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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

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ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

(21 November - 16 December 1988)

DE RHAM THEORY.

A. Assadi
Max-Planck-Institut für Mathematik
Bonn
Fed. Rep. Germany

and

University of Wisconsin
Madison, Wisconsin
U.S.A.

These are preliminary lecture notes, intended only for distribution to participants.

De Rham Theory

Amir Assadi
(Max-Planck-Institut, Bonn)
&
(University of Wisconsin, Madison)

Outline: 1) Review of constructions
2) Basic Properties
3) Relation with Topology

References: (a) Singer–Thorpe: Lectures on Topology and Geometry, Springer-Verlag
(b) Bott–Tu: Differential Forms in Algebraic Topology
(c) Griffiths–Harris: Principles of Algebraic Geometry.

Notation: X^n = C^∞-manifold of dimension n. T^*X = cotangent bundle, $C^\infty(X, \Lambda^k T^*X)$ = the $\mathbb{R}$-vector space of C^∞ differential k-forms. Let $C^0_{dR} = C^\infty(X, \mathbb{R})$, $C^k_{dR} = C^\infty(X, \Lambda^k T^*X)$.

We have the exterior differential $d: C^k_{dR} \to C^{k+1}_{dR}$, locally expressed via: $f \in C^0_{dR}$, $f: X \to \mathbb{R}$, $df = \sum \frac{\partial f}{\partial x_i} dx_i$ and $\omega \in C^k_{dR}$ $\omega = \sum_I a_I dx_I$ $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, then

$d\omega = \sum_{i,I} \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I$. This gives a sequence of $\mathbb{R}$-vector spaces and $\mathbb{R}$-linear maps:

$$(*) \quad 0 \to C^0_{dR} \xrightarrow{d} C^1_{dR} \xrightarrow{d} C^2_{dR} \to \cdots \to C^n_{dR} \to 0$$

Since $dd(\omega) = 0$, $d \circ d = 0$ for the composition of any two consecutive differentials. Hence $(*)$ is a cochain complex.

Def$^{\underline{n}}$: $$H^k_{dR}(X) \equiv \frac{\operatorname{Ker}(d: C^k_{dR} \to C^{k+1}_{dR})}{\operatorname{Im}(d: C^{k-1}_{dR} \to C^k_{dR})} = \frac{\text{Closed Forms}}{\text{Exact Forms}}$$ is called the k-th de Rham cohomology

Remark: Let $\mathcal{C}$ = category of C^∞-manifolds and C^∞-maps, and let $\mathcal{V}_{\mathbb{R}}$ = category of graded $\mathbb{R}$-vector spaces and graded $\mathbb{R}$-linear maps. Then $H^*_{dR}(\cdot) = \bigoplus_{i \geq 0} H^i_{dR}(\cdot)$ defines a functor from $\mathcal{C}$ to $\mathcal{V}_{\mathbb{R}}$. If $f: X^n \to Y^m$ is a C^∞-map of smooth manifolds, then we get $f^*: C^k_{dR}(Y) \to C^k_{dR}(X)$ by pulling back differential forms; f^* commutes with the exterior differentials, so that we get a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & C^k_{dR}(Y) \arrow[r, "d"] \arrow[d, "f^*"] & C^{k+1}_{dR}(Y) \arrow[r] \arrow[d, "f^*"] & \cdots \\
\arrow[r] & C^k_{dR}(X) \arrow[r, "d"] & C^{k+1}_{dR}(X) \arrow[r] & \cdots
\end{tikzcd}$$

Thus, f^* is a cochain homomorphism, and as such, it induces a homomorphism on cohomology $f^*: H^k_{dR}(Y) \longrightarrow H^k_{dR}(X)$. So, if ω represents $[\omega] \in H^k_{dR}(Y)$, then $f^*(\omega)$ represents $f^*[\omega] \in H^k_{dR}(X)$. This can be checked to be compatible with compositions of smooth maps: $(g\circ f)^* = f^*\circ g^*$ and $(\text{identity})^* = \text{identity}$, thus establishing the functoriality.

Corollary: If $f: X \longrightarrow Y$ is a C^∞-diffeomorphism, then $f^*: H^k_{dR}(Y) \longrightarrow H^k_{dR}(X)$ is an $\mathbb{R}$-linear isomorphism for each $k \geq 0$.

2. Properties:

(a) If $X \amalg Y$ is the disjoint union of n-dimensional C^∞-manifolds, then $H^k_{dR}(X \amalg Y) \cong H^k_{dR}(X) \oplus H^k_{dR}(Y)$.

(b) $H^*_{dR}(\{\text{point}\}) = \begin{cases} 0 & k\neq 0 \\ \mathbb{R} & k=0 \end{cases}$. Also $\underset{\substack{\parallel \\ \{f: X\to\mathbb{R} \mid df=0\} = \text{constant functions} \cong \mathbb{R}}}{H^0_{dR}(X)} = \mathbb{R}$ if X is connected.

(c) Ring structure: The wedge product of forms:

$$C^i_{dR}(X)\otimes C^j_{dR}(X) \longrightarrow C^{i+j}_{dR}(X) \qquad \omega\otimes\eta \longmapsto \omega\wedge\eta$$

satisfies: $\omega\wedge\eta = (-1)^{\deg\omega \deg\eta}\,\eta\wedge\omega$, and the Leibnitz rule $d(\omega\wedge\eta) = (d\omega)\wedge\eta + (-1)^{\deg\omega}\omega\wedge d\eta$. Therefore, we have a product on the level of cohomology which is up to a (± 1) sign commutative:

$$[\omega]\cdot[\eta] = [\omega\wedge\eta] \qquad H^i_{dR}(X)\otimes H^j_{dR}(X) \longrightarrow H^{i+j}_{dR}(X).$$

Thus, $H^*_{dR}(X)$ is a graded $\mathbb{R}$-algebra.

(d) Künneth Formula: Let p_1 and p_2 be the projections onto the 1^{st} and 2^{nd} factors in the diagram:

$$\begin{tikzcd}
& X^n\times Y^m \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X & & Y
\end{tikzcd}$$

Then $p_1^*: C^i_{dR}(X) \longrightarrow C^i_{dR}(X\times Y)$ and $p_2^*: C^j_{dR}(Y) \longrightarrow C^j_{dR}(X\times Y)$. The wedge product in $C^*_{dR}(X\times Y)$ gives rise to an external product:

$$C^i_{dR}(X)\otimes C^j_{dR}(Y) \longrightarrow C^{i+j}_{dR}(X\times Y) \quad \text{via}$$

$\omega\otimes\eta \longmapsto (p_1^*\omega)\wedge(p_2^*\eta)$. We check that

This operation passes to cohomology using (c) above:

$$H^i_{dR}(X)\otimes H^j_{dR}(Y) \longrightarrow H^{i+j}_{dR}(X\times Y).$$ Summing over all i and j with $i+j=k$, we get a homomorphism

$$\lambda : \bigoplus_{i+j=k} H^i_{dR}(X)\otimes H^j_{dR}(Y) \longrightarrow H^{i+j}_{dR}(X\times Y) = H^k_{dR}(X\times Y).$$

<u>Künneth Formula</u>: λ is an isomorphism for each $k\geq 0$.

(e) <u>Poincaré Lemma</u>: If $U =$ interior (D^n), then $H^i_{dR}(U) = \begin{cases} 0 & i>0 \\ \mathbb{R} & i=0 \end{cases}$. Therefore $H^i_{dR}(\mathbb{R}^n) = \begin{cases} 0 & i>0 \\ \mathbb{R} & i=0 \end{cases}$ also since $\mathbb{R}^n$ and U are diffeomorphic.

<u>Proof</u>: Consider the de Rham complex for U:

$$0 \longrightarrow C^0_{dR} \longrightarrow C^1_{dR} \longrightarrow C^2_{dR} \longrightarrow \cdots \longrightarrow C^k_{dR} \longrightarrow C^{k+1}_{dR} \longrightarrow \cdots$$

To show that $H^i_{dR}(U)=0$ for $i>0$, we construct a "contracting cochain homotopy" just as in algebraic topology: $h_i : C^i_{dR} \longrightarrow C^{i-1}_{dR}$ such that $h_{i+1}\circ d + d\circ h_i =$ $=$ identity. (More precisely $h_{i+1}\circ(d)_i + (d)_{i-1}\circ h_i =$ identity.

Now, if ω is a <u>closed</u> i-form, i.e. $d\omega = 0$, then $\omega = h_{i+1}(d\omega) + d(h_i(\omega)) = d(h_i(\omega)) \equiv$ exact, for each $i>0$. Thus $H^i_{dR}(U)=0$ for $i>0$. For $i=0$, by 2(b) on page 3, $H^0_{dR}(U)=\mathbb{R}$. The formula for h_i is given by the following: In local coordinates, $\omega = g\, dx_{j_1}\wedge\cdots\wedge dx_{j_k}$. Set $\mu = x_{j_1}(dx_{j_2}\wedge\cdots\wedge dx_{j_k}) - x_{j_2}(dx_{j_1}\wedge dx_{j_3}\wedge\cdots) + \cdots + (-1)^{\ell} x_{j_\ell}(dx_{j_1}\wedge\cdots\wedge\widehat{dx_{j_\ell}}\wedge\cdots\wedge dx_{j_k}) + \cdots + (-1)^k x_{j_k}\, dx_{j_1}\wedge\cdots\wedge dx_{j_{k-1}}$. (i.e. drop this factor)

Compute that $d\omega = k\, dx_{j_1}\wedge\cdots\wedge dx_{j_k}$. Set $h_k(\omega)(x) = \left(\int_0^1 t^k g(tx)\,dt\right)\mu$, and compute (as in Singer-Thorpe page 130) $h_{k+1}(d\omega) + d(h_k(\omega)) = \omega$.

(f) <u>The Mayer-Victories Principle</u>: Let U_+ and U_- be two open subsets of X such that $X = U_+\cup U_-$, and consider the commutative diagram of <u>inclusions</u>:

$$\begin{tikzcd} & U_+\cap U_- \arrow[dl, "j_+"'] \arrow[dr, "j_-"] & \\ U_+ \arrow[dr, "\lambda_+"'] & & U_- \arrow[dl, "\lambda_-"] \\ & X & \end{tikzcd}$$

Using a partition of unity subordinate to the cover $\{U_+, U_-\}$ of X, we show that the homomorphism:

$$C^k_{dR}(U_+)\oplus C^k_{dR}(U_-) \xrightarrow{j_+^* - j_-^*} C^k_{dR}(U_+\cap U_-)$$ is

surjective. We obtain a short exact sequence of cochain complexes by taking direct sum of the following exact sequences:

$$0 \longrightarrow C^k_{dR}(X) \xrightarrow{\lambda_+^*\oplus\lambda_-^*} C^k_{dR}(U_+)\oplus C^k_{dR}(U_-) \xrightarrow{j_+^*-j_-^*} C^k_{dR}(U_+\cap U_-)\longrightarrow 0$$

$$\omega\longmapsto(\lambda_+^*\omega,\ \lambda_-^*\omega) \qquad (\alpha,\beta)\longmapsto j_+^*\alpha - j_-^*\beta$$

This gives rise to a long exact sequence of cohomology groups:

$$\cdots\longrightarrow H^k_{dR}(X) \xrightarrow{\lambda_+^*\oplus\lambda_-^*} H^k_{dR}(U_+)\oplus H^k_{dR}(U_-)\xrightarrow{j_+^*-j_-^*} H^k_{dR}(U_+\cap U_-)\xrightarrow{\delta} H^{k+1}_{dR}(X)\xrightarrow{\lambda_+^*\oplus\lambda_-^*}\cdots$$

g) Homotopy Invariance: Let σ_0 and $\sigma_1 : X\longrightarrow X\times\mathbb{R}$ i.e. the inclusions $\sigma_0(x)=(x,0)$, $\sigma_1(x)=(x,1)$. Then $\sigma_0^*\equiv\sigma_1^*: H^k_{dR}(X\times\mathbb{R})\longrightarrow H^k_{dR}(X)$ (and it is an isomorphism) for all $k\geq 0$. We use this, as in elementary algebraic topology to show that if $f,g: X\longrightarrow Y$ are homotopic (as C^∞ maps), then $f^*=g^*: H^k(Y)\longrightarrow H^k(X)$ for all $k\geq 0$.

(h) Examples of Computations:
$S^0=\{x_+\}\sqcup\{x_-\}$. Therefore $H^i_{dR}(S^0)=\begin{cases}\mathbb{R}\oplus\mathbb{R} & i=0\\ 0 & i>0\end{cases}$

Let $\varepsilon>0$, $(-\varepsilon,\varepsilon)\subset\mathbb{R}$. Then $H^i_{dR}(S^0\times(-\varepsilon,\varepsilon))\cong H^i((-\varepsilon,\varepsilon))\oplus H^i((-\varepsilon,\varepsilon))=\begin{cases}\mathbb{R}\oplus\mathbb{R} & i=0\\ 0 & i>0\end{cases}$. Suppose by induction we have computed the cohomology of S^k. Since

$$S^k\times(-\varepsilon,\varepsilon)\ \underset{\text{inclusion}}{\overset{\text{projection}}{\rightleftarrows}}\ S^k$$

describe a homotopy equivalence, we conclude that $H^*_{dR}(S^k\times(-\varepsilon,\varepsilon))\cong H^*_{dR}(S^k)$. To compute $H^*_{dR}(S^{k+1})$, use the Mayer-Vietoris decomposition of the covering of S^{k+1} by upper and lower hemispheres which slightly overlap: $S^{k+1}=U_+\cup U_-$
$U_+\cong$ interior $D^{k+1}\cong U_-$, $U_+\cap U_-\cong S^k\times(-\varepsilon,\varepsilon)$

Induction hypothesis $H^i_{dR}(S^k)=\begin{cases}\mathbb{R} & i=0,k\\ 0 & i\neq 0,k\end{cases}$ for $k\geq 1$.

$$\xrightarrow{\delta} H^i_{dR}(S^{k+1})\longrightarrow H^i_{dR}(U_+)\oplus H^i_{dR}(U_-)\longrightarrow H^i_{dR}(U_+\cap U_-)\xrightarrow{\delta} H^{i+1}_{dR}(S^{k+1})\longrightarrow\cdots$$

Using the Poincaré lemma, the induction hypothesis (for $k=1$ use the Mayer-Vietoris sequence for S^1 and calculation for S^0), and the above sequence, we prove:

$$H^i_{dR}(S^{k+1})=\begin{cases}0 & i\neq 0,k+1\\ \mathbb{R} & i=0,k+1\end{cases}.$$

Next, we use this computation, to compute the cohomology of "handles" $X = S^k \times \mathring{D}^{n-k}$, where $\mathring{D}^{n-k}$ = interior (D^{n-k}). Using the Künneth-formula or homotopy invariance:

$$H^i_{dR}(S^k \times \mathring{D}^{n-k}) = \begin{cases} 0 & i \neq 0, k \\ \mathbb{R} & i = 0, k \end{cases}$$

The key tool to compute $H^*_{dR}(X)$ for many other manifolds, in particular compact connected manifolds, is the decomposition of X into "handles". This is the subject of Morse Theory, which is one step farther from advanced calculus.

References: J. Milnor (1) Morse Theory and (2) Lectures on the h-Cobordism Theorem.

Basic Result (due to Smale). Any compact manifold X^n, $\partial X = \emptyset$, has a handle-body decomposition. That is, there is a finite sequence of open subspaces $W_i \subset X^n$, $W_0 = $ interior D^n, $W_N = X$, $W_0 \subseteq W_1 \subseteq \dots \subseteq W_N = X$, such that $W_{i+1} = W_i \cup \underbrace{(\mathring{D}^{k_i} \times \mathring{D}^{n-k_i})}_{U_i} = W_i \cup U_i$

$U_i \cong \mathring{D}^n$ and $W_i \cap U_i = W_i \cap \mathring{D}^{k_i} \times \mathring{D}^{n-k_i} = S^{k_i-1} \times \mathring{D}^{n-k_i}$

$W_i \cap U_i = S^{k_i-1} \times \mathring{D}^{n-k_i}$. Although this result,

is not stated in the above references (Milnor) it can be easily deduced from the arguments and proofs there. Thus, we can treat de Rham cohomology in the category of compact smooth manifolds the same way that we treat singular cohomology in the category of finite CW complexes.

3. Relation with Topology

The main result is the Isomorphism of de Rham Cohomology and singular cohomology. Since all the well-known ordinary cohomology theories (e.g. singular, simplicial, cellular, Čech, ...) agree for C^∞-manifolds, we can use any one of the above to compute $H^*_{dR}(X)$, and vice versa.

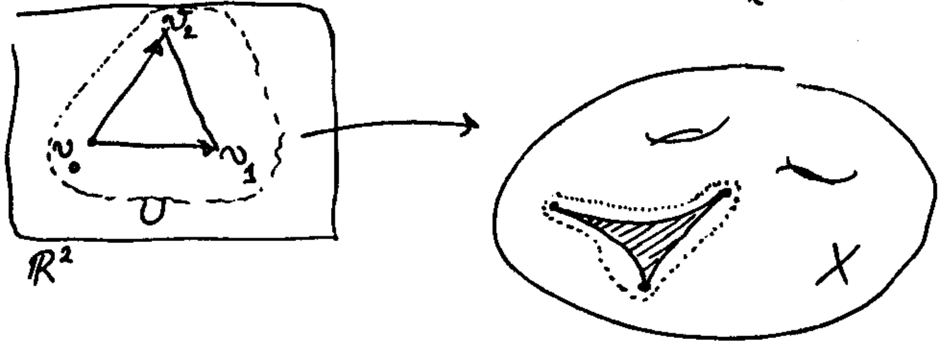
Assume that X is compact, connected, $\partial X = \emptyset$.

de Rham's Theorem: There exists a cochain homomorphism

$$g : C^*_{dR}(X) \longrightarrow C^*_{singular}(X; \mathbb{R}) = \text{singular cochains with values in } \mathbb{R}.$$

obtained by integrating forms. g induces an isomorphism $g^* : H^*_{dR}(X) \longrightarrow H^*(X; \mathbb{R})$ which is functorial for the category of smooth manifolds. Further, g^* is compatible with the $\mathbb{R}$-algebra structure of $H^*_{dR}(X)$ and $H^*(X; \mathbb{R})$ (using cup products) as well as Künneth isomorphisms.

Definition of $\mathcal{I}$: Let $\sigma_k = \langle v_0, v_1, \dots, v_k \rangle$ be the k-simplex, and $(r_1, \dots, r_k)$ the coordinate system given by the basis $\{v_1 - v_0, v_2 - v_0, \dots, v_k - v_0\}$ for $\mathbb{R}^k$.



From a technical point of view, we must use singular simplices $h : \sigma_k \longrightarrow X$ which can be extended to a smooth map $\bar{h} : U \longrightarrow X$, where U is some open neighborhood of σ_k in $\mathbb{R}^k$. Thus, we should consider "the smooth singular chain complex" $C^{smooth}_{singular}(X)$ consisting of integral linear combination of singular simplices which satisfy the above smooth extension requirement. However, because we can approximate any continuous map by a smooth one, it is possible (and elementary) to show that both chain complexes give the same homology groups. For the

sake of exposition, we suppress this technicality.

Given $\omega \in C^k_{dR}(X)$ and $\sigma_k \xrightarrow{h} X$, let

$(\mathcal{I} \quad)(h) = \int_{\sigma_k} h^*\omega$. In local coordinates,

$h^*\omega = g(r_1, \dots, r_k)\, dr_1 \wedge \dots \wedge dr_k$ and $\mathcal{I}(\omega)(h) = \int_{\sigma_k} g(r_1, \dots, r_k)\, dr_1 \dots dr_k$

If $\alpha = \sum a_i h_i$, $a_i \in R$, $h_i : \sigma_k^i \longrightarrow X$,

then set $(\mathcal{I}(\omega))(\alpha) = \sum a_i (\mathcal{I}(\omega))(h_i)$.

$\mathcal{I}$ satisfies:

(a) $\mathcal{I}(\omega)$ defines a singular cochain, (it changes sign if we change the orientation of σ_k)

(b) $\mathcal{I}$ is a cochain homomorphism, i.e. $\delta \circ \mathcal{I} = \mathcal{I} \circ d$ where $\delta : C^k_{sing} \longrightarrow C^{k+1}_{sing}$ is the coboundary.

This is a consequence of the Stoke formula:

$$(\mathcal{I} \circ d)(\omega)(h) = \int_\sigma h^*(d\omega) = \int_\sigma d(h^*\omega) = \int_{\partial\sigma} h^*\omega =$$

$$= (\mathcal{I}(\omega))(h|\partial\sigma) = \delta(\mathcal{I}(\omega))(h).$$

Therefore $h \longmapsto \int_{\sigma_k} h^*\omega$ defines a cochain homomorphism $\mathcal{I} : C^k_{dR} \longrightarrow C^k_{sing}$.

(see pages 164–173 of Singer–Thorpe for a different treatment of a weaker version of de Rham's theorem).

Consequently, we have a homomorphism of cohomology groups:

$$\vartheta^* : H^k_{dR}(X) \longrightarrow H^k(X;\mathbb{R}) .$$

To prove that ϑ^* is an isomorphism, we use our previously developed technique:

Step 1: Poincaré lemma $\Rightarrow$ ϑ^* is isomorphism for $U = \mathring{D}^n$ or $\mathbb{R}^n$.

Step 2. Functoriality of ϑ can be checked directly from the definition. $\Rightarrow$ ϑ is compatible with Mayer-Victoris sequences,

$$\begin{tikzcd}
{} \arrow[r, "\delta"] & H^k_{dR}(X) \arrow[r] \arrow[d, "\vartheta_X"] & H^k_{dR}(U_+)\oplus H^k_{dR}(U_-) \arrow[r] \arrow[d, "\vartheta_+\oplus\vartheta_-"] & H^k_{dR}(U_+\cap U_-) \arrow[r, "\delta"] \arrow[d, "\vartheta_0"] & H^{k+1}_{dR}(X) \arrow[r] \arrow[d, "\vartheta_X"] & {} \\
{} \arrow[r, "\delta'"] & H^k(X;\mathbb{R}) \arrow[r] & H^k(U_+;\mathbb{R})\oplus H^k(U_-;\mathbb{R}) \arrow[r] & H^k(U_+\cap U_-;\mathbb{R}) \arrow[r, "\delta'"] & H^{k+1}(X;\mathbb{R}) &
\end{tikzcd}$$

Step 3 Steps (1) & (2) $\Rightarrow$ de Rham isomorphism ϑ^* for S^k and $S^{k-1}\times \mathring{D}^{n-k}$, and $U = \mathring{D}^n = \mathring{D}^k \times \mathring{D}^{n-k}$

Step 4: Use (3) above, handle-body decomposition of page 9, and induction, to prove that de Rham isomorphism for W_i implies ϑ^* is an isomorphism for $W_{i+1} = W_i \cup \mathring{D}^{k_i}\times \mathring{D}^{n-k_i}$. This step uses M-V sequence and the 5-lemma. Let $H^k(\cdot)$ = singular cohomology with R-co

$$\begin{tikzcd}
{} \arrow[r, "\delta"] \arrow[d, "\cong"] & H^k_{dR}(W_{i+1}) \arrow[r] \arrow[d, "\vartheta^*"] & H^k_{dR}(U)\oplus H^k_{dR}(W_i) \arrow[r] \arrow[d, "\cong", "\text{Induction \& Poincaré lemma}"'] & H^k_{dR}(U\cap W_i) \arrow[r, "\delta"] \arrow[d, "\cong", "\text{Step 3}"'] & H^{k+1}_{dR}(\;) \arrow[d] \\
{} \arrow[r, "\delta'"] & H^k(W_{i+1}) \arrow[r] & H^k(U)\oplus H^k(W_i) \arrow[r] & H(U\cap W_i) \arrow[r, "\delta'"] & H^{k+1}(\;)
\end{tikzcd}$$

Since $U\cap W_i \cong S^{k-1}\times \mathring{D}^{n-k}$, we use step 3 and the 5-lemma to work our way up the M-V ladder and prove $\vartheta^* : H^k_{dR}(W_{i+1}) \longrightarrow H^k(W_{i+1})$ is an isomorphism for $k=0,1,2,\dots$

This proves the induction step, hence the theorem.

The proofs of other parts are scattered in the literature, but mostly in Bott–Tu's book. Although this proof is not in Bott–Tu or other references, the definition of ϑ is the same. Therefore, the reader can use the above reference to adapt their proofs to this situation. ∎