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YANG-MILLS THEORY AND THE TOPOLOGY OF 4-DIMENSIONAL MANIFOLDS

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PRELIMINARY VERSION

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This decade has seen tremendous advances in our understanding of four dimensional manifolds. The vigorous work of the 60's and 70's led to an understanding of higher dimensional manifolds in terms of their algebraic topology. The 4 dimensional case is quite different in that these algebraic methods do not suffice. Rather, techniques and ideas from differential geometry—and most strikingly from high energy physics—have yielded new information beyond what can be culled from algebro-topological methods. The early studies of the *classical* Yang-Mills equations in the late 70's used algebro-geometric methods and applied only to special 4-manifolds [ADHM], [AHS]. Gradually techniques from the theory of partial differential equations were brought into play, leading to general results applicable to a wide variety of 4-manifolds [U1], [U2], [T1]. In his 1982 thesis [D1] Simon Donaldson used these results to produce new obstructions (not coming from algebraic topology) to the existence of smooth structures on topological 4-manifolds. Since then Donaldson has systematically developed the theory in quite general form [D2], [D3], [D4]. Many applications of the theory have been to algebraic surfaces [D5], [FM1], [FM2]. There are topological applications as well [FS1], [FS2], [Ma], [Ru]. Recently, Andreas Floer [F] has shown how to attach a new sort of cohomology theory to homology 3-spheres; the resulting cohomology groups play a principal role in the theory on manifolds with boundary.[1] Atiyah's exposition of these ideas [A] inspired Ed Witten [W1] to place Donaldson's theory firmly in the context of *quantum* Yang-Mills theory. Thus whereas the classical equations of gauge theory inspired this decade of mathematical work, it is the quantum gauge theory which emerges as the central idea. Clearly, the intuitions and techniques of quantum physics will play a primary role in future developments.

The reader will quickly gather from our brief history that this subject is extremely rich and draws on many fields. We cannot possibly do justice to all of this material. Instead, our aims are quite modest. We begin in §1 with a recitation of major results close to the central development of the theory. These

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[1] It is important to extend Floer's work to arbitrary 3-manifolds.

statements never mention internal concepts (connections, instantons, etc.)—the ultimate testimony to t[...] theory's strength. Section 2 is an introduction to the basic configuration space attached to a 4-manifold X the space $\mathcal{C}_X$ of connections up to gauge transformations. Of primary importance are cohomology clas[...] attached to surfaces in X. The moduli space of instantons is, roughly speaking, a homology class in $\mathcal{C}$[...] The justification of this assertion forms the guts of the theory. We summarize some highlights in §3. Final[...] in §4 we show how to derive some of the results of §1 from the theory.

CONTENTS:

§1 Topology of 4-Manifolds

notopy and homeomorphism

One of the basic invariants of a topological space X is its *fundamental group* $\pi_1 X$.[1] It depends on ıoice of basepoint in X, and is the group of loops in X starting and ending at the basepoint, up to ıotopy. If all such loops are contractible, then $\pi_1 X = 0$ and X is *simply connected*. There are two nected (topological) 1-manifolds—the real line $\mathbb{R}$ and the circle S^1—and these are distinguished by their damental groups $\pi_1\mathbb{R} = 0$ and $\pi_1 S^1 = \mathbb{Z}$. The situation is more interesting in 2 dimensions. Suppose X is *losed* (compact without boundary) connected 2-manifold. The abelianization of $\pi_1 X$ is the first homology up $H_1(X)$, and it determines the manifold:

1)

X^2	$H_1(X)$
S^2	0
$T^2 = S^1 \times S^1$	$\mathbb{Z}^2$
$T^2 \# T^2$	$\mathbb{Z}^4$
$\mathbb{RP}^2$	$\mathbb{Z}/2\mathbb{Z}$
$\mathbb{RP}^2 \# \mathbb{RP}^2$	$\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$

ıe symbol '#' denotes *connected sum*, which is defined for any two manifolds X, Y of the same dimension. is formed by cutting a disk out of each of X and Y and gluing together the remains via an orientation versing diffeomorphism of the boundary spheres. Any closed connected 2-manifold is a connected sum tori T^2 and projective spaces $\mathbb{RP}^2$. In this reckoning the sphere S^2 is the empty connected sum. The assification of manifolds in 3 dimensions is not known. The (original) Poincaré conjecture states that a osed oriented connected simply connected 3-manifold is the 3-sphere S^3, but this is yet to be proved. Any ıitely presented group is the fundamental group of a closed 4-manifold X, which makes π_1 an ineffective ıvariant in isolation. For simplicity we will usually assume $\pi_1 X = 0$.

Now we introduce the *intersection form* I_X attached to an oriented 4-manifold X. It is the basic lgebraic invariant. Let

$$H_X = H^2(X)/\text{torsion} \tag{1.2}$$

ıe the "free part" of the second cohomology group of X. (If $\pi_1 X = 0$, then $H_X = H^2(X)$.) The intersection orm is the cup product

$$I_X : H_X \otimes H_X \longrightarrow H^4(X) \cong \mathbb{Z}, \tag{1.3}$$

[1] An even more basic invariant is $\pi_0 X$, the set of path components. For the most part we will work with connected spaces.

where the last isomorphism uses the orientation. It is a symmetric nondegenerate pairing. Passing to real coefficients we use differential 2-forms to represent elements of $H_X \otimes \mathbb{R}$, and then I_X is the pairing

$$\alpha \otimes \beta \longmapsto \int_X \alpha \wedge \beta. \tag{1.4}$$

The basic invariants of (1.4) are the rank $b_2(X) = \operatorname{rank}(H_X)$ and the dimensions $b_2^+(X)$ (resp. $b_2^-(X)$) of the maximal subspaces of $H_X \otimes \mathbb{R}$ on which (1.4) is positive (resp. negative) definite. We have

$$b_2(X) = b_2^+(X) + b_2^-(X), \tag{1.5}$$

$$\operatorname{Sign}(X) = b_2^+(X) - b_2^-(X), \tag{1.6}$$

where $\operatorname{Sign}(X)$ is the signature of X. The form I_X is termed *even* if $I_X(a,a)$ is an even integer for all $a \in H_X$. If X is simply connected, then I_X is even if and only if X is a spin manifold. Here is a table of standard examples:

(1.7)

X^4	$b_2(X)$	$b_2^+(X)$	$b_2^-(X)$	I_X
S^4	0	0	0	0
$S^2 \times S^2$	2	1	1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbb{CP}^2$	1	1	0	(1)
$\overline{\mathbb{CP}^2}$	1	0	1	(−1)
$K3$	22	3	19	$-2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

In this table $\overline{\mathbb{CP}^2}$ denotes $\mathbb{CP}^2$ with the opposite orientation. Also, $K3$ is the quartic surface in $\mathbb{CP}^3$. Finally, E_8 is the 8×8 Cartan matrix attached to the Lie algebra E_8:

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \tag{1.8}$$

If X and Y are simply connected, then for connected sums we have

$$I_{X \# Y} = I_X \oplus I_Y. \tag{1.9}$$

A basic result of Whitehead and Milnor asserts that in the simply connected case, the intersection fo is a complete *homotopy* invariant.

Theorem 1.10 (Whitehead-Milnor [Wh], [Mi]). *Suppose X_1, X_2 are connected, simply connected, closed topological 4-manifolds. Let $\theta: I_{X_1} \to I_{X_2}$ be an isomorphism of the intersection pairings. Then there is a homotopy equivalence $\tilde{\theta}: X_1 \to X_2$ which induces θ. Furthermore, every nondegenerate symmetric bilinear form (over $\mathbf{Z}$) occurs as I_X for some X.*

The seminal work of Michael Freedman [Fr] extends Theorem 1.10 (Whitehead-Milnor) to a classification by *homeomorphism* type. We state his result as follows.

Theorem 1.11 (Freedman). *Suppose X_1, X_2 are connected, simply connected, closed smooth 4-manifolds. Let $\theta: I_{X_1} \to I_{X_2}$ be an isomorphism of the intersection pairings. Then there is a homeomorphism $\tilde{\theta}: X_1 \to X_2$ which induces θ.*

In fact, for *topological* manifolds there is an additional $\mathbf{Z}/2\mathbf{Z}$ invariant, discovered by Kirby and Siebenmann, which suffices for the homeomorphism classification.

In short, the classical invariants of algebraic topology classify topological 4-manifolds.

Smooth structures on 4-manifolds

Although our main concern is compact 4-manifolds, there is no better place to begin than with the existence of an exotic differentiable structure on $\mathbf{R}^4$. In any dimension but four the standard smooth structure on flat space is known to be the only one. However, Donaldson's first result Theorem 1.13 combined with Freedman's work shows that there are other smooth structures on $\mathbf{R}^4$ [FU,§1]. More recent work underscores this aspect of four dimensional space.

Theorem 1.12 (Gompf-Taubes [G], [T1]). *There are uncountably many fake $\mathbf{R}^4$'s.*

That is, there are uncountable many diffeomorphism classes of 4-manifolds homeomorphic to $\mathbf{R}^4$. In fact, there is a 2-parameter family of such manifolds. We will not discuss the proof of Theorem 1.12 in these notes.

By general principles there can only be countably many differentiable structures on a *compact* topological 4-manifold. However, whereas in lower dimensions any topological manifold carries a unique differentiable structure, there are some compact 4-manifolds with no smooth structures. For example, it follows from Freedman's work that there is a compact simply connected topological 4-manifold with intersection form E_8. But a classical theorem of Rohlin[3] implies that this manifold carries no smooth structure. Donaldson extended this nonexistence result to many other manifolds constructed by Freedman.

Theorem 1.13 (Donaldson [D1], [D3]). *Let X be a closed oriented smooth 4-manifold whose intersection form is definite. Then the intersection form is diagonalizable.*

[3] Rohlin's theorem states that the signature of a *spin* 4-manifold is divisible by 8. Recall that if X is simply connected, then X is spin if and only if I_X is an even form.

It then follows from Theorem 1.11 and the classification of quadratic forms that if X is simply connected, then X is homeomorphic to a connected sum of $\mathbf{CP}^2$'s or to a connected sum of $\overline{\mathbf{CP}^2}$'s. Of course, any symmetric bilinear form over the reals is diagonalizable, but over the integers there are many definite nondiagonalizable forms. The forms E_8 and $E_8 \oplus E_8$ are two examples. (Recall from (1.7) that $E_8 \oplus E_8$ is part of the intersection pairing of $K3$. This relates to the construction of exotic $\mathbf{R}^4$'s.) The original proof [D1] of Theorem 1.13 uses a bordism argument; it forms the subject of [FU]. This proof only deals with the simply connected case. The proof in [D2] uses a homology argument and is more in line with the general theory. One goal of these notes is to present the main ideas of this proof.

Donaldson also proves a nonexistence theorem for certain indefinite forms. Recall that $b_2^+(X)$ is the rank of the positive part of the intersection pairing I_X.

Theorem 1.14 (Donaldson [D2]). *Let X be a closed oriented smooth 4-manifold with even intersection form. Suppose $b_2^+(X) \le 2$ and $H_1(X;\mathbf{Z})$ has no 2-torsion. Then X is homeomorphic to S^4, $S^2 \times S^2$, or to the connected sum $(S^2 \times S^2) \# (S^2 \times S^2)$.*

Donaldson actually computes the intersection form; Freedman's work (Theorem 1.11) then gives the homeomorphism type. Of course, Theorem 1.13 is slightly stronger in case $b_2^+(X) = 0$ (the definite case) as there is no hypothesis on the first homology group.

Complex surfaces give a rich supply of oriented 4-manifolds. The classification of algebraic surfaces as complex manifolds is known, and questions about their diffeomorphism type have been around for quite some time.[4] A recent result of Donaldson answers some of these questions.

Theorem 1.15 (Donaldson [D4]). *If a smooth simply connected algebraic surface X is decomposable as a smooth connected sum $X = X_1 \# X_2$, then one of X_1 and X_2 has negative definite intersection form.*

We remark that the *blow-up* of an algebraic surface X has the diffeomorphism type of $X \# \overline{\mathbf{CP}^2}$.

There is a family of algebraic surfaces, the *Dolgachev surfaces* $S_{p,q}$ (p,q relatively prime), each of which is homeomorphic to $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}^2}$.

Theorem 1.16 (Donaldson [D5]). *$S_{2,3}$ is not diffeomorphic to $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}^2}$.*

Theorem 1.16 was extended in many ways by Friedman and Morgan [FM1], [FM2]. One of their conclusions is

Theorem 1.17 (Friedman-Morgan [FM2]). *There is an infinite (countable) number of inequivalent smooth structures on $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}^2}$.*

This is a great contrast with higher dimensions where a compact topological manifold carries a finite number of inequivalent smooth structures. One can speculate [FM1] that the classification of algebraic surfaces

[4] See [FM1] for a survey.

diffeomorphism type is closer to the classification as complex manifolds than to the classification by meomorphism type.

pen problems

There are still open problems concerning the existence and uniqueness of smooth structures on closed nanifolds.[5]. The existence of differentiable structures on simply connected manifolds would be settled by conjecture of Kas and Kirby.

ONJECTURE 1.18 ("THE 11/8 CONJECTURE"). *Suppose X is a closed, oriented, simply connected, smooth manifold with even indefinite intersection form. Then*

$$\frac{b_2(X)}{|\operatorname{Sign}(X)|} \geq \frac{11}{8}.$$

ie bound is sharp—it is realized by any connected sum of $K3$ surfaces. In fact the conjecture is equivalent the assertion that every such manifold is *homeomorphic* to a connected sum of $K3$ and $S^2 \times S^2$.

The most well-known open problem about uniqueness is the four dimensional Poincaré conjecture.

ONJECTURE 1.19. *There is a unique smooth structure on* S^4.

Finally, one can ask whether algebraic surfaces form the building blocks for oriented 4-manifolds, just tori form the building blocks for oriented 2-manifolds.

UESTION 1.20. *Is every closed, oriented, smooth, simply connected 4-manifold a connected sum of algebraic rfaces and their conjugates?*

The conjugate surface has the reverse orientation. Also, S^4 is the empty connected sum.) As Theorem 1.17 dicates there is an infinite number of building blocks. Friedman and Morgan [FM1] credit René Thom ith the speculation that Question 1.20 has an affirmative answer.

[5] A more extensive discussion is given in the survey article [FM1]

§2 Spaces of Connections

Principal bundles and gauge transformations

Let X be a compact oriented 4-manifold. Then for any connected Lie group G we consider principal G bundles $\pi: P \to X$. Thus P is a smooth manifold on which G acts freely (on the right) with quotient X. The group G acts simply transitively on the fiber $\pi^{-1}(x) = P_x$ over $x \in X$—given $p_1, p_2 \in P_x$ there is a unique $g \in G$ with $p_2 = p_1 g$. The simplest example is the trivial bundle $P = X \times G$, with G acting on the second factor. More generally any P is *locally* trivial. That is, about each point in X there is a neighborhood U so that the restriction of P to U is trivial: $\pi^{-1}(U) \cong U \times G$ as G-spaces. A trivialization is equivalent to a local section $s: U \to P$. (A *section* s is a smoothly varying choice $s(x) \in P_x$ for each $x \in X$. In other words, $\pi \circ s$ is the identity.) For then

$$\begin{aligned} U \times G &\longrightarrow \pi^{-1}(U) \\ (x, g) &\longmapsto s(x)g \end{aligned} \tag{2.1}$$

is the corresponding trivialization.

The topological classification of principal bundles depends on the nature of G. For example, if G is a finite group then P is a (principal) covering space of X. If G is abelian these covering spaces are classified by elements of $H^1(X;G)$; the classification is more complicated for nonabelian finite G. If $G = \mathbb{T}$ is the circle group, then principal $\mathbb{T}$ bundles are in 1:1 correspondence with elements of $H^2(X;\mathbb{Z})$. When G is a connected, simply connected, compact, simple Lie group there is a single characteristic class in $H^4(X;\mathbb{Z}) \cong \mathbb{Z}$ which classifies G bundles. The case of most interest to us is $G = SU_2$, the group of 2×2 unitary matrices of determinant 1:

$$SU_2 = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1 \right\}. \tag{2.2}$$

As a smooth manifold SU_2 is diffeomorphic to the 3-sphere. Associated to a principal SU_2 bundle $P \to X$ is a rank 2 complex vector bundle $E \to X$ with trivialized determinant line bundle $\det E \to X$. (This is the line bundle $\bigwedge^2 E \to X$.) In general, a rank 2 complex vector bundle E has Chern classes $c_1(E) \in H^2(X;\mathbb{Z})$ and $c_2(E) \in H^4(X;\mathbb{Z})$, but the condition on the determinant means $c_1(E) = 0$. Under the isomorphism $H^4(X;\mathbb{Z}) \cong \mathbb{Z}$ given by the orientation, the second Chern class $c_2(E)$ goes over into an integer k.

To summarize, specifying the Lie group $G = SU_2$ and an integer k determines a unique equivalence class of principal SU_2 bundles over any given compact oriented 4-manifold X.[6]

[6] More generally, we can take as starting data any compact Lie group G, though it may not be so easy to specify a unique isomorphism class of G bundles on each closed oriented 4-manifold. Recent work in three dimensional topology and two dimensional physics [W2] suggests that this more general framework is desirable. We will develop some aspects of the theory in this generality, but the analysis has only been worked out in the literature for $G = SU_2$ or $G = SO_3$. The difficulties in extending to more complicated Lie groups are nontrivial.

Now if P, P' are $G = SU_2$ bundles over X with the same Chern class, then there exist diffeomorphisms

$$\varphi\colon P \to P' \tag{2.3}$$

which commute with the G action and induce the identity map on the base $X = P/G = P'/G$. If $P = P'$ then the set of such maps forms a group under composition. This is the group $\mathcal{G}_P$ of *gauge transformations*. It is the basic symmetry of the theory, once a bundle $P \to X$ is fixed, and all of our constructions will be *gauge invariant*, or at least account for this symmetry. One can give $\mathcal{G}_P$ the structure of an infinite dimensional Lie group. The terminology comes from the case of a trivial bundle. For a trivialization $P \cong X \times G$ allows us to express any global section $s\colon X \to P$ as a map $\tilde{s}\colon X \to G$. This is a (global) *gauge*, and any other gauge $\tilde{t}\colon X \to G$ is related by the equation

$$\tilde{t}(x) = \tilde{\varphi}(x)\tilde{s}(x). \tag{2.4}$$

Here $\tilde{\varphi}\colon X \to G$ operates on the *left* and so commutes with the *right* G action on P. In other words, $\mathcal{G}_P$ is the mapping space $\mathrm{Map}(X, G)$. This picture applies locally on any principal G bundle $P \to X$: A gauge transformation is a group element at each point of X operating by left multiplication on P, though now the copies of the group twist as we move on X. More precisely, the automorphisms of the fiber P_x which commute with the right G action form a group G_x. To each a point of P_x is associated an identification $P_x \sim G$, and so an isomorphism $G_x \cong G$. These identifications depend on the point in P_x. (However, there is a *canonical* identification of the center of G_x with the center of G.) As x varies the groups G_x form a bundle of groups $G_P \to X$, and the group $\mathcal{G}_P$ of gauge transformations is the group of sections of G_P. On the infinitesimal level the Lie algebra $\mathfrak{g}_x$ of G_x is (noncanonically) isomorphic to the Lie algebra $\mathfrak{g}$ of G. As x varies these Lie algebras $\mathfrak{g}_x$ form a bundle of Lie algebras $\mathfrak{g}_P \to X$, and the Lie algebra of *infinitesimal gauge transformations* is the algebra of sections of this bundle. More geometrically, an infinitesimal gauge transformation is a vector field along the fibers of $P \to X$ which is invariant under the G action on P. Notice that (local) trivializations of P induce (local) trivializations of G_P and $\mathfrak{g}_P$.

Let $\rho\colon G \to \mathrm{Aut}(V)$ be a representation of G on a vector space V. Then for each $x \in X$ the space of G-equivariant maps $f\colon P_x \to V$ form a vector space V_x. The G-equivariance of f is the condition

$$f(p \cdot g) = \rho(g^{-1})f(p), \quad g \in G, p \in P_x. \tag{2.5}$$

To each point of P_x is associated an isomorphism $V_x \cong V$. The vector spaces V_x fit together to form a vector bundle $V_P \to X$. This is called the associated bundle construction. A global section of V_P is then a G-equivariant map $P \to V$. For example, $\mathfrak{g}_P$ is the vector bundle associated to the adjoint representation

$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$. Finally, gauge transformations act on an associated bundle V_P. If $\varphi_x \in G_x$ and $f \in V$ then

$$\varphi_x f(p) = f(\varphi_x^{-1}p) = \rho_x(\varphi_x)f(p), \tag{2.6}$$

where $\rho_x\colon G_x \to \mathrm{Aut}(V_x)$ is the representation induced by ρ.

Connections and curvature

The (real) cohomology of a manifold is a global invariant; locally a manifold has no cohomology. Sti differential geometers study cohomology via local objects—differential forms. Similarly, we use connectio to study principal bundles. Let $\pi\colon P \to X$ be a G bundle. The infinitesimal $\mathfrak{g}$ action gives a canonic identification of $\mathfrak{g}$ with the tangent space to P along the fibers of π. Thus for each $p \in P$ there is an exa sequence

$$0 \to \mathfrak{g} \to T_pP \xrightarrow{\pi_*} T_xX \to 0, \tag{2.7}$$

where $x = \pi(p)$. A *horizontal space at p* is a splitting of (2.7); a *connection* is a G-invariant splitting of t corresponding sequence of vector bundles over P:

$$0 \to P \times \mathfrak{g} \to TP \xrightarrow{\pi_*} \pi^*TX \to 0. \tag{2.8}$$

As a map $T_xX \to T_pP$ a splitting gives horizontal lifts for tangent vectors to X. As a map $TP \to$ connection is a $\mathfrak{g}$-valued 1-form A on P satisfying

$$g^*A = (\mathrm{Ad}\, g^{-1})A, \quad g \in G, \tag{2.9}$$

$$A = \mathrm{id} \ \text{ on } \ker \pi_*. \tag{2.10}$$

Note that (2.9) is a linear condition, but (2.10) is an affine condition. That is, if A, A' are connections th their difference α satisfies (2.9) and

$$\alpha = 0 \text{ on } \ker \pi_*. \tag{2.11}$$

The set of all such α is a linear space. It is the space $\Omega^1_X(\mathfrak{g}_P)$ of 1-forms on X with values in the bundle Lie algebras defined in the last section.[7] So the space $\mathcal{A}_P$ of all connections is an affine space with associa vector space $\Omega^1_X(\mathfrak{g}_P)$.

[7] Recall that $\Omega^0_X(\mathfrak{g}_P)$, the space of sections of $\mathfrak{g}_P$, is the Lie algebra of infinitesimal gauge transformations. More gener elements of $\Omega^k_X(\mathfrak{g}_P)$ are $\mathfrak{g}$-valued k-forms α on P satisfying (2.9) and the condition that α annihilate vectors in $\ker \pi_*$.

The derivative of functions on X is an operator

$$\Omega^0_X \xrightarrow{d} \Omega^1_X \tag{2.12}$$

into 1-forms on X. Let $A \in \mathcal{A}_P$ be a connection on P. Then (2.12) extends to an operator

$$\Omega^0_X(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1_X(\mathfrak{g}_P) \tag{2.13}$$

as follows. An element of $\Omega^0_X(\mathfrak{g}_P)$ is a $\mathfrak{g}$-valued function ξ on P satisfying (2.9). Then

$$d_A\xi = d\xi + [A,\xi] \tag{2.14}$$

is a 1-form on P satisfying (2.9) and (2.11). Note that if $\mathfrak{g}$ is abelian, then $d_A = d$. The operator d_A obeys the Leibnuits rule

$$d_A(f\xi) = df \cdot \xi + f \cdot d_A\xi, \quad f \in \Omega^0_X,\ \xi \in \Omega^0_X(\mathfrak{g}_P). \tag{2.15}$$

The same formula (2.14) extends to the whole de Rham complex:

$$0 \to \Omega^0_X(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1_X(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^2_X(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^3_X(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^4_X(\mathfrak{g}_P) \to 0. \tag{2.16}$$

In general $d_A^2 \neq 0$. Rather it is multiplication by an element $F_A \in \Omega^2_X(\mathfrak{g}_P)$, the *curvature* of A. The computation

$$\begin{aligned} d_A^2\xi &= d_A(d\xi + [A,\xi]) \\ &= d^2\xi + d[A,\xi] + [A,d\xi] + [A,[A,\xi]] \\ &= [dA,\xi] - [A,d\xi] + [A,d\xi] + [A,[A,\xi]] \\ &= [dA + \frac{1}{2}[A,A],\xi] \end{aligned} \tag{2.17}$$

shows

$$F_A = dA + \frac{1}{2}[A,A]. \tag{2.18}$$

(At the last stage of (2.17) we use the Jacobi identity.) Locally, F_A is a $\mathfrak{g}$-valued 2-form.

Next we must understand the effect of gauge transformations on these constructions. Let $\varphi\colon P \to P$ be a gauge transformation and A a connection. Define a new connection $A \cdot \varphi$ by

$$A \cdot \varphi = (\varphi)^* A. \tag{2.19}$$

Then $A \cdot \varphi$ satisfies (2.9) and (2.10), so is a new connection on P. Thus (2.19) defines a right action of $\mathcal{G}_P$ on $\mathcal{A}_P$. Our major object of study is the quotient

$$\mathcal{C}_X = \mathcal{A}_P/\mathcal{G}_P, \tag{2.20}$$

the space of connections modulo gauge equivalence. Notice that we write $\mathcal{C}_X$ rather than $\mathcal{C}_P$. This is because for any principal G bundle $P' \to X$ which is isomorphic to P there is a *canonical* identification

$$\mathcal{A}_P/\mathcal{G}_P \cong \mathcal{A}_{P'}/\mathcal{G}_{P'} \tag{2.21}$$

(cf. (2.3)). So $\mathcal{C}_X$ just depends on X and the fixed isomorphism class of G bundles. For $G = SU_2$ the space $\mathcal{C}_X = \mathcal{C}_X(k)$ just depends on the integer k (though we often delete 'k' from the notation). For any connection $A \in \mathcal{A}_P$, we denote by $\bar{A}$ its projection in $\mathcal{C}_X$.

From (2.14) we compute

$$\begin{aligned} d_{A\cdot\varphi} &= \varphi^{-1} \circ d_A \circ \varphi, \\ &= d + \varphi^{-1} \circ A \circ \varphi + \varphi^{-1} d\varphi, \end{aligned} \tag{2.22}$$

where the gauge transformation φ acts on $\mathfrak{g}P$ via the adjoint representation (cf. (2.6)). Hence

$$A \cdot \varphi = \varphi^{-1} A \varphi + \varphi^{-1} d\varphi, \tag{2.23}$$

which can be derived directly from (2.9). The infinitesimal version of (2.23) is a vector field v_ξ on $\mathcal{A}_P$ for each infinitesimal gauge transformation $\xi \in \Omega^0_X(\mathfrak{g}_P)$. Set $\varphi = e^{t\xi}$ in (2.23), differentiate in t, and use (2.14) to compute

$$v_\xi(A) = d_A\xi. \tag{2.24}$$

The behavior of curvature under gauge transformations can be deduced from (2.18) or more easily from the first line of (2.22) and the fact that $F_A = d_A^2$. In any case

$$F_{A\cdot\varphi} = \varphi^{-1} F_A \varphi. \tag{2.25}$$

We want to study gauge invariant concepts, i.e., concepts associated to $\mathcal{C}_X$. For example, the equation

$$F_A = 0 \tag{2.26}$$

is gauge invariant, so that the solutions form a subset of $\mathcal{C}_X$. These are the (gauge equivalence classes of) *flat* connections. Our main object of study is the space of half-flat or anti-self-dual connections on a Riemannian 4-manifold. Another way to get gauge invariant expressions out of curvature is by taking a trace. For $G = SU_2$ we have the Chern-Weil formula

$$k = \frac{1}{8\pi^2}\int_X \operatorname{tr}(F_A \wedge F_A). \tag{2.27}$$

Locally, F_A is a 2×2 matrix of 2-forms, so the matrix product $F_A \wedge F_A$ is a 2×2 matrix of 4-forms. Taking the trace gives an ordinary 4-form. It is invariant under gauge transformations, since a gauge transformation conjugates F_A by (2.25), and the trace is conjugation invariant. Equation (2.27) is a topological constraint on the possible curvatures of a connection on P. Similar formulæ can be written for other groups.

Now let $\rho\colon G \to \operatorname{Aut}(V)$ be a representation of G and V_P the associated vector bundle. Then we define a complex

$$\Omega^0_X(V_P) \xrightarrow{d_A} \Omega^1_X(V_P) \xrightarrow{d_A} \Omega^2_X(V_P) \xrightarrow{d_A} \Omega^3_X(V_P) \xrightarrow{d_A} \Omega^4_X(V_P) \tag{2.28}$$

analogous to (2.16) by the formula

$$d_A\xi = d\xi + \dot\rho(A)\xi \tag{2.29}$$

analogous to (2.14). Here $\xi \in \Omega^k_X(V_P)$ is a G-equivariant V-valued k-form on P which annihilates vectors in $\ker \pi_*$. Also, $\dot\rho\colon \mathfrak{g} \to \operatorname{End}(V)$ is the induced representation of the Lie algebra. The operator d_A is the *covariant derivative*. Its square is multiplication by $\dot\rho(F_A)$, the curvature of V_P, which is an element of $\Omega^2_X(\operatorname{End} V_P)$.

The space $\mathcal{C}_X$

In this section we begin to study the basic *configuration space*

$$\mathcal{C}_X = \mathcal{A}_P/\mathcal{G}_P. \tag{2.30}$$

Our first goal is to understand its singularities. To this end we introduce an auxiliary space $\mathcal{C}'_X$. Fix a basepoint $x_0 \in X$ and let $\mathcal{G}'_P$ be the subgroup of $\mathcal{G}_P$ consisting of gauge transformations which are the identity on P_{x_0}. There is an exact sequence

$$1 \to \mathcal{G}'_P \to \mathcal{G}_P \xrightarrow{\mathrm{ev}_{x_0}} G_{x_0} \to 1, \tag{2.31}$$

where ev_{x_0} is evaluation at the basepoint. Define

$$\mathcal{C}'_X = \mathcal{A}_P/\mathcal{G}'_P. \tag{2.32}$$

Then

$$\mathcal{C}_X = \mathcal{C}'_X/G_x. \tag{2.33}$$

Our justification for introducing $\mathcal{C}'_X$ is the following

Lemma 2.34. *$\mathcal{G}'_P$ acts freely on $\mathcal{A}_P$ if X is connected.*

Proof: Suppose $\varphi \in \mathcal{G}'_P$, $A \in \mathcal{A}_P$, and $A\cdot\varphi = A$. Then from (2.23) we deduce

$$\varphi^{-1}d\varphi = A - \varphi^{-1}A\varphi. \tag{2.35}$$

Consider (2.35) as a first order differential equation for φ. The right hand side is 0 at x_0, since $\varphi(x_0) = \mathrm{id}$. So $\varphi(x) \equiv \mathrm{id}$ is a solution of (2.35) which is correct at one point, hence by general properties of first order differential equations is correct everywhere. (This uses the connectivity of X.) Hence $\varphi(x) \equiv \mathrm{id}$ and $\mathcal{G}'_P$ acts freely.

We will not worry about the technicalities of smooth structures on infinite dimensional manifolds in these lectures. Suffice it to say that the spaces we deal with may be treated as finite dimensional manifolds. Proper treatment is given in any standard reference, e.g. [FU,§3]. There a more precise statement is proved.

Proposition 2.36. *$\mathcal{C}'_X$ is a smooth manifold.*

This does not follow from Lemma 2.34 since $\mathcal{G}'_P$ is not compact. Rather, a slice theorem must be proved. We defer to [FU,§3] for details.

For each $A \in \mathcal{A}_P$ let $\Gamma_A \subset \mathcal{G}_P$ be the isotropy group of the $\mathcal{G}_P$ action on $\mathcal{A}_P$ at A:

$$\Gamma_A = \{\varphi \in \mathcal{G}_P : A\cdot\varphi = A\}. \tag{2.37}$$

It is immediate that

$$\Gamma_{A\cdot\varphi} = \varphi\Gamma_A\varphi^{-1}. \tag{2.38}$$

Lemma 2.34 implies that Γ_A projects isomorphically onto a subgroup of G_{x_0} under the sequence (2.31).[6] We record the following general fact about these stabilizers.

[6] In fact, the image in G_{x_0} is the centralizer of the holonomy subgroup at x_0.

Lemma 2.39. *Let Z be the center of G. Then $Z \subset \mathcal{G}_P$ and $Z \subseteq \Gamma_A$ for all A.*

Proof: If $z \in Z$ then for any $x \in X$ the map $p \mapsto pz$ is an automorphism of P_x, so lies in G_x. Elements in the image of the induced inclusion $Z \hookrightarrow \mathcal{G}_P$ are called *global gauge transformations*. It follows directly from (2.35) that they lie in every stabiliser Γ_A.[9]

Now specialise to $G = SU_2$. Then $Z = \{\pm 1\}$. The possible Γ_A are the subgroups of SU_2 conjugate to either $\{\pm 1\}$, $\mathbb{T}$, or SU_2, where $\mathbb{T}$ is the circle group of diagonal matrices in SU_2.[10] We discuss each case separately.

Let $\mathcal{A}_P^* \subset \mathcal{A}_P$ be the subspace of $A \in \mathcal{A}_P$ with $\Gamma_A = \{\pm 1\}$. These are the *irreducible connections*. The image of Γ_A in G_{x_0} is $\{\pm 1\}$, so that $G_{x_0}/\{\pm 1\} \cong SO_3$ acts freely on $\mathcal{C}_X'^* = \mathcal{A}_P^*/\mathcal{G}_P'$. Hence there is a principal bundle

$$SO_3 \to \mathcal{C}_X'^* \to \mathcal{C}_X^* \tag{2.40}$$

and $\mathcal{C}_X^*$ is a smooth manifold. It is not hard to see that $\mathcal{C}_X^*$ is homotopy equivalent to $\mathcal{C}_X$, since the complement $\mathcal{C}_X \setminus \mathcal{C}_X^*$ of *reducible connections* has infinite codimension. Still, the reducible connections play an important role in the theory.

If $\Gamma_A = SU_2$ then $F_A = 0$. This follows from (2.25), as the only element of the Lie algebra $\mathfrak{g} = \mathfrak{su}_2$ invariant under conjugation by SU_2 is 0. So the subspace of connections with $\Gamma_A = SU_2$ are the flat connections. The Chern-Weil formula (2.27) implies that these connections only exist if $k = 0$. Flat connections modulo gauge equivalence are given by representations $\pi_1 X \to SU_2$ up to conjugacy. If X is simply connected then this is a single point: the trivial representation. We denote it by

$$\{\theta\} \in \mathcal{C}_X(0). \tag{2.41}$$

Finally, if $\Gamma_A = \mathbb{T}$ than A reduces to a connection on a $\mathbb{T}$ bundle Q over X. (This is a general fact: A reduces to a connection on a bundle whose structure group is the centraliser of the image of Γ_A in G_{x_0}, i.e., to a bundle whose structure group is the holonomy group.) A standard computation with characteristic classes shows

$$c_1(Q)^2 = -c_2(P). \tag{2.42}$$

In terms of the intersection pairing, if $c = c_1(Q)$ then

$$I_X(c,c) = -k. \tag{2.43}$$

[9] If Z is not discrete then there is an infinite dimensional group of global gauge transformations—simply let z vary with $x \in X$. However, only the constants are in Γ_A (if X is connected).
[10] Other subgroups of SU_2, such as cyclic groups of even order > 2, cannot occur since Γ_A is (isomorphic to) the centraliser of a subgroup—the holonomy subgroup—of SU_2.

Note that if c satisfies (2.43), then so does $-c$. Conversely, any solution $c \in H^2(X;\mathbb{Z})$ to (2.43) determines a $\mathbb{T}$ bundle $Q \to X$, and there is an inclusion

$$\mathcal{C}_c = \mathcal{A}_Q/\mathcal{G}_Q \hookrightarrow \mathcal{C}_X \tag{2.44}$$

by extending the structure group.

We determine the local structure of $\mathcal{C}_X$ near these reducible connections. (Recall that $\mathcal{C}_X$ is smooth near irreducible connections.) Let $a \in \mathcal{A}_Q$ be a connection on the $\mathbb{T}$ bundle Q; then any other connection is $a + i\alpha$ for $\alpha \in \Omega_X^1$ an ordinary 1-form. The gauge transformation e^{if}, $f \in \Omega_X^0$, maps $a + i\alpha$ to $a + i(\alpha + df)$. So $\mathcal{C}_c$ is an affine space with associated vector space $\Omega_X^1/d\Omega_X^0$. Now extend a to an SU_2 connection $A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$. Under the action of the diagonal matrices $\mathbb{T} \subset SU_2$ there is a decomposition

$$\mathfrak{g} = \mathfrak{su}_2 \cong \mathfrak{h} \oplus \mathfrak{m}, \tag{2.45}$$

where $\mathfrak{h}$ consists of diagonal matrices (the Lie algebra of $\mathbb{T}$) and $\mathfrak{m} = \left\{\begin{pmatrix} 0 & b \\ -\bar b & 0 \end{pmatrix}\right\}$ has a natural complex structure (which identifies the matrix above with $b \in \mathbb{C}$). The element $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathbb{T}$ acts on $\mathfrak{m}$ as multiplication by λ^2. There is a corresponding decomposition

$$\mathfrak{g}_P \cong \mathfrak{h}_P \oplus \mathfrak{m}_P \tag{2.46}$$

of the adjoint bundle.[11] Since $\mathbb{T}$ acts trivially on $\mathfrak{h}$, the bundle $\mathfrak{h}_P$ is trivial. Thus the tangent space $T_A\mathcal{A}_P \cong \Omega_X^1(\mathfrak{g}_P)$ decomposes as

$$\Omega_X^1(\mathfrak{g}_P) \cong \Omega_X^1 \oplus \Omega_X^1(\mathfrak{m}_P). \tag{2.47}$$

By (2.24) the subspace of (2.47) corresponding to the action of infinitesimal gauge transformations is the image of

$$\begin{array}{ccc} \Omega_X^0(\mathfrak{g}_P) & \xrightarrow{d_A} & \Omega_X^1(\mathfrak{g}_P) \\ \| & & \| \\ \Omega_X^0 \oplus \Omega_X^0(\mathfrak{m}_P) & \xrightarrow{d \oplus d_A} & \Omega_X^1 \oplus \Omega_X^1(\mathfrak{m}_P) \end{array} \tag{2.48}$$

The constants in Ω_X^0 are in the kernel of d_A, corresponding to the stabiliser $\Gamma_A = \mathbb{T}$. Therefore, the tangent space $T_A\mathcal{C}_X$ in the space of connections modulo gauge equivalence is the quotient of

$$\frac{\Omega_X^1}{d\Omega_X^0} \oplus \frac{\Omega_X^1(\mathfrak{m}_P)}{d_A\Omega_X^0(\mathfrak{m}_P)} = H_{\mathbb{R}} \oplus H_{\mathbb{C}} \tag{2.49}$$

[11] This can be stated in terms of vector bundles. If E is the rank 2 bundle associated to P, then the reducible connection A gives a decomposition $E \cong L \oplus L^{-1}$. Then $\mathfrak{h}_P$ is the trivial bundle, whereas $\mathfrak{m}_P \cong L^2$ as complex line bundles.

by the Γ_A action. Now $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \Gamma_A$ acts trivially on the real Hilbert space $H_{\mathbb{R}}$, and as multiplication by λ^2 on the complex Hilbert space $H_{\mathbb{C}}$. Hence

$$T_{\tilde{A}}\mathcal{C}_X = H_{\mathbb{R}} \times (H_{\mathbb{C}}/\Gamma_A), \tag{2.50}$$

and the second factor is a cone on the infinite projective space $\mathbb{CP}^{\infty}$. Finally, a slice theorem asserts that a neighborhood of A in $\mathcal{C}_X$ is modeled on (2.50).[12]

Our final observation is that under (2.44) the image of $\mathcal{C}_c$ is the same as the image of $\mathcal{C}_{-c}$. In other words, connections of Chern class c are SU_2 gauge equivalent to connections of Chern class $-c$. This is most easily seen in terms of vector bundles. If $E \cong L \oplus L^{-1}$, then

$$v_L \oplus v_{L^{-1}} \longmapsto -v_{L^{-1}} \oplus v_L \tag{2.51}$$

defines a gauge transformation of E which maps the connection $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ to $\begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}$.

The theory with $G = SO_3$ is slightly simpler, since SO_3 has trivial center so that $\Gamma_A = \{\pm 1\}$ cannot occur. However, the singularities are more intricate for more complicated groups G since there are more possibilities for the stabilizer subgroups Γ_A.

[12] Here we omit the analysis referred to after the statement of Proposition 2.38.

§3 The Moduli Space of Instantons

For the rest of our work we take $G = SU_2$.[13] In the sequel X is a connected, simply connected, closed oriented, smooth 4-manifold.

The anti-self-dual equations

Consider Euclidean 4-space $V = \mathbb{E}^4$. If e_1, e_2, e_3, e_4 is the standard oriented orthonormal basis, with dual basis e^1, e^2, e^3, e^4, then the volume form is

$$\text{vol} = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \in {\textstyle\bigwedge}^4 V^*, \tag{3.1}$$

and the $*$ operator

$$*: {\textstyle\bigwedge}^k V^* \longrightarrow {\textstyle\bigwedge}^{4-k} V^*, \quad k = 0, 1, \dots, 4, \tag{3.2}$$

is defined by the equation

$$(\alpha, \beta)\,\text{vol} = \alpha \wedge *\beta, \quad \alpha, \beta \in {\textstyle\bigwedge}^k V^*. \tag{3.3}$$

An easy computation shows

$$*^2 = 1 \text{ on } {\textstyle\bigwedge}^2 V^*, \tag{3.4}$$

so we can decompose

$${\textstyle\bigwedge}^2 V^* = {\textstyle\bigwedge}^2_+(V^*) \oplus {\textstyle\bigwedge}^2_-(V^*) \tag{3.5}$$

into *self-dual* and *anti-self-dual* forms. If X is an oriented Riemannian 4-manifold there is a corresponding decomposition on 2-forms:

$$\Omega^2_X = (\Omega^2_X)_+ \oplus (\Omega^2_X)_-. \tag{3.6}$$

This extends to forms with values in a vector bundle. Let

$$P_+ = \frac{1}{2}(1 + *): \Omega^2_X \longrightarrow (\Omega^2_X)_+ \tag{3.7}$$

[13] With little change there is an analogous theory for $G = SO_3$.

be orthogonal projection onto the self-dual forms.

Now let $P \to X$ be a principal SU_2 bundle, A a connection on P, and $F_A \in \Omega^2_X(\mathfrak{g}_P)$ its curvature. The ***anti-self-dual equation*** is

$$P_+F_A = 0, \tag{3.8}$$

or equivalently

$$*F_A = -F_A. \tag{3.9}$$

Solutions to (3.8) are connections whose curvature is anti-self-dual. We term these solutions *instantons*. There are several elementary observations. First, from (2.18) we see that (3.8) is a nonlinear first order differential equation for A. Then from (2.25) we see that the set of solutions is preserved by gauge transformations. Finally, if A is a solution to (3.8), then from (2.27)

$$k = \frac{1}{8\pi^2}\int_X \mathrm{tr}(F_A \wedge F_A) = \frac{1}{8\pi^2}\int_X -\mathrm{tr}(F_A \wedge *F_A) = \frac{1}{8\pi^2}\int_X |F_A|^2 \,\mathrm{vol} \geq 0. \tag{3.10}$$

Here we use (3.3) and the fact that $(\xi_1,\xi_2) = -\mathrm{tr}(\xi_1\xi_2)$ is a positive definite inner product on $\mathfrak{su}_2$. It follows from (3.10) that if $k=0$ the instantons are precisely the flat connections (2.26).

Therefore, for each $k \geq 0$ and each Riemannian metric g on X there is a *moduli space*

$$\mathcal{M}_X(k,g) \subset \mathcal{C}_X(k). \tag{3.11}$$

If X is simply connected, as we are assuming, then

$$\mathcal{M}_X(0,g) = \{\theta\} \tag{3.12}$$

is the (equivalence class of the) trivial connection θ (2.41).

We begin our study of these moduli spaces by computing the linearization of (3.8) for a fixed metric g. Suppose A is anti-self-dual and $\alpha \in \Omega^1_X(\mathfrak{g}_P)$. Then from (2.18) and (2.14) we compute

$$P_+F_{A+\alpha} = P_+d_A\alpha + \frac{1}{2}P_+[\alpha,\alpha]. \tag{3.13}$$

So the linearization of (3.8) is the equation

$$P_+d_A\alpha = 0. \tag{3.14}$$

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Now P_+d_A fits into the elliptic complex

$$0 \to \Omega^0_X(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1_X(\mathfrak{g}_P) \xrightarrow{P_+d_A} \Omega^2_X(\mathfrak{g}_P)_+ \to 0. \tag{3.15}$$

Ellipticity means that there is a finite dimensional linear space $H^1(A)$ of solutions $\alpha \in \Omega^1_X(\mathfrak{g}_P)$ to the equations

$$P_+d_A\alpha = 0, \tag{3.16}$$

$$d_A^*\alpha = 0. \tag{3.17}$$

Note that $H^1(A) = H^1(A,g)$ depends on the metric g. To see the geometric meaning of equation (3.17) we recall formula (2.24). It states that d_A in (3.15) is the infinitesimal action of gauge transformations $\mathcal{G}_P$ on connections $\mathcal{A}_P$. Put differently, the image of d_A is the tangent to the orbit of A under $\mathcal{G}_P$. Now solutions to (3.17) form the orthogonal complement to the image of d_A, so are perpendicular to the gauge orbit. This is a local slice of the gauge action if A is irreducible [FU,Theorem 3.2]; it projects 1:1 onto the tangent space $T_{\bar{A}}\mathcal{C}_X$ under the projection $\mathcal{A}_P \to \mathcal{C}_X$. (The irreducibility of A means that $\mathcal{C}_X$ is smooth at the image $\bar{A}$.) So $H^1(A)$ projects 1:1 onto a linear subspace of $T_{\bar{A}}\mathcal{C}_X(k)$ if A is irreducible. We will see that if $\mathcal{M}_X(k,g)$ is smooth, then this is its tangent space at $\bar{A}$.

Let $H^0(A)$ be the space of solutions to

$$d_A\xi = 0, \quad \xi \in \Omega^0_X(\mathfrak{g}_P), \tag{3.18}$$

and $H^2(A)$ the space of solutions to

$$(P_+d_A)^*\tau = 0, \quad \tau \in \Omega^2_X(\mathfrak{g}_P)_+. \tag{3.19}$$

Ellipticity of (3.15) implies that these are finite dimensional. Also, $H^0(A)$ is independent of g, whereas $H^2(A) = H^2(A,g)$ depends on g. Set

$$h^i(A) = \dim H^i(A). \tag{3.20}$$

Then the Atiyah-Singer index theorem calculates

$$d_X(k) = h^1(A) - h^0(A) - h^2(A) = 8k - 3(1 + b_2^+(X)). \tag{3.21}$$

This formula is purely topological. The right hand side is independent of A; it depends only on the second Chern class k and on $b_2^+(X)$. We will interpret $H^0(A)$ in the next section and $H^2(A)$ in the following section. The basic result is that if the moduli space $\mathcal{M}_X(g,k)$ is smooth, then its dimension is $d_X(k)$.

The most basic example is the $k = 1$ moduli space $M_{S^4}(1)$ on the round sphere S^4. According to (3.21) it is a 5 dimensional manifold. In fact, it is an open 5-ball. It has been described in detail in many sources. ([AHS] seems to be the original, cf. [FU,§6].) Let x be a quaternionic coordinate on $\mathbb{R}^4 \cong \mathbb{H}$, which we identify with $S^4 \setminus \{pt\}$. The Lie algebra $\mathfrak{su}_2$ can be identified with the Lie algebra of imaginary quaternions. That understood, there is a family of instantons on $\mathbb{R}^4$,

$$A_\lambda = \operatorname{Im}\left(\frac{\bar{x}\,dx}{\lambda^2 + |x|^2}\right), \quad \lambda \in (0,\infty), \tag{3.22}$$

with curvature

$$F_\lambda = \frac{\lambda^2\,d\bar{x} \wedge dx}{(\lambda^2 + |x|^2)^2}. \tag{3.23}$$

These extend easily to $k = 1$ instantons on S^4. Note that as $\lambda \to 0$ the curvature F_λ concentrates near $x = 0$. (As $\lambda \to \infty$ the curvature concentrates near ∞.) By translation we obtain families centered at any pair of antipodal points. Therefore, the 5 parameters of $M_{S^4}(1)$ are the centers and the scale λ.

Reducible connections

At $A \in \mathcal{A}_P$ the infinitesimal action of gauge transformations is

$$\Omega^0_X(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1_X(\mathfrak{g}_P). \tag{3.24}$$

Hence the kernel of d_A, which we denote $H^1(A)$ (cf. (3.18)), is the Lie algebra of the stabilizer group Γ_A. We classified these stabilizers in §2. If A is irreducible then $\Gamma_A = \{\pm 1\}$, so that $H^1(A) = 0$. If A is the trivial connection θ, then $\Gamma_A = SU_2$, and $H^1(A)$ is a 3 dimensional space. (Recall that this only occurs if $k = 0$.) The remaining case is $\Gamma_A = \mathbb{T}$; then $H^1(A)$ is 1 dimensional.

Suppose that A is an instanton with $\Gamma_A = \mathbb{T}$. Then A is the SU_2 extension of a connection a on a $\mathbb{T}$ bundle Q whose Chern class c satisfies (2.43). Since $\mathbb{T}$ is abelian, its curvature $\omega = F_a$ is an ordinary 2-form on X, and (3.8) asserts that ω is anti-self-dual. Furthermore, $d\omega = 0$ by differentiating (2.18). Hence $d{*}\omega = 0$ also, and ω is harmonic. Hodge theory asserts that ω is the unique harmonic form in its cohomology class. So the set of instantons on Q is the set of connections a on Q whose curvature is ω. Now if a is an instanton on Q and $\alpha \in \Omega^1_X$, then the curvature of the connection $a + \alpha$ is $\omega + d\alpha$. So the space of instantons on Q can be identified with the space of closed 1-forms on X. But since X is simply connected, $\alpha = df$ for some function f. Then a is gauge equivalent to $a + \alpha$ via the $\mathbb{T}$ gauge transformation $e^{if} \in \mathcal{G}_Q$. Thus there is a single gauge equivalence class of instanton in $\mathcal{A}_Q$. Finally, the SU_2 gauge transformation (2.51) maps instantons with Chern class c to instantons with Chern class $-c$.

We summarize in

PROPOSITION 3.25. *Let A be an instanton on the SU_2 bundle $P \to X$ with Chern class k. If $\Gamma_A = \{\pm 1\}$ then $h^1(A) = 0$ and A is irreducible. If $\Gamma_A = SU_2$ then $h^1(A) = 3$, $k = 0$, and A is gauge equivalent to the trivial connection θ. If $\Gamma_A = \mathbb{T}$ then $h^1(A) = 1$ and $F_A = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$, where ω is an anti-self-dual 2-form. There is a finite number of gauge orbits of these reducible instantons on P, one orbit for each pair $\pm c$ of solutions to (2.43) whose harmonic representatives are anti-self-dual.*

If I_X is negative definite then every harmonic 2-form is anti-self-dual, so we have merely to count pairs of solutions to (2.43). If I_X is not negative definite, then we will see that for a generic metric there are no anti-self-dual 2-forms (Proposition 3.34).

Generic metrics

Fix a Chern class $k > 0$. Let Met_X be the space of Riemannian metrics on X. Then for each $g \in \mathrm{Met}_X$ we have the instanton moduli space

$$M_X(g) \subset \mathcal{C}_X. \tag{3.26}$$

(We omit 'k' from the notation here.) The union over g yields a subset

$$\mathcal{N}_X \subset \mathcal{C}_X \times \mathrm{Met}_X \tag{3.27}$$

consisting of pairs (A, g) such that F_A is anti-self-dual for the metric g. Let $\mathcal{N}_X^*$ be the subset where $\Gamma_A = \{\pm 1\}$, i.e., the irreducible instantons.

THEOREM 3.28 [FU,§3]. *$\mathcal{N}_X^*$ is a smooth manifold.*

We do not assert that $M_X(g)^*$ is smooth for each g. But it will follow from Theorem 3.28 that $M_X(g)^*$ is smooth for a dense set of metrics.

We first explain how smoothness of the moduli space is related to $H^2(A, g)$, defined in (3.19). Namely, suppose A is an instanton for a metric g, and assume that A is irreducible. Thus $H^0(A) = 0$. Then we claim that $M_X(k, g)$ is smooth at $\bar{A}$ if $H^2(A, g) = 0$. (Recall that $\mathcal{C}_X$ is smooth at $\bar{A}$ since A is irreducible.) For near A the moduli space is the kernel of the operator

$$S_{A,g}(\alpha) = d_A^*\alpha \oplus P_+ F_{A+\alpha}, \quad \alpha \in \Omega^1_X(\mathfrak{g}_P). \tag{3.29}$$

If α is in the kernel of S, then $A + \alpha$ is an instanton. Here we use the slice theorem to identify $\mathcal{C}_X$ near $\bar{A}$ with the kernel of d_A^*. We seek solutions with α small. But the linearization of (3.29),

$$dS(\alpha) = d_A^*\alpha \oplus P_+ d_A\alpha, \tag{3.30}$$

s surjective, since the cokernel $H^0(A) \oplus H^2(A,g)$ vanishes. Therefore, the implicit function theorem implies hat the kernel of S is smooth near $\alpha = 0$. Also, the tangent space to $M_X(k,g)$ at $\bar{A}$ is the kernel $H^1(A,g)$ of the linearization (3.30). By (3.21) this has dimension $d_X(k)$.

There is no a priori guarantee that $H^2(A,g)$ vanishes for any metric g. However, it can be shown that f we let the metric g be a variable in (3.29), then the new operator $S_A(g,\alpha)$ has no cokernel anywhere. In ther words, at any instanton A with respect to the metric g, the differential of S with respect to g maps nto $H^2(A,g)$. This argument is [FU,Theorem 3.4], which relies on $G = SU_2$. Theorem 3.28 then follows rom the implicit function theorem.

Our main interest is in fixed metrics, and so in

COROLLARY 3.31. *There is an open dense set of metrics g for which $M_X(k,g)^*$ is smooth for all k.*

Ve term this the set of "generic metrics." To prove Corollary 3.31 we consider the projection

$$\pi : \mathcal{N}_X^* \longrightarrow \mathrm{Met}_X . \tag{3.32}$$

A short calculation shows that the cokernel of π at $(A,g) \in \mathcal{N}_X^*$ is isomorphic to $H^2(A,g)$. The infinite imensional version of Sard's theorem, due to Smale, asserts that π has a dense set of regular values. At these etrics $H^2(A,g)$ vanishes for all instantons A. Also, $h^2(A,g) = \dim H^2(A,g)$ is an upper semicontinuous unction, so vanishes on open sets. Thus the set of generic metrics for fixed k is open and dense. Since the ntersection of any countable number of open dense sets is open and dense, we have such a set of generic etrics for all k simultaneously.

We need to know the dependence of the moduli space on the metric. Suppose g_0 and g_1 are generic etrics, and $g_t\ (0 \le t \le 1)$ an arbitrary path of metrics between them. Then a refinement of Sard's theorem tates that arbitrarily small perturbations $\tilde{g}_t$ of g_t fixing the endpoints have the property that $\pi^{-1}(\tilde{g}_t) \subset \mathcal{N}_X^*$ s a smooth manifold. Its boundary is the disjoint union $M_X(k,g_0)^* \amalg M_X(k,g_1)^*$. Hence

COROLLARY 3.33. *The moduli spaces $M_X(k,g_t)^*$ for different generic metrics g_0, g_1 are bordant.*

We turn now to reducible connections. If $k = 0$ then the moduli space is $M_X(0,g) = \{\theta\}$ for any etric g by (3.12). The remaining case is $\Gamma_A = \mathbb{T}$ (and $k > 0$). The case where I_X is *not* negative definite s easiest.

PROPOSITION 3.34 [FU,COROLLARY 3.21]. *If $b_2^+(X) > 0$ then for an open dense set of metrics there are o reducible instantons.*

The curvature of a reducible instanton is an anti-self-dual 2-form whose cohomology class lies in the lattice $\pi i H^2(X;\mathbb{Z}) \subset H^2(X;\mathbb{R})$. So Proposition 3.34 asserts that there are no such forms for an open dense set of etrics. This makes good intuitive sense: A metric determines a decomposition

$$H^2(X;\mathbb{R}) \cong H^+ \oplus H^- \tag{3.35}$$

by identifying $H^2(X;\mathbb{R})$ with harmonic forms and decomposing via the $*$ operator. Now if $\dim H^- < \dim H^2(X;\mathbb{R})$, then a subspace of dimension $\dim H^-$ will not intersect a lattice. Therefore, to prove Proposition 3.34 one has to show that generic perturbations of the metric put the space H^- of anti-self-dual forms into general position. In [FU] Proposition 3.34 is derived from Theorem 3.28.

This leaves the case where I_X is negative definite. Then a reducible instanton A cannot be perturbed away. But according to Proposition 3.25 $\bar{A}$ is an isolated point of $M_X(k,g)$. The discussion surrounding (2.50) asserts that a neighborhood of $\bar{A}$ in $\mathcal{C}_X$ looks like $H_{\mathbb{R}} \times (H_{\mathbb{C}}/\Gamma_A)$. Note that $H_{\mathbb{R}}$ corresponds to directions in the space of reducible connections, which $M_X(k,g)$ intersects transversely. Now a discussion similar to the irreducible case shows that $H^2(A,g) = 0$ is the good situation. Then a neighborhood of $\bar{A}$ in $M_X(k,g)$ is modeled on $H^1(A,g)/\Gamma_A$, which is a cone on a complex projective space of complex dimension $h^1(A,g)/2$. One can prove that this good case is generic.

THEOREM 3.36 [FU,§4]. *If I_X is negative definite, then there is an open dense set of metrics g for which $M_X(k,g)$ is smooth at irreducible instantons, and is locally a cone on a complex projective space near a reducible instanton.*

We will often implicitly fix a generic "background" metric g and delete 'g' from the notation.

Compactness and ends

To carry out intersection theory in $\mathcal{C}_X$, our ultimate goal, we need to know whether or not the moduli spaces $M_X(k)$ are compact. In fact, $M_X(k)$ is usually not compact, but its structure near infinity (the *ends*) is well-understood. The basic results are due to Uhlenbeck [U2] and require detailed analysis of the nonlinear anti-self-dual equation (3.8). There are two difficulties: the conformal invariance and the gauge symmetry. The conformal invariance necessitates special *a priori* estimates to control the noncompactness. But the equations are only elliptic modulo the gauge symmetry. Here Uhlenbeck uses novel techniques to deal with the symmetry. We briefly review some of the major analytic ideas, leaving detailed estimates to the references. (*In this preliminary version we only review the analytic ideas in a simpler situation.*)

Consider first a linear elliptic differential equation

$$P\psi = 0 \tag{3.37}$$

on a compact manifold X. Ellipticity implies that the space of solutions is finite dimensional. But it is a linear space, hence never compact (unless trivial). The failure of compactness is due to scaling: If ψ solves (3.37) then so does $\lambda\psi$ for any $\lambda > 0$. Another basic property of elliptic equations is the smoothness (regularity) of distributional solutions. The proof goes roughly as follows. Introduce Banach spaces H_s, $s \in \mathbb{R}$, with the properties:

$$H_s \subset H_{s'} \text{ if } s \geq s'; \tag{3.38}$$

$$\bigcap_s H_s = C^\infty; \tag{3.39}$$

$$\text{The inclusion } H_s \longrightarrow H_{s'} \text{ is compact if } s > s'. \tag{3.40}$$

Heuristically, H_s is the space of functions with s derivatives (perhaps in some integral sense). Then if P is elliptic and nonnegative, the basic elliptic estimate asserts that if $\mu > 0$, then

$$P + \mu \colon H_s \longrightarrow H_{s-r} \text{ is invertible} \tag{3.41}$$

for some r and all s.[14] Now if $\psi \in H_s$ solves (3.37), then by (3.41) we can find $\tilde{\psi} \in H_{s+r}$ with

$$(P + \mu)\tilde{\psi} = \mu\psi, \tag{3.42}$$

and again by (3.41) we deduce $\tilde{\psi} = \psi$ in H_s. Hence ψ is an H_{s+r} function, and now iterating we find $\psi \in H_s$ for all s, whence by (3.39) ψ is smooth. This argument is called *elliptic regularity*.

Now suppose

$$Q \colon H_s \longrightarrow H_{s-r'} \tag{3.43}$$

is a nonlinear (continuous) operator, and assume that $r' < r$.[15] Then solutions to the nonlinear partial differential equation

$$(P + Q)\psi = 0 \tag{3.44}$$

are again smooth. For if $\psi \in H_s$ is a solution, then we can find $\tilde{\psi} \in H_{s+(r-r')}$ which solves

$$(P + \mu)\tilde{\psi} = -(Q - \mu)\psi, \tag{3.45}$$

from which we deduce $\tilde{\psi} = \psi$ in H_s as before. This shows $\psi \in H_{s+(r-r')}$; bootstrapping this procedure we conclude $\psi \in C^\infty$. Furthermore, if we can show that the space of solutions to (3.44) is bounded in some H_s (s sufficiently large), then the space of solutions is compact. This follows from (3.40). To establish the

[14] r is the order of P.

[15] It is worth pointing out that if $P + Q$ comes from a variational problem, this hypothesis implies that the energy functional is "Palais-Smale."

boundedness in H_s we must analyze closely the problem at hand, beyond merely demonstrating ellipticity, to produce good *a priori* estimates.

Unfortunately, the anti-self-dual Yang-Mills equations are not of this type; rather, $r' = r$. This reflects the fact that the equations are *conformally invariant*. That is, if $f \in \Omega^0_X$ and A is an instanton for the metric g, then A is also an instanton for the metric $e^f g$. Furthermore, the equations are elliptic only after dividing by the gauge symmetry. So our heuristics above do not apply directly. Nonetheless, they give some flavor of the following regularity theorem.

THEOREM 3.46 (UHLENBECK [FU,§8]). *If A is an instanton, then there exists a gauge transformation φ such that $A \cdot \varphi$ is smooth.*

We emphasize that the techniques used in the proof go far beyond the simple analysis when $r' < r$. Theorem 3.46 is proved together with estimates from which one deduces compactness results, to which we turn next.

We have already seen that $M_{S^4}(1)$ is *not* compact; it is the open 5-ball. It is easily compactified by adding the boundary 4-sphere, which we identify with the original space S^4. These boundary instantons correspond to $\lambda = 0$ in (3.22), which means that the curvature (3.23) is a δ-form supported at the origin. This is not a rigorous picture, but the intuition is invaluable: The instantons at ∞ are particles. These are the true instantons, completely localized to in instant in space(time). Notice that away from the origin these point instantons are flat.

A similar picture works on arbitrary X. Suppose $\{\tilde{A}_j\} \subset M_X(1)$ is a sequence of instantons. Then either a subsequence converges, or there is a point $x \in X$ such that a subsequence converges to a flat instanton on $X \setminus \{x\}$ together with a particle at x. Furthermore, a particle at x looks the same as a particle on S^4. That is, if we conformally rescale X by enlarging a neighborhood of x according to the scale of A_j, then the rescaled Riemannian manifolds X_j converge to $T_xX \cong \mathbb{R}^4$ and the rescaled instantons A_j converge to the standard $\lambda = 1$ instanton (3.22). This is the basic result of Uhlenbeck [FU,§8], which extends to higher k as well.

THEOREM 3.47 (UHLENBECK). *Let $\tilde{A}_j$ be a sequence in $M_X(k)$. Then after passing to a subsequence we can find lifts A_j, points $x_1, \dots, x_\ell \in X$, and weights $k_i \in \mathbb{Z}^+$ such that for some $A \in M_X(k - \sum k_i)$ we have $A_j \to A$ on compact subsets of $X \setminus \{x_1, \dots, x_\ell\}$.*

Informally, A_j converges to an instanton A of charge $k - \ell$ together with ℓ point particles. Notice that the total charge is conserved. Put differently, we can compactify $M_X(k)$ by adjoining these extra limit points:

$$\overline{M_X(k)} \subset M_X(k) \cup M_X(k-1) \times X \cup M_X(k-2) \times S^2(X) \cup \cdots \cup \{\theta\} \times S^k(X), \tag{3.48}$$

where $S^\ell(X)$ is the ℓ^{th} symmetric product of X.

This description is slightly deceiving as a parameter count with (3.21) quickly reveals. For example, consider $M_X(2)$ and suppose $b_2^+(X) = 0$. Then $\dim M_X(2) = 13$, whereas $\dim S^2(X) = 8$. Where are the extra 5 parameters near instantons which are approximately 2 particles? The scales λ of the concentrated instantons account for 2 of the parameters. The remaining 3 are a relative phase in $SO_3 = SU_2/\text{center}$ between the 2 particles. This is an important point: The instantons are *nonabelian* particles, so feel this relative phase between any pair of particles.

Suppose now $b_2^+(X) = 1$. Then $\dim M_X(2) = 10$ and we seem to be short 3 parameters in the moduli space (near the end where 2 particles are forming). In fact, the nonzero self-dual form is an obstruction to the existence of particles, and there is only a 5 dimensional subset of $X \times X$ on which 2 particles can form. This phenomenon was studied in detail by Taubes [T2] (cf. [D2]), and the resulting picture of the moduli space near its ends is crucial in applications.

Orientation

We must orient the moduli spaces $M_X(k)$ to carry out oriented intersection theory, and so obtain integer invariants (rather than $\mathbb{Z}/2\mathbb{Z}$ invariants). The orientability of $M_X(1)$ is discussed in [D1] and [FU §5]. Donaldson [D3] deals with the general case, orienting all $M_X(k)$ simultaneously. Here we review the main ideas of his construction.

An orientation of a manifold M is a trivialization of its determinant bundle $\det TM$.[16] This is a real line bundle over M, which is trivial if and only if its restriction to every circle in M is trivial.[17] If M is orientable, there are two possible orientations on each component.

More generally, if

$$0 \to E^0 \to E^1 \to \cdots \to E^k \to 0 \tag{3.49}$$

is a complex of vector bundles over a manifold C, then there is a determinant line bundle

$$\det E^\bullet = (\det E^0) \otimes (\det E^1)^{-1} \otimes \cdots \otimes (\det E^k)^{(-1)^k} \tag{3.50}$$

defined over C. This bundle is canonically trivial if (3.49) is exact. In general there is a natural isomorphism

$$\det E^\bullet \cong \det H^\bullet(E^\bullet), \tag{3.51}$$

where $H^\bullet(E^\bullet)$ is the (trivial) complex of homology groups. Using (3.51) we extend the determinant construction to the case where E^j is infinite dimensional but the homology is still finite dimensional. This

[16] The determinant of a finite dimensional vector space V is a line—its top exterior power $\det V$. The nonzero elements of $\det V$ fall into two components. An orientation of V is a choice of one of these components.

[17] Formally, the obstruction to orientability is measured by the first Stiefel-Whitney class $w_1(M) \in H^1(M; \mathbb{Z}/2\mathbb{Z})$.

applies to elliptic complexes, where the Atiyah-Singer index theory provides techniques for computing (3.51) topologically. Finally, since we deal with real vector bundles up to topological isomorphism, we can omit the inverses in (3.50)—a real line bundle is isomorphic to its dual.

We work with the moduli spaces of irreducible instantons.[18] If $\tilde{A} \in M_X(k)$ is irreducible, then the tangent space can be identified as

$$T_{\tilde{A}} M_X(k) = H^1(A) \tag{3.52}$$

(cf. (3.16) and (3.17)). On the other hand, recall that at A we have the elliptic complex (3.15) with homology groups $H^\bullet(A)$. At an irreducible connection $H^0(A) = H^2(A) = 0$, and so

$$\begin{aligned} \det H^1(A) &\cong \det H^\bullet(A) \\ &= \det H^0(A) \otimes \det H^1(A) \otimes \det H^2(A). \end{aligned} \tag{3.53}$$

By the previous discussion, the bundle on the right hand side of (3.53) extends to a real line bundle $L_X(k) \to C_X(k)$, the determinant line bundle of (3.15). Since orientations pull back, an orientation of $L_X(k)$ induces an orientation of $M_X(k)$. So it suffices to orient $L_X(k)$.

Consider first $k = 0$. Then $\mathfrak{g}_P$ in (3.15) is the trivial bundle $X \times \mathfrak{su}_2$, and relative to a basis of $\mathfrak{su}_2$ the complex (3.15) degenerates to 3 copies of the self-dual complex

$$0 \to \Omega_X^0 \to \Omega_X^1 \to (\Omega_X^2)_+ \to 0. \tag{3.54}$$

The determinant of the homology of (3.54) is the real line

$$L_X = \det H^0(X) \otimes \det H^1(X) \otimes \det H_+^2(X). \tag{3.55}$$

This is independent of the connection A, so $L_X(0)$ is the trivial bundle

$$L_X(0) = C_X(0) \times L_X. \tag{3.56}$$

Hence to orient $L_X(0)$ we need to fix an orientation of L_X. Donaldson terms this choice a ***homology orientation*** of X. Of course, there is a canonical isomorphism $H^0(X) \cong \mathbb{R}$ by constant functions, and if X is simply connected $H^1(X) = 0$. Hence we have only to orient $H_+^2(X)$. Note that $H^0(X)$ and $H^1(X)$ are independent of the metric, whereas $H_+^2(X)$ depends on the metric (cf. (3.35)). But the intersection form I_X is positive definite on $H_+^2(X)$, and the set of all subspaces on which I_X is positive definite is contractible. Hence the choice of metric is irrelevant, and a single choice of homology orientation works for all metrics simultaneously.

[18] As discussed previously the reducible instantons are isolated singular points in the moduli space. We will apply intersection theory to these by "chopping off" the moduli space near these singular points.

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