-Bott connection

$\Rightarrow R^\nabla = d\, \partial \log h = -\partial\bar\partial \underbrace{\log h}_{\text{"Kähler potential"}}$

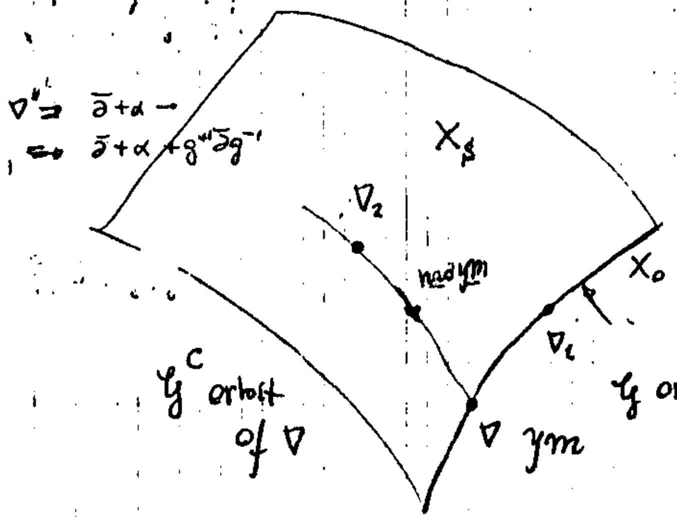
Hence holomorphic structures on $L \to \Sigma_g$ are in 1-1 correspondence with "$\bar\partial$ operators"

i.e. operators of the form

$\bar\partial + \underbrace{\alpha(z) d\bar z}_{\alpha \in \Omega^{(0,1)}(\Sigma_g)}$

two holomorphic structures are equivalent when they are obtained from each other by the action of the complex gauge group $\mathcal{G}^c = \{$ nonzero functions $\}$

(moduli space $\mathcal{M}_{Hol}$) $\qquad \Omega^1(\mathcal{Y}) \simeq \Omega^{(0,1)}(\Sigma_g)$



a is orthogonal to $T_\nabla \mathcal{G}^c$ iff a is harmonic: $\Delta a = 0$

hence $T_\nabla \mathcal{M}_{Hol} \simeq T_\nabla \mathcal{M}$

and proceeding as before $\mathcal{M}_{Hol} \simeq \mathcal{M} \simeq Jac(\Sigma_g)$

since $\mathcal{G}^c$ is retractible to $\mathcal{G}$.

if $X_0 = \mathcal{G}$ orbit of ∇ ym $=$ ym connections

$X_s = \mathcal{G}^c X_0$ (saturation of X_0 w.r.to $\mathcal{G}^c$)

$J(\Sigma) \simeq \underbrace{X_0 / \mathcal{G}}_{\text{symplectic structure on } J(\Sigma)} \simeq \underbrace{X_s / \mathcal{G}^c}_{\text{Kähler structure on } J(\Sigma)}$

compare with

(21)

The Kähler structure is

$$\omega(\alpha,\beta) = \frac{i}{2}\int_\Sigma \bar\alpha\wedge\beta \qquad \alpha,\beta\in\Omega^{(0,1)}$$

if $\{\alpha_i,\beta_i\}$ are harmonic 1 forms dual to $\{a_i\, b_i\}$ then the basis of hol 1 forms $\{\omega_1\ldots\omega_g\}$ is actually given by

$$\left\{\begin{array}{l}\alpha_i = A_{ij}\,\omega_j + \bar A_{ij}\bar\omega_j\\ \beta_i = B_{ij}\,\omega_j + \bar B_{ij}\bar\omega_j\end{array}\right. \qquad A = -\bar Z\,(Z-\bar Z)^{-1}, \quad B = (Z-\bar Z)^{-1}$$

$dz = dq + \tau dp$ holom

$d\bar z = dq + \bar\tau dp$ antihol

$*dq = dp$

$\tau = -i$

$a = dq$

$dq - i\,dp \in \Omega^{(0,1)}$

and we recover (*) upon restricting the Kähler structure to $T_p\mathcal{M}_{Hol}$, or rather, upon rewriting the symplectic structure in terms of the abelian differentials $\{\omega_i\}_{i=1,\ldots g}$.

(if $g=1$ $\quad \tau = i$

$$a\in\Omega^1(\gamma) \to a - i*a \in \Omega^{(0,1)}$$

$$\nabla\gamma \to \nabla(b_1) + *\nabla(b_2)$$
$$db_1 + *\,db_2$$

$(\gamma = b_1 + ib_2)$

$\Rightarrow$ a orthogonal to $\{\nabla\gamma\}$ iff a is harmonic)

$\left(\langle a, db_1 + *\,db_2\rangle = 0 \quad \forall\, b_j\ j=1,2\right.$ iff $\Delta a = 0$ /

(M1)

- Construction of multi instantons

$(c_1(E)=0 \quad c_2(E) = \frac{1}{8\pi^2}\int \mathrm{tr}\, F\wedge F = -k$

As already remarked, a ... ization of the procedure devised for S^2 yields the basic instanton on S^4 and "multi instantons" as well.

Recall the formula for the Grassmannian connection:

$$\nabla f = P\,df$$

Let $Q = 1-P = P^\perp$

Let $B = Q\,dQ$

Define $\nabla_B f = (d + Q\,dQ)f$

∇_B extends ∇ since if $Pf = f$:

$$\nabla_B f = df + Q\,dQ\,f = df + Q\,(d(Qf) - Q\,df)$$
$$= df - Q^2 df = (1-Q)f \quad (Q^2 = Q)$$
$$= P\,df.$$

curvature: $\nabla_B^{\,2} = dB + B^2$

But $dB = dQ\,dQ$

$$B^2 = Q\,dQ\,Q\,dQ = Q\,(dQ^2 - Q\,dQ)\,dQ =$$
$$= Q\,dQ\,dQ - Q\,dQ\,dQ = 0$$

so $\quad \nabla_B^{\,2} = dB$

upon restriction:

$$P\,dB\,P = P\,dQ\,dQ\,P = \quad (Q = 1-P)$$
$$= P\,dP\,dP\,P = \nabla^2$$

(M2)

Let $Q = vv^*$ $\quad v^*v = I$

($v = \begin{pmatrix} 0 & 0 \\ 0 & v_0 \end{pmatrix}$ in block form)

$\Rightarrow u^*v = 0 \qquad (\Rightarrow v^*u = 0)$

then $\nabla^2 = P\,dQ\,dQ\,P = P\,(d(vv^*)\,d(vv^*))\,P$

$= uu^*\,(dv\,v^* + v\,dv^*)(dv\,v^* + v\,dv^*)\,uu^*$

$= P\,dv\,dv^*\,P$

if v is not orthogonal $v = w\rho$ $\quad \rho = (v^*v)^{\frac{1}{2}}$

w orthogonal (polar decomposition)

$Q = ww^* = v\rho^{-2}v^*$

$\Rightarrow \nabla^2 = P\,dv\,\rho^{-2}\,dv^*\,P$

Apply this to $S^4 \simeq P_1(H) = H^2 - 0/\sim$

$(x, y) \sim (x', y')$ iff $\begin{matrix} x' = xq \\ y' = yq \end{matrix}$ $\quad q \neq 0$

(right quaternionic space)

Let

$v(x, y) = Cx + Dy$ $\quad$ a $(N+1 \times N)$

matrix of maximal rank $\forall (x, y)$:

the N columns are independent and generate a N dimensional subspace which only depends on $xy^{-1} \in S^4$

(M3)

Any column gives rise to a tautological line bundle on $S^4 \simeq P_1(H)$, with

$c_2 = +1$

$\Rightarrow$ the total bundle $E^\perp$ has Chern class $= +N$. Let E be the orth bundle to $E^\perp$.

it is a quaternionic line bundle, having $c_2 = -N$

taking an affine coordinate $x = (\alpha, 1)$ $\quad (y = 1)$

we have

$dv = C\,dx$

$dv^* = d\bar{x}\,C^*$

So

$$\nabla^2 = P\,C\,dx\,\underbrace{(\bar{x}C^* + D^*)(Cx + D)}_{\diamond}\,d\bar{x}\,C^*$$

if $\diamond$ is real, then we get a self dual expression!

To be more explicit, choose u: $\begin{matrix} u^*u = 1 \\ u^*v = 0 \end{matrix}$

$A = u^*du$

(these conditions are preserved if $u \to uq$ $\quad q \in Sp(1)$

(transformations of H which preserve norms)

$Sp(1) \simeq SU(2)$

write

$v(x, y) = \begin{pmatrix} C_0 x + D_0 y \\ C_1 x + D_1 y \end{pmatrix}$ $\quad$ $C_0 D_0$ $1 \times N$ matrices, $C_1 D_1$ $N \times N$ matrices

(M4)

without loss of generality assume C_1 non singular

by transforming:

$$v \to R v S \qquad S \text{ real } k\times k \text{ matrix}, \quad R \in Sp(\ldots)$$

get $C_1 = -I \quad C_0 = 0$

So $v(x) = \begin{pmatrix} \Lambda \\ B - x I_{k\times k} \end{pmatrix}$ $\quad \Lambda$ $1\times k$ matrix, B $k\times k$ matrix (assume $B - xI$ non singular)

$\Lambda = (\lambda_1 \ldots \lambda_k)$

Now real entries

$$\begin{cases} \Lambda^* \Lambda + B^* B \text{ real} \\ B^* x + \bar{x} B \text{ real} \quad \forall x \in H \end{cases}$$

The second condition is equivalent to have B symmetric

Correct check using: $\quad (x = 1, i, j, k)$

Let $u = \begin{pmatrix} -1 \\ U \end{pmatrix} \sigma$ $\quad U$ $k \times 1$ matrix, $\sigma > 0$

$$U = \begin{pmatrix} U_1 \\ \vdots \\ U_k \end{pmatrix}$$

$$\begin{cases} u^* v = 0 \\ u^* u = 1 \end{cases}$$

become $\quad -\Lambda + U^*(B - xI) = 0$

$$\sigma^{-2} = I + U^* U$$

At points x such that $B - xI$ is non singular we have:

$$U^* = \Lambda (B - xI)^{-1}$$

(M5)

Then $A = u^* du$ becomes

$$A = \sigma U^* dU \sigma + \sigma^{-1} d\sigma =$$

$$= \sigma^2 U^* dU + \sigma^{-1} d\sigma$$

using $\sigma = (1 + U^* U)^{-\frac{1}{2}}$

$$A = (1 + U^*U)^{-1} U^* dU + (1+U^*U)^{\frac{1}{2}} d(1+U^*U)^{-\frac{1}{2}}$$

$$= (1+U^*U)^{-1} U^* dU + (1+U^*U)^{\frac{1}{2}} \left(-\tfrac{1}{2}\right) (1+U^*U)^{-\frac{3}{2}} (dU^*U + U^*dU)$$

$$= +\frac{1}{2} (1+U^*U)^{-1} (-dU^*U + U^*dU)$$

$$= \operatorname{Im} \frac{U^* dU}{(1 + U^*U)} = \sum_{j=1}^{k} \frac{U_j^* dU_j}{(1 + \sum U_j^* U_j)}$$

Note for the moment: $\quad U = \Lambda (B - xI)^{-1} =$

$$= (\lambda_1 (b_1 - x)^{-1}, \ldots, \lambda_k (b_k - x)^{-1})$$

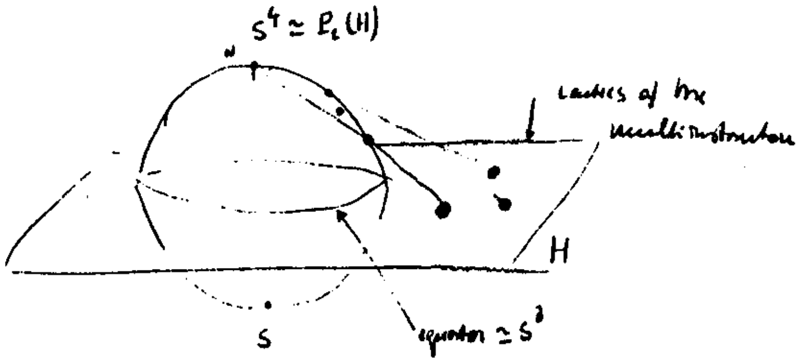
(Compare with the Anti-instanton $A = \operatorname{Im} \dfrac{\bar{y}\, dy}{1+|y|^2} = \operatorname{Im} \dfrac{\bar{\xi}^{-1} d\xi}{1+|\xi|^2}$

$$y \sim \lambda_1 (b - x)^{-1} \sim \xi^{-1}$$

(to get instantons take U^*)

The t'Hooft solutions

$B = \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_k \end{pmatrix}$ $\quad \Lambda = (\lambda_1 \ldots \lambda_k)$ $\quad \lambda_j$ real, positive



The Atiyah Hitchin Drinfeld Manin theorem says that (Λ, B) is gauge equivalent to (Λ', B')

iff $\Lambda' = q\Lambda T \qquad B' = T^{-1} B T$

$q \in Sp(1) \quad T \in O(k)$ (the only if part is easy)

effective parameters: $\underbrace{4k}_{\lambda} + \underbrace{4 \tfrac{1}{2} k(k+1)}_{B} = 2k^2 + 6k$

correspond to our initial choice of Λ and B

The equations involved in

$$B^*B + \lambda^*\lambda = \text{real}$$

are $3\,\frac{k(k+1)}{2}$ since $(B^*B)^* = B^*B$, $(\Lambda^*\Lambda)^* = \Lambda^*\Lambda$ the diagonal terms are already real

Diagonal terms $(B^*B + \lambda^*\lambda)_{ii} =$

$$= \sum_j (B^*)_{ij} B_{ji} + |\lambda_i|^2$$

$$= \sum_j \overline{B}_{ji} B_{ji} + |\lambda_i|^2$$

are real

$SU(2) \simeq Sp(1)$ has dimension 3 $(\simeq S^3)$

$O(k)$ has dim $\frac{1}{2} k(k-1)$

we expect:

$$\text{\# eff. parameters} = 2k^2 + 6k - \frac{3k(k-1)}{2} - 3 - \frac{1}{2}k(k+1)$$

$$= 2k^2 + 6k - 2(k-1)k - 3 = 8k - 3$$

which can be deduced from an index theorem

If we confine ourselves to diagonal solutions

we get $\underbrace{4k}_{\Lambda} + \underbrace{4k}_{B} - 3 = 8k - 3$

but we can't realise (*) $B^*B + \lambda^*\lambda$ real if Λ is [illegible]

They are only local parameters

(see Atiyah-Jones)

(mean a 't Hooft solutions one can use $\lambda_1 \ldots \lambda_k\ b_1 \ldots b_k$ to realise (*) real.