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SOME TOPOLOGICAL APPLICATIONS OF HARMONIC MAPPINGS.

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Some Topological Applications of Harmonic Mappings
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One of the oldest problems in topology is to decide whether there exist continuous mappings between two topological spaces inducing a given homomorphism between their homology groups. One concrete example would be the following: suppose M and N are compact oriented manifolds of the same dimension, when does there exist a continuous map $f : M \longrightarrow N$ of non-zero degree? This question has been formulated by M. Gromov in the following very suggestive way: if one says $M \geq N$ if such a mapping exists, then $\geq$ defines an ordering on manifolds, where intuitively $M \geq N$ means that M is more complicated than N, and the question is to compute this ordering. For surfaces the answer is easy: $M \geq N$ if and only if $genus(M) \geq genus(N)$. But for higher dimensional manifolds very little is known, and the methods of algebraic topology do not give very much information. There are necessary conditions, for instance an inequality for the Betti numbers: $b_i(M) \geq b_i(N)$. While this inequality gives the complete picture for surfaces, it has very weak consequences in higher dimensions.

The purpose of this lecture is to indicate the proof of a general theorem, which was proved jointly with J. Carlson, concerning the existence of mappings of non-zero degree between two very well-known and extensively studied classes of manifolds. For the domain manifold M we will always take a compact Kähler manifold, for the target manifold N a compact locally symmetric space of "non-compact type". The latter means that the universal cover of N is a manifold of the form G/K, where G is a semi-simple Lie group with no compact factors and K is a maximal compact subgroup of G. It is well known that G/K has an essentially unique metric invariant under G, and that this metric is complete and of non-positive curvature (cf [H]). It is also known, but more difficult to see, that G/K covers plenty of compact manifolds N (cf [B]). Thus N can be represented in the form $N = \Gamma\backslash G/K$ for a co-compact, discrete subgroup of G. It will be useful to keep three concrete examples in mind:

1. Real hyperbolic space $H^n_{\mathbb{R}} = SO(n,1)/SO(n)$,

2. Complex hyperbolic space $H^n_{\mathbb{C}} = SU(n,1)/U(n)$, which is the same as the unit ball in $\mathbb{C}^n$ with its Bergmann metric,

3. The space P_n of positive definite symmetric matrices with determinant 1; as a homogeneous space $P_n = SL(n,\mathbb{R})/SO(n)$.

Note that 2. has an invariant Kähler metric (ie, is a Hermitian symmetric space), while for $n > 2$, 1. and 3. are not Hermitian symmetric. Also $H^2_{\mathbb{R}} = H^1_{\mathbb{C}} = P_2$.

Theorem 1: *Let M and $N = \Gamma\backslash G/K$ be as above, and suppose that G/K does not contain the hyperbolic plane as a factor. Then a continuous map $\phi : M \longrightarrow N$ of non-zero degree exists if and only if G/K is Hermitian symmetric and there exists a holomorphic map $f : M \longrightarrow N$ (relative to an invariant complex structure on G/K) of non-zero degree.*

In other words, the only mappings of non-zero degree that can exist are "obvious" ones. If G/K is Hermitian symmetric, the above statement is (a consequence of) a famous theorem of Siu [Si]. Thus to prove Theorem 1, one only has to show that if G/K is not Hermitian symmetric, then a continuous map of non-zero degree cannot exist. If $G/K = H^n_{\mathbb{R}}$, this is a consequence of a theorem of Sampson [Sa]. The general statement is proved in [CT] and was motivated by the theorems of Siu and Sampson.

The proof of the theorem is based on harmonic mappings. Recall that a smooth mapping from M to N is called harmonic if it is an extreme value for the energy functional [ES]

$$E(f) = \frac{1}{2}\int_M \|df\|^2. \tag{1}$$

Since N has non-positive curvature, the fundamental existence theorem of Eells and Sampson [ES] implies that any continous map from M to N is homotopic to an essentially unique harmonic mapping. Thus the classification of homotopy classes of mappings from M to N is essentially reduced to the classification of harmonic mappings. The latter was for a very long time an intractable problem, until the following breakthrough was made by Siu [Si] for harmonic mappings $f : M \longrightarrow N$, where M is compact Kähler and N is Hermitian symmetric. Recall first that the Euler-Lagrange equation for (1) is

$$tr\nabla df = 0 \tag{2}$$

where ∇df denotes the second fundamental form of f, a section of the bundle $S^2T^*M \otimes f^*TN$, and tr denotes contraction with the metric in M. Now in a Kähler manifold M the bundle S^2T^*M splits as a direct sum of two sub-bundles, namely the tensors invariant and anti-invariant under the complex structure J. The complexification of the first bundle is the space of 1,1 symmetric tensors on M, and we will use the superscript 1,1 to denote the J-invariant component of a tensor. Using this notation, we can state Siu's main discovery as follows: if G/K is Hermitian symmetric, then any harmonic map $f : M \longrightarrow N$ is pluriharmonic, ie, satisfies

$$(\nabla df)^{1,1} = 0, \tag{3}$$

and from this he derived, in particular, the complex-analyticity of harmonic mappings of non-zero degree referred to above. If $dim_{\mathbb{C}}(M) = 1$ (3) is equivalent to (2), but in higher dimensions (3) is an over-determined system much stronger than (2).

Now in [Sa] Sampson proved that (3) also holds for all symmetric spaces G/K (not necessarily Hermitian), and also pointed out the following very useful consequence of (3): For a given $x \in M$, $T_{f_x}N$ can be identified with the summand $\mathfrak{p}$ in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra of G. Let $W = d_x f(T^{(1,0)}M) \subset \mathfrak{p}^{\mathbb{C}}$. Then W is abelian:

$$[W, W] = 0 \qquad (4).$$

Thus one should be able to reduce many questions on harmonic mappings to algebraic questions on abelian subalgebras of $\mathfrak{p}^{\mathbb{C}}$. For instance, $d_x f(TM^{\mathbb{C}}) = W + \overline{W}$, thus $d_x f(TM)$ is the subspace of real vectors in $W + \overline{W}$, thus one gets the estimate on the rank of $d_x f$:

$$rank_{\mathbb{R}}(d_x f) \leq 2\, dim_{\mathbb{C}}(W) \qquad (5)$$

For the general definition of the Cartan decomposition see [H]. We illustrate it with the example of $P_n = SL(n, \mathbb{R})/SO(n)$. In this case $\mathfrak{g}$ is the algebra of matrices with trace zero, $\mathfrak{k}$ is the Lie subalgebra of skew-symmetric matrices, $\mathfrak{p}$ is the subspace of symmetric matrices, and the Cartan decomposition is the familiar direct sum decomposition

$$X = \frac{1}{2}(X - X^t) + \frac{1}{2}(X + X^t).$$

Thus for $G/K = P_n$, Sampson's theorem says that for each $x \in M$, W is an abelian space of complex symmetric matrices.

Theorem 1 follows from the following statement, which for simplicity we state only in the case that G/K is irreducible, ie, G is a simple Lie group:

Theorem 2: *Let G be a simple non-compact group not isomorphic to $SL(2, \mathbb{R})$, and let $W \subset \mathfrak{p}^{\mathbb{C}}$ be abelian. Then $dim(W) \leq \frac{1}{2} dim(\mathfrak{p}^{\mathbb{C}})$, and equality holds if and only if G/K is Hermitian symmetric and W is the space of 1,0 tangent vectors to one of the two invariant complex structures on G/K.*

Namely, a mapping of non-zero degree must have maximum rank at some point, and the only way this is allowed by (5) and Theorem 2 is that G/K be Hermitian symmetric and f statisfy the Cauchy-Riemann equations.

We indicate the proof of Theorem 2 for the space P_n, $n > 2$. The abelian space W can be decomposed as a direct sum $W = W_s \oplus W_n$ of semi-simple and nilpotent matrices respectively (Jordan decomposition). The space W_s can be simultaneously diagonalized, and then the space W_n lies in the centralizer of W_s, which consists of block-diagonal matrices, each of the k blocks being a nilpotent subspace of symmetric n_i by n_i matrices, $n_1 + ... + n_k = n$. Here k is the number of distinct eigenvalues of the typical matrix in W_s and n_i is the multiplicity of the i^{th} eigenvalue. Since there are no real symmetric nilpotent

matrices, none of these spaces can have real points, therefore their dimension is at most half of the dimension of the corresponding block, and some simple arithmetic shows that one cannot get to half the dimension unless $W_s = 0$ and there is only one block. But a nilpotent space of half the dimension is equivalent to an invariant complex structure: since it has no real points, one gets a direct sum decomposition $\mathfrak{p}^{\mathbb{C}} = W \oplus \overline{W}$, and since W is isotropic for the Killing form, it follows that this complex structure is isometric, hence invariant, which is impossible for $P_n, n > 2$. A similar argument works for any G, using the root space decomposition of G/K, see [CT] for details.

We give an immediate application of Theorem 2 to a well known problem in topology and geometry, namely to find restrictions on the homotopy types of compact Kähler manifolds. Suppose that $rank(G) = rank(K)$ and $V \subset K$ is the centralizer of a torus in K; a concrete example would be $G = SO(2p, q)$, $K = SO(2p) \times SO(q)$ and $V = U(p) \times SO(q)$. It is known that G/V is a homogeneous complex manifold [GS], hence $\Gamma \backslash G/V$ is a complex manifold which fibres over $N = \Gamma \backslash G/K$ with fibre not homologous to zero, hence the map induced on homology by the projection is surjective. Just as Theorem 1 was deduced from Theorem 2, one sees that if G/K is not Hermitian symmetric, then no Kähler manifold can map to N surjectively in homology. Hence $N = \Gamma \backslash G/K$ cannot be homotopy equivalent to a Kähler manifold. It is interesting to observe that here the non-linear harmonic equation restricts the homotopy type of a Kähler manifold, much in analogy with the way the linear harmonic theory has been used to derive similar conclusions, cf[Mo]. For $G = SO(2p, q)$, G/K is not Hermitian symmetric precisely when $p \neq 1$ and $q \neq 2$.

Theorem 2 is a very crude estimate of the rank of harmonic mappings and just the beginning of a serious application of (3) and (4) to the classification problem of harmonic mappings of Kähler manifolds to locally symmetric spaces. One would like to know for each G/K the maximum dimension of an abelian subalgebra of $\mathfrak{p}^{\mathbb{C}}$, call this number $\alpha(G/K)$. One would also like to know the possible conjugacy classes of maximal abelian subspaces, and to use their classification to totally classify harmonic mappings. For instance, in [Sa] Sampson proves that $\alpha(H^n_{\mathbb{R}}) = 1$, hence in this case the rank of any harmonic mapping $f : M \longrightarrow N$ is at most 2. In [CT] we developed this ideas further for the rank one symmetric spaces, were able to give a complete classification of the harmonic mappings, and in particular sharpen Sampson's theorem for $H^n_{\mathbb{R}}$ to the statement that any harmonic mapping must factor through a map of M to a hyperbolic Riemann surface S and a map of S into N. This statement has a very interesting consequence, touching on another classical problem: find restrictions on the possible fundamental groups of compact Kähler manifolds:

Theorem 3: *Let Γ be the fundamental group of a compact manifold of constant negative curvature and dimension at least three. Then Γ is not isomorphic to the fundamental group* *of a compact Kähler manifold.*

Theorem 3 is proved by contradiction: suppose there is a compact Kähler manifold M with fundamental group Γ, take a harmonic representative $f : M \longrightarrow N = \Gamma\backslash H^n_{\mathbb{R}}$ of the classifying map of the universal cover of M (since N is the Eilenberg-MacLane space of Γ), and the above factorization makes Γ isomorphic to a subgroup of the fundamental group of a surface, which is impossible.

Finally, we say a word on how one computes $\alpha(G/K)$. The problem is to find the maximum dimension of various spaces of commuting matrices. The classical results here are due to Schur [Sc] and Malcev [Ma], for all complex matrices and for all the simple complex Lie algebras respectively. Their methods can be suitably modified to apply here, and we mention only the result for P_n: a straight-forward modification of Schur's method easily gives $\alpha(P_n) \leq \left[\frac{n^2+2n}{8}\right]$, for $n > 3$. Moreover for n even, say $n = 2k$ one can see that this bound is sharp by considering the following abelian space of complex symmetric matrices (cf [Sa]):

$$\begin{pmatrix} A & iA \\ iA & -A \end{pmatrix}$$

where A is an arbitrary complex symmetric k by k matrix. Moreover all abelian spaces of this maximum dimension are conjugate, and can be realized in an obvious way by harmonic mappings of compact Kähler manifolds, since they correspond to totally geodesic embeddings of the Siegel upper half-plane of genus k in P_n (cf[Sa]). It turns out that in many cases the bound on rank arising from (5) is sharp, but in other cases it is not.

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