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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

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# CLASSICAL GAUGE THEORY

D.S. Freed
Department of Mathematics
The University of Chicago
Chicago, Illinois 60637
U.S.A.

# Classical Gauge Theory

DANIEL S. FREED

Department of Mathematics
University of Chicago

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In this lecture we introduce the geometric setup of the classical nonabelian gauge theories.

CONTENTS:


## Hodge theory

The Yang-Mills equations are a nonlinear generalization of Hodge theory, and in fact reduce to Hodge theory in the special case of a circle bundle. We begin, then, with a quick review. Let $M$ be a compact oriented $n$-manifold. The smooth differential $p$-forms, $\Omega^p$, are defined, and these fit together to form the *de Rham complex*

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n, \tag{1}$$

where $d$ is the exterior differentiation of E. Cartan. The de Rham cohomology groups are the cohomology groups of this complex:

$$H^p_{DR} = \frac{\operatorname{Ker} d : \Omega^p \to \Omega^{p+1}}{\operatorname{Im} d : \Omega^{p-1} \to \Omega^p}$$

Suppose $M$ is endowed with a Riemannian structure. Then we can define the energy of a form

$$\mathcal{E}(\alpha) = \int_M |\alpha|^2 * 1, \tag{2}$$

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This lecture was given long ago (September, 1983) in Berkeley when the author was a graduate student. The author now wishes to dissociate himself from any obscurities and burdensome notation!

where $*1$ is the volume form (the notation will be explained later). If we restrict $\mathcal{E}$ to a given cohomology class $\{\alpha + d\beta\}$, we can compute the equations for a critical point:

$$\begin{aligned} 0 = \frac{d}{dt}\bigg|_{t=0} \mathcal{E}(\alpha + td\beta) = \frac{d}{dt}\bigg|_{t=0} \int_M (\alpha + td\beta, \alpha + td\beta) * 1 \\ = 2\int_M (\alpha, d\beta) * 1 \\ = 2\int_M (d^*\alpha, \beta) * 1. \end{aligned}$$

Thus the Euler-Lagrange equation is:

$$d * \alpha = 0. \tag{3}$$

Of course, $\alpha$ is also closed,

$$d\alpha = 0, \tag{4}$$

and we can combine (3) and (4) to obtain the single second order linear elliptic equation

$$\Delta\alpha = (dd^* + d^*d)\alpha = 0. \tag{5}$$

Since $\mathcal{E}$ is a convex functional, any critical point is a minimum.

HODGE THEOREM. *In each cohomology class there is a unique form satisfying* (5).

### Principal Bundles

The Yang-Mills Lagrangian, which generalizes (2) in a nonlinear framework, arises most naturally in the geometry of principal bundles. Let $G$ be a compact Lie group, and $P$ a principal $G$-bundle over a manifold $M$. This means that $P$ is a locally trivial smooth fiber bundle with fiber $G$, and $G$ acts on $P$ via a *right* action $P \times G \to P$. All geometric objects associated to $P$ must respect this group action in some fashion. The simplest example of a principal bundle is the product bundle $P_0 = M \times G$. A principal bundle has no global sections unless it is a product; $P_0$ clearly has the global section

$$i_0 : m \mapsto (m, e) \tag{6}$$

where $e \in G$ is the identity element. Twisted $G$-bundles are *locally* isomorphic to products.

Any time $G$ acts on another space $F$ on the *left*, we can form a new bundle by replacing the fiber $G$ of $P$ with $F$. This is the *associated bundle* construction which we now recall. Let $\rho : G \to \mathrm{Aut}(F)$ be the action of $G$ on $F$. Here "Aut($F$)" is used loosely to describe the automorphisms of the structure of $F$. So if $F$ is a vector space, then Aut($F$) is the group of linear automorphisms of that space; if $F$ is a Lie algebra, then Aut($F$) is the group of Lie algebra automorphisms; if $F$ is a differentiable manifold, then Aut($F$) is the group of diffeomorphisms of $F$; etc. To form the associated bundle, first stick a copy of $F$ over every point of $P$, i.e., form $P \times F$, and then identify $\langle p, x\rangle \in P \times F$ with $\langle pg, \rho(g^{-1})x\rangle$. This collapses the fiber of $P \times F \rightarrow X$ down to $F$, and so the associated bundle is

$$P \times_G F = P \times F / \langle p, g\rangle \sim \langle p \cdot g, \rho(g^{-1})x\rangle.$$

EXAMPLES:

1. $F = G$ and $\rho(g)g' = gg'g^{-1}$ is conjugation. The associated bundle is denoted $\operatorname{Aut} G$, and now there is a group structure in each fiber.
2. $F = \mathfrak{g}$ the Lie algebra of $G$, and $\rho(g)Z = \operatorname{Ad}(g)Z$. The associated bundle of Lie algebras is denoted $\operatorname{ad} G$.
3. $\rho : G \rightarrow \operatorname{Aut}(V)$ is a linear representation. The associated bundle $\xi$ is a vector bundle.

Finally, it is clear that a section of $P \times_G F$ can be realized by a map $s : P \rightarrow F$ satisfying

$$s(p \cdot g) = \rho(g^{-1}) \cdot s(p). \tag{7}$$

The basic geometric ingredient in the Yang-Mills Lagrangian is a *connection* in $P$, or, in the physicists' terminology, a *vector potential*. In $P$ the vertical tangent space $VT_pP$ is well-defined, and via the infinitesimal action of $G$ it can be canonically identified with the Lie algebra $\mathfrak{g}$. A horizontal complement $H_p$ at each point comprises a connection. As stated above, there must be respect for the group action, which in this case means

$$H_{p \cdot g} = (R_g)_* H_p, \tag{8}$$

where $R_g : P \rightarrow P$ is right multiplication by $g \in G$. Alternatively, we can define the connection by the projection $\omega_p$ onto the vertical $VT_pP$ with kernel $H_p$. Then since $VT_pP \approx \mathfrak{g}$, $\omega$ is a $\mathfrak{g}$-valued 1-form on $P$:

$$\omega : T_pP \rightarrow \mathfrak{g}.$$

Furthermore, $\omega$ satisfies

$$\begin{aligned} \omega|_{VTP} &= \mathrm{id} \\ (R_g)^* \omega &= \operatorname{Ad} g^{-1} \cdot \omega \end{aligned} \tag{9}$$

The second property follows from (8). In the product situation $P_0$, we can pull $\omega$ down to the base via the section (6). Hence a connection in this case is described by

$$A = i_0^*\omega \in \Omega^1(\mathfrak{g}) = \mathfrak{g}\otimes\Omega^1, \tag{10}$$

a Lie-algebra valued 1-form on $M$. If $Z_\alpha$ is a basis of $\mathfrak{g}$ and $x^i$ are coordinate on $M$, then

$$A = A_i^\alpha Z_\alpha \otimes dx^i$$

carries two sets of indices. Notice that on $P_0$ there is a canonical product connection (for which $A\equiv 0$). The description (10) of a connection holds locally for twisted bundles.

Let $\mathcal{A}$ be the space of all connections on $P$, and suppose $\omega,\omega'\in\mathcal{A}$. Then by (9) the difference $\eta=\omega'-\omega$ satisfies

$$\eta|_{VTP} = 0,$$
$$(R_g)^*\eta = \operatorname{Ad} g^{-1}\cdot\eta$$

The first condition implies that $\eta$ descends to the base, and the second implies that $\eta$ takes values in the bundle $\operatorname{ad} P$ (cf. (7)). So $\eta\in\Omega^1(\operatorname{ad}P)$, and $\mathcal{A}$ is an affine space whose space of directions is $\Omega^1(\operatorname{ad}P)$. In the product situation, $\operatorname{ad}P$ is trivial, so that $\Omega^1(\operatorname{ad}P)\doteq\Omega^1(\mathfrak{g})$, and our description (10) of connections reflects the existence of a distinguished origin in $\mathcal{A}$, the product connection.

Let $\rho: G\to \operatorname{Aut}(V)$ be a linear representation of $G$, and $\xi$ the associated vector bundle. A connection on $P$ induces a connection on $\xi$ which can also be described by the associated *covariant derivative*

$$D:\Omega^0(\xi)\to\Omega^1(\xi). \tag{11}$$

In the product situation, we write

$$D = d + A, \tag{12}$$

which means

$$Df = df + A_i^\alpha\dot\rho(Z_\alpha)f\otimes dx^i, \tag{13}$$

where $f\in\Omega^0(\xi)$ is a section of $\xi$ (a map $M\to V$ in this product case), and $\dot\rho:\mathfrak{g}\to\operatorname{End}(V)$ is the representation of $\mathfrak{g}$ induced by $\rho$. The $d$ in (12) can be thought of as the canonical product connection. In fact, in the twisted situation if we fix a base connection $D_0$, any other connection can be written

$$D = D_0 + A.$$

Note that on ad $P$ (13) is the equation

$$Df = df + [A, f].$$

The covariant derivative (11) can be extended to

$$D : \Omega^p(\xi) \to \Omega^{p-1}(\xi)$$

by

$$D(f \otimes \alpha) = Df \wedge \alpha + f \otimes d\alpha$$

for $f \in \Omega^0(\xi)$, $\alpha \in \Omega^p$. Then (1) generalizes to the elliptic complex

$$\Omega^0(\xi) \xrightarrow{D} \Omega^1(\xi) \xrightarrow{D} \Omega^2(\xi) \xrightarrow{D} \cdots \xrightarrow{D} \Omega^n(\xi). \tag{14}$$

Strictly speaking, (14) is not a complex—it is no longer true that $D^2 = 0$. In fact, $D^2$ is multiplication by the *curvature*, a notion we now define.

In the context of connections on principal bundles, the curvature $\Omega$ of a connection $\omega$ measures the failure of $\{H_p\}$ to be an integrable distribution of $n$-planes. Think about the Frobenius Theorem to see that

$$\Omega(X, Y) = -\omega([\widetilde{X}, \widetilde{Y}]) \tag{15}$$

measures this failure; here $\widetilde{X}$ is the horizontal part of $X \in TP$, and $X, Y$ are extended to local vector fields so that the bracket is defined. The right hand side of (15) is independent of the extensions. It can also be checked that

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]. \tag{16}$$

The $\mathfrak{g}$-valued 2-form $\Omega$ satisfies

$$\iota(X)\Omega = 0 \qquad \text{for } X \in VTP,$$
$$(R_g)^*\Omega = \operatorname{Ad} g^{-1} \cdot \Omega,$$

which implies that $\Omega$ is the pullback of a 2-form on the base with value in ad $P$, i.e., $\Omega$ is a section of $\Omega^2(\operatorname{ad} P)$. Here $\iota(X)$ is contraction with $X$. In the product case $P_0 = M \times G$ we can apply the section $i_0$ to (16) to obtain

$$F = i_0^*\Omega = dA + \frac{1}{2}[A \wedge A].$$

This can be made more explicit. Let

$$F = \sum_{i<j} F_{ij}\, dx^i \wedge dx^j = \frac{1}{2} F_{ij}\, dx^i \wedge dx^j;$$

Then

$$F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + [A_i, A_j],$$

where

$$A_i = \sum_\alpha A_i^\alpha Z_\alpha.$$

As mentioned above, $D^2$ in (14) also gives the curvature. For $f \in \Omega^0(\xi)$, in the product case we compute

$$\begin{aligned} D^2 f &= D(df + \dot\rho(A)f) \\ &= d(\dot\rho(A)f) + \dot\rho(A)\wedge df + \dot\rho(A)\wedge\dot\rho(A)f \\ &= \dot\rho(dA)f - \dot\rho(A)\wedge df + \dot\rho(A)\wedge df + \frac{1}{2}\dot\rho([A\wedge A])f \\ &= \dot\rho(dA + \frac{1}{2}[A\wedge A])f \\ &= \dot\rho(F)f. \end{aligned} \tag{17}$$

The curvature of $\xi$ is $\dot\rho(F) \in \Omega^2(\operatorname{End}\xi)$.

We will need to know how the curvature changes with the connection. Thus for $\eta \in \Omega^1(\operatorname{ad} P)$, thinking of $\eta$ as a $\mathfrak{g}$-valued 1-form on $P$,

$$\begin{aligned} \Omega_{\omega+\eta} &= d(\omega+\eta) + \frac{1}{2}[(\omega+\eta)\wedge(\omega+\eta)] \\ &= d\omega + d\eta + \frac{1}{2}[\omega\wedge\omega] + [\omega\wedge\eta] + \frac{1}{2}[\eta\wedge\eta] \\ &= \Omega_\omega + D\eta + \frac{1}{2}[\eta\wedge\eta]. \end{aligned} \tag{18}$$

Here, as in (12),

$$D\eta = d\eta + [\omega\wedge\eta].$$

A computation similar to (17) gives an alternative derivation of (18); there the result is expressed in the notation

$$F_{D+A} = F_D + DA + \frac{1}{2}[A\wedge A].$$

### Gauge Transformations

The group of symmetries of $P$ is the set of maps

$$\phi : P \to P$$

satisfying

(1) $\phi(p \cdot g) = \phi(p) \cdot g \quad \text{for } g \in G$;

(2) $\phi$ covers the identity on $M$,

i.e., the group of bundle automorphisms of $P$ covering the identity on the base $M$. We call this the group of *gauge transformations*. Let

$$\phi(p) = p \cdot s(p)$$

for $s(p) \in G$. Now the $G$-equivariance becomes

$$\begin{aligned} p \cdot gs(p \cdot g) &= \phi(p \cdot g) \\ &= \phi(p) \cdot g \\ &= p \cdot s(p)g \\ &= p \cdot gg^{-1}s(p)g, \end{aligned}$$

so that

$$s(p \cdot g) = g^{-1}s(p)g.$$

Hence $s$ is a section of $\operatorname{Aut} P$, and the group of gauge transformations can be identified with $C^\infty(\operatorname{Aut} P)$. Denote this (very large infinite dimensional) group by $\mathcal{G}$. In the product situation $P_0$, gauge transformations are simply maps $s : M \to G$, and these act by *left* multiplication on sections of $P_0$. Such *global* gauge transformations exist in the twisted case if $G$ has a nontrivial center $C$. For since $G$ acts trivially on $C$ by conjugation, the bundle

$$P \times_G C \subset P \times_G G$$

is trivial. Global sections of $P \times_G C$ define global gauge transformations. If the center is discrete (e.g., the center $\mathbf{Z}/n\mathbf{Z}$ of $SU(n)$), then these global gauge transformations are constant. When $G$ is abelian (e.g., $G = U(1)$), then $\operatorname{Aut} P = M \times G$ and all gauge transformations are global. Finally, we remark that $C^\infty(\operatorname{ad} P)$ is the Lie algebra of $\mathcal{G}$.

Gauge transformations act on connections by either pushforward or pullback. We choose pullback since this gives a right action of $\mathcal{G}$ on $\mathcal{A}$, and the fibration $\mathcal{G} \to \mathcal{A} \to \mathcal{A}/\mathcal{G}$ is *essentially* a principal bundle. (It is not a principal bundle since the action of $\mathcal{G}$ is not effective if $G$ has a center, and is not free on all connections in $\mathcal{A}$. So $\mathcal{G} \to \mathcal{A} \to \mathcal{A}/\mathcal{G}$ must be modified slightly to obtain a principal fibration.) Let $s \in \mathcal{G}$ and define the bundle automorphism

$$\phi(p) = p \cdot s(p).$$

If $\omega$ is a connection 1-form, the pullback connection is $\phi^*\omega$. Now

$$(19) \qquad (\phi^*\omega_p)(X) = \omega_{p\cdot s(p)}(\phi_* X) = \mathrm{Ad}\, s(p)^{-1}\omega_p\Big((R_{s(p)^{-1}})_*\phi_* X\Big).$$

Let $\psi_t$ be a curve on $P$ with tangent $X$. So $\psi_0 = p$ and $\dot\psi_0 = X$. Thus

$$\begin{aligned}(R_{s(p)^{-1}})_*\phi_* X &= \frac{d}{dt}\Big|_{t=0}(R_{s(p)^{-1}} \circ \phi)(\psi_t)\\ &= \frac{d}{dt}\Big|_{t=0}\psi_t \cdot s(\psi_t)s(p)^{-1}\\ &= X + ds(X)s(p)^{-1}.\end{aligned}$$

Here we have implicitly identified $VT_pP \approx \mathfrak{g}$. Plugging back into (19),

$$(\phi^*\omega_p)(X) = \mathrm{Ad}\, s(p)^{-1}\cdot\omega_p(X) + \mathrm{Ad}\, s(p)^{-1}\cdot ds(X)s(p)^{-1},$$

or

$$(20) \qquad \phi^*\omega = s^{-1}ds + \mathrm{Ad}\, s^{-1}\omega.$$

We will also denote $\phi^*\omega$ by $s^*\omega$. In the product case we obtain

$$(21) \qquad s^*A = s^{-1}ds + s^{-1}As.$$

This can also be derived by thinking of the action of $s$ on some associated vector bundle (the representation $\rho : G \to \mathrm{Aut}(V)$ induces $\mathrm{Aut}\,\rho : \mathrm{Aut}\, G \to \mathrm{Aut}(\mathrm{Aut}\, V)$ which allows us to think of $s$ as a section of $\mathrm{Aut}(\xi)$), and computing the effect on the covariant derivative $D = d + A$. Omitting $\rho$ from the notation, for $f \in \Omega^0(\xi)$ we have

$$(s^*D)f = s^{-1}\circ(d+A)\circ s(f) = s^{-1}sdf + s^{-1}ds(f) + s^{-1}As(f).$$

So

$$(22) \qquad s^*D = d + s^{-1}ds + s^{-1}As,$$

and (21) follows. This calculation carries over to the twisted situation where we fix a base connection $D_0$ and write an arbitrary connection as $D_0 + A$.

The action of $s \in \mathcal{G}$ on curvature is computed by

$$\begin{aligned}(\phi^*\Omega_p)(X,Y) &= \Omega_{p\cdot s(p)}(\phi_* X, \phi_* Y)\\ &= \operatorname{Ad} s(p)^{-1}\Omega\Big((R_{s(p)^{-1}})_*\phi_* X, (R_{s(p)^{-1}})_*\phi_* Y\Big)\\ &= \operatorname{Ad} s(p)^{-1}\Omega(X,Y),\end{aligned}$$

since $\Omega$ vanishes on vertical vectors. So

$$\phi^*\Omega = \operatorname{Ad} s^{-1}\Omega. \tag{23}$$

In the product situation,

$$s^* F = s^{-1} F s. \tag{24}$$

We can check (24) directly:

$$F_{s^* D} = (s^{-1} D s)^2 = s^{-1} D^2 s = s^{-1} F_D s.$$

In making the computation we have multiplied elements of $\mathfrak{g}$ by implicitly using a representation $\dot{\rho} : \mathfrak{g} \to \operatorname{End}(V)$.

### The Yang-Mills Equations

We are finally ready to set up the calculus of variations problem leading to the Yang-Mills equations. Assume that $M$ is compact oriented and is endowed with a Riemannian structure. Since $G$ is compact, minus the Killing form is an inner product on $\mathfrak{g}$. Then we define a functional $\mathcal{YM}$ on the space of connections $\mathcal{A}$ by

$$\mathcal{YM}(\omega) = \int_M |\Omega_\omega|^2 * 1. \tag{25}$$

Here $\Omega = \Omega_\omega \in \Omega^2(\operatorname{ad} P)$ is the curvature of $\omega \in \mathcal{A}$, and $|\Omega|$ is computed using the Riemannian metric on the base and minus the Killing from on $\mathfrak{g}$. Note immediately that for any gauge transformation $s$,

$$|s^*\Omega| = |\operatorname{Ad} s^{-1}\Omega|$$

by (23), and since the Killing form is Ad-invariant,

$$|s^*\Omega| = |\Omega|.$$

So $\mathcal{YM}$ is $\mathcal{G}$ invariant and actually defines a function on $\mathcal{A}/\mathcal{G}$. Furthermore, the Euler-Lagrange equations of $\mathcal{YM}$ are $\mathcal{G}$ invariant. Before computing these equations, we remark that in the case of Hodge theory we restricted the functional (2) to a fixed cohomology class. In the Yang-Mills setup, the fixed topology of the bundle P obviate the need to restrict $\mathcal{YM}$. Now we compute the Euler-Lagrange equations assuming that $M$ is closed (compact without boundary):

$$
\begin{aligned}
0 = \frac{d}{dt}\Big|_{t=0} \mathcal{YM}(\omega + t\eta) &= \frac{d}{dt}\Big|_{t=0} \int_M |\Omega + tD\eta + \frac{t^2}{2}[\eta\wedge\eta]|^2 * 1 \\
&= 2\int_M (\Omega, D\eta) * 1 \\
&= 2\int_M (D^*\Omega, \eta) * 1
\end{aligned} \tag{26}
$$

for all $\eta \in \Omega^1(\mathrm{ad}\, P)$. So the equations for a critical point are

$$D^*\Omega = 0, \tag{27}$$

a second order equation in $\omega$ called the *Yang-Mills equation*. The Bianchi identity can be written

$$D\Omega = 0, \tag{28}$$

and (27) and (28) together say that Yang-Mills curvatures are the degree two harmonic elements in the complex

$$\Omega^0(\mathrm{ad}\, P) \xrightarrow{D} \Omega^1(\mathrm{ad}\, P) \xrightarrow{D} \Omega^2(\mathrm{ad}\, P) \xrightarrow{D} \cdots \xrightarrow{D} \Omega^n(\mathrm{ad}\, P). \tag{29}$$

This is the analogy to Hodge theory (cf. (1)–(4)). In fact, if $G = U(1)$ we recover the de Rham complex form (29), and (27) and (28) reduce to (3) and (4). We discuss this in greater detail later. Notice, however, that $\mathcal{YM}$ is not a convex functional, and not all critical points are guaranteed to be minima.

If $M$ is a manifold with boundary, then there is an extra term from the integration by parts in (26). Let $j : \partial M \hookrightarrow M$ denote the inclusion of the boundary, and assume that $\partial M$ is oriented, so that there is a global section $\nu$ of the normal bundle to $\partial M$ in $M$. Then the boundary term in (26) appears in

$$\frac{1}{2}\frac{d}{dt}\Big|_{t=0} \mathcal{YM}(\omega + t\eta) = \int_M (D^*\Omega, \eta) * 1 \pm \int_{\partial M} (\iota(\nu) j^*\Omega, j^*\eta) * 1 \tag{30}$$

where $\iota(\cdot)$ is contraction of a vector field with a form. This boundary term is most easily computed using the star operator, as discussed in the next section. There are two cases which arise frequently for which this boundary term disappears.

1. We restrict to connections fixed on $\partial M$. Then $j^*\eta = 0$ for all variations considered.

2. A connection $\omega$ on $P$ with curvature $\Omega$ restricts to a connection $j^*\omega$ on $j^*P$ with curvature $j^*\Omega$. If we restrict to connections $\omega$ for which the restricted connection is flat, $j^*\Omega = 0$, then again the boundary term vanishes.

In both cases we obtain the Yang-Mills equation (27), but now over a restricted set of connections.

Finally, in the product situation $D = d + A$, we write (27) as

$$d^*dA + \frac{1}{2}d^*[A \wedge A] + [A \lrcorner dA] + \frac{1}{2}[A \lrcorner [A \wedge A]] = 0. \tag{31}$$

(Here $\lrcorner$ denotes inner product on forms.) In this form the cubic nonlinearity is apparent. Also the gauge symmetry manifests itself in that (31) is not elliptic; tangents to the orbits of $\mathcal{G}$ on $\mathcal{A}$ provide a kernel for the symbol of (31). To compensate one often restricts to connections $\{A : d^*A = 0\}$ perpendicular to the orbit. Then a solution $A$ satisfies the elliptic equation

$$\Delta A + \text{lower order } = 0.$$

### The Star Operator

Let $V^n$ be an oriented inner product space. The volume form vol is a distinguished element of $\Lambda^n V$. We define

$$* : \Lambda^p V \longrightarrow \Lambda^{n-p} V$$

by

$$\alpha \wedge *\beta = (\alpha, \beta)\,\text{vol} \tag{32}$$

for all $\alpha, \beta \in \Lambda^p V$. In terms of an oriented orthonormal basis $e_1, \cdots, e_n$,

$$*(e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_{j_1} \wedge \cdots \wedge e_{j_{n-p}} \tag{33}$$

where $\{i_1, \cdots, i_p, j_1, \cdots, j_{n-p}\}$ is an even permutation of $\{1, 2, \cdots, n\}$. It is easy to check that

$$*^2 = (-1)^{p(n-p)} \quad \text{on } \Lambda^p V. \tag{34}$$

Note that $*1 = \text{vol}$, explaining our notation for the volume form in integrals.

EXAMPLE: In the four dimensions, writing $e_{i_1\cdots i_p} = e_{i_1}\wedge\cdots\wedge e_{i_p}$,

$$*e_{12} = e_{34},$$
$$*e_{13} = -e_{24},$$
$$*e_{14} = e_{23}.$$

We can also define the $*$ operator for oriented vector spaces whose inner product is not positive definite. If $e_1,\cdots,e_n$ is an oriented orthonormal basis, where now $(e_i,e_i)=\pm1$, then denoting $\#(e_{i_1},\ldots,e_{i_p}) =$ number of $e_{i_j}$ with $(e_{i_j},e_{i_j})=-1$, and $\# = \#(e_1,\ldots,e_n)$, we have

$$*(e_{i_1}\wedge\cdots\wedge e_{i_p}) = (-1)^{\#(e_{i_1},\ldots,e_{i_p})}e_{j_1}\wedge\cdots\wedge e_{j_{n-p}}, \tag{35}$$

and

$$*^2 = (-1)^{p(n-p)+\#}. \tag{36}$$

EXAMPLE: In the Lorentz metric $-(f_1,f_2)=(f_2,f_2)=(f_3,f_3)=(f_4,f_4)=1$,

$$*f_{12} = -f_{34}$$
$$*f_{13} = f_{24}$$
$$*f_{14} = -f_{23}$$
$$*f_{34} = f_{12}$$
$$*f_{24} = -f_{13}$$
$$*f_{23} = f_{14}.$$

Also, $*^2=-1$ on $\Lambda^2 V^4_{\text{Lorentz}}$.

Suppose $M$ is an oriented Riemannian manifold. Then $*$ can be used to express the $L^2$ inner product:

$$(\alpha,\beta)_{L^2} = \int_M (\alpha,\beta)*1 = \int_M \alpha\wedge*\beta \quad \text{for } \alpha,\beta\in\Omega^p. \tag{37}$$

Then

$$|\Omega|^2 * 1 = \Omega\wedge*\Omega,$$

so (25) can be written

$$\mathcal{YM}(\omega) = \int_M \Omega_\omega\wedge*\Omega_\omega. \tag{38}$$

The $*$ operator can also be used to express the adjoint of $d$.

LEMMA. $d^* = (-1)^{np+n+1} * d *$ on $\Omega^p$.

PROOF: For $\alpha \in \Omega^p$, $\beta \in \Omega^{p-1}$,

$$\int_M (\alpha, d\beta) * 1 = \int_M d\beta \wedge *\alpha.$$

But

$$d(\beta \wedge *\alpha) = d\beta \wedge *\alpha + (-1)^{p-1}\beta \wedge d * \alpha$$

integrates to zero if $\partial M = \emptyset$. So

$$\begin{aligned}\int_M (\alpha, d\beta) * 1 &= (-1)^p \int_M \beta \wedge d * \alpha \\ &= (-1)^p \int_M (\beta, *^{-1} d * \alpha) * 1 \\ &= (-1)^p (-1)^{(p-1)(n-p+1)} \int_M (\beta, *d * \alpha) * 1.\end{aligned}$$

Hence

$$d^*\alpha = (-1)^{np+n+1} * d * \alpha.$$

Note that if $n$ is *even*,

$$d^* = - * d*$$

for forms of all degrees. The formula for $d^*$ extends to $D^*$ trivially, so the Yang-Mills equation (27) can be written

$$D * \Omega = 0.$$

Also, the formula

$$(39) \qquad d(\eta \wedge *\Omega) = D\eta \wedge *\Omega - \eta \wedge D * \Omega$$

holds, and we leave it to the reader to derive (30) from (39).

**Self-duality and Conformal Invariance**

In four dimensions there is an extra feature to the Yang-Mills equation which allows us to obtain solutions from a first-order equation. First, let $V$ be an oriented four-dimensional inner product space. Then

$$* : \Lambda^2 V \to \Lambda^2 V$$

satisfies $*^2 = +1$. So

$$\Lambda^2 V = \Lambda^2_+ V \oplus \Lambda^2_- V$$

splits into the direct sum of the $+1$ and $-1$ eigenspaces. (This corresponds to the Lie algebra decomposition $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.) Note that $\{e_{12} + e_{34}, e_{13} - e_{24}, e_{14} + e_{23}\}$ is a basis of $\Lambda^2_+ V$, and $\{e_{12} - e_{34}, e_{13} + e_{24}, e_{14} - e_{23}\}$ is a basis of $\Lambda^2_- V$. This decomposition extends to oriented Riemannian 4-manifolds. We call forms in $\Omega^2_+$ *self-dual* and forms in $\Omega^2_-$ *anti-self-dual*. The curvature $\Omega \in \Omega^2(\operatorname{ad} P)$ of a connection decomposes

$$\Omega = \Omega_+ + \Omega_-$$

If $\Omega$ is self-dual or anti-self-dual,

$$*\Omega = \pm\Omega. \tag{40}$$

Then since

$$D\Omega = 0$$

by Bianchi, we have

$$D * \Omega = 0,$$

and $\Omega$ automatically satisfies the Yang-Mills equation. We will see in the next section that (anti-) self-dual solutions are minima of $\mathcal{YM}$. Note that (40) is a first order equation.

There is another special feature that the four dimensional equations enjoy: conformal invariance. Suppose $g \to \kappa^2 g$ is a conformal change of metric. Then

$$\text{vol} \to \kappa^4 \ \text{vol},$$

and under this change the inner product on 2-forms is multiplied by $\kappa^{-4}$. So the right-hand side of

$$\alpha \wedge *\beta = (\alpha, \beta) \ \text{vol}$$

is conformally invariant. This proves that the $*$ operators on 2-forms is conformally invariant, and it follows that the full Yang-Mills equation

$$D * \Omega = 0$$

as well as the (anti-) self-dual equation

$$*\Omega = \pm\Omega$$

are conformally invariant in four dimensions.

## A Topological Invariant

Recall that if $P$ is a $G$ bundle over a compact manifold $M$, $\Omega$ is the curvature of a connection $\omega$ on $P$, and $\varphi : \mathfrak{g}^{\ell} \to \mathbf{R}$ is an Ad-invariant polynomial on $\mathfrak{g}$, then the cohomology class $\varphi(\Omega)$ is a *characteristic class* of the bundle $P$, in particular is a topological invariant of $P$ independent of the choice of connection. The Killing form

$$\varphi(X,Y) = \mathrm{tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$$

is an Ad-invariant polynomial, and so

$$p(\Omega) = \mathrm{tr}(\mathrm{ad}\,\Omega \wedge \mathrm{ad}\,\Omega)$$

is a 4-form whose cohomology class is a topological invariant.

Write

$$\Omega = \Omega_+ + \Omega_-$$

as before. Since $\Omega_+$ and $\Omega_-$ are perpendicular,

$$p(\Omega) = \mathrm{tr}(\mathrm{ad}\,\Omega_+ \wedge \mathrm{ad}\,\Omega_+) + \mathrm{tr}(\mathrm{ad}\,\Omega - \wedge\, \mathrm{ad}\,\Omega_-) = -|\Omega_+|^2 + |\Omega_-|^2.$$

If $M$ is four dimensional, we can integrate to obtain the topological invariant

$$TI = \int_M (|\Omega_-|^2 - |\Omega_+|^2) * 1. \tag{41}$$

But

$$\mathcal{YM}(\omega) = \int_M (|\Omega_-|^2 + |\Omega_+|^2) * 1 \tag{42}$$

Hence

$$\mathcal{YM}(\omega) \geq |TI|$$

with equality if and only if

$$*\Omega = (\mathrm{sign}\, TI)\Omega.$$

For $TI > 0$ self-dual solutions are minimizing, and for $TI < 0$ anti-self-dual solutions are minimizing. Note that there is no guarantee that the topological lower bound $|TI|$ is attained. The invariant $TI$ is the first Pontrjagin number of $\mathrm{ad}\, P$ up to a numerical factor.

**Line Bundles**

We now come full circle and show that when $G = U(1)$, the Yang-Mills setup reduces to Hodge theory. For then ad $P$ is trivial and $\Omega^p(\text{ad}\,P) \approx \Omega^p$, so that the curvature is an ordinary 2-form on $M$. Furthermore, ad $P$ inherits the trivial connection from $P$ no matter what the connection is on $P$, and so the Yang-Mills equations reduce to

$$df = 0$$
$$d^*f = 0$$

for $f \in \Omega^2$. We have recovered the equations (3) and (4) for harmonic forms. Note that there are always solutions to these equations by the Hodge theorem.

EXAMPLE: Over a Lorentz manifold, choosing an orthonormal local frame for which $-|\theta^1|^2 = |\theta^2|^2 = |\theta^3|^2 = |\theta^4|^2 = 1$, we obtain

$$f = \begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} \\ & 0 & f_{23} & f_{24} \\ & & 0 & f_{34} \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ & 0 & -B_z & B_y \\ & & 0 & B_x \\ & & & 0 \end{pmatrix}$$

$$*f = \begin{pmatrix} 0 & -f_{34} & f_{24} & -f_{23} \\ & 0 & f_{14} & -f_{13} \\ & & 0 & f_{12} \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ & 0 & E_z & -E_y \\ & & 0 & E_x \\ & & & 0 \end{pmatrix}$$

Now $df = 0$ is the condition

$$E_{x;y} - E_{y;x} - B_{z;t} = 0, \quad \text{etc.},$$

which is

(43)
$$\frac{d\vec{B}}{dt} = \nabla \times \vec{F}, \qquad \nabla \cdot \vec{B} = 0.$$

Similarly, $D * f = 0$ becomes

(44)
$$\frac{d\vec{F}}{dt} = \nabla \times \vec{B}, \qquad \nabla \cdot \vec{E} = 0.$$

Equations (43) and (44) are Maxwell's equations, demonstrating that Yang-Mills generalizes Maxwell. A self-dual solution satisfies $*f = if$ (remember $*^2 = -1$ in the Lorentz metric!) which is

$$\vec{E} = -i\vec{B},$$

and an anti-self-dual solution satisfies $*f = -if$, or

$$\vec{E} = i\vec{B}.$$