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COLLEGE

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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# ALGEBRAIC TOPOLOGY.

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## ALGEBRAIC TOPOLOGY

### §1. Introduction.

In topology, one is interested in topological spaces and continuous maps. Algebraic topology is the study of topological invariants of a space. By this we mean that for any space $X$, we associate an invariant $I(X)$ which can be a set, a group or other structure and it is an invariant if whenever $X$ and $Y$ are homeomorphic, $I(X)$ and $I(Y)$ are isomorphic.

It has turned out that the best way to describe invariants is via functors. A functor $M$ associates to every space $X$ a set (a group, etc.) $M(X)$ and to every map $f : X \to Y$ a map of sets (groups, etc.) $M(f) : M(X) \to M(Y)$ so that

(1) If $f : X \to X$ is the identity
then $M(f) : M(X) \to M(X)$ is the identity

(2) If $X \xrightarrow{f} Y \xrightarrow{g} Z$
then $M(g \circ f) = M(g) \circ M(f)$

If $M$ is a functor, then $M(X)$ is a topological invariant, for if $X$ and $Y$ are homeomorphic, then there exist $X \xrightarrow{f} Y, Y \xrightarrow{g} X$ with $g \circ f = 1_X$, $f \circ g = 1_Y$ and hence $M(X) \xrightarrow{M(f)} M(Y)$ $M(Y) \xrightarrow{M(g)} M(X)$ satisfy $M(g) \circ M(f) = M(g \circ f) = M(1_X) = I_{M(X)}$, $M(f) \circ M(g) = M(f \circ g) = M(1_Y) = I_{M(Y)}$ and thus $M(f)$ and $M(g)$ are inverses of each other, establishing an isomorphism $M(f) : M(X) \approx M(Y)$.

Recall that if $A, B$ and $C$ are sets and $B^A = \{f : A \to B\}$ we have the exponential law:

$$(B^A)^C \approx B^{A \times C}$$

as sets.

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One defines $\phi : (B^A)^C \to B^{A\times C}$ by

$$\phi(f)(a,c) = f(c)(a)$$

and $\quad \Phi : B^{A\times C} \to (B^A)^C$

by $\quad \Phi(g)(c)(a) = g(a,c)$

then $\phi$ and $\Phi$ are inverses of each other.

We denote by $\mathrm{Map}(X,Y)$ the set of maps $f : X \to Y$ with the compact open topology: A subbasis consists of the subsets $\{C,U\}$ of $\mathrm{Map}(X,Y)$, where $C$ is compact in $X$ and $U$ open in $Y$ and $\{C,U\} = \{f : X \to Y | f(C) \subset U\}$. The exponential law now says:

If $Y$ is locally compact Hausdorff, $X$ Hausdorff, then

$$\mathrm{Map}(X \times Y, Z) \approx \mathrm{Map}(X, \mathrm{Map}(Y,Z))$$

(homeomorphism).

We now define the most important topological invariant; the set of path components $\Pi_0 X$ in a space $X$. Let $I$ denote the closed interval $[0,1]$. Then a <u>path</u> in $X$ is a map $\alpha : I \to X$, with $\alpha(0) =$ initial point, $\alpha(1) =$ end point. We define an equivalence relation in $X$, $x \sim y$ iff there exists a path $\alpha$ with initial point $x$ and end point $y$. The equivalence classes are called path components in $X$. If $x \in X$, $[x]$ will denote its path-component, and $\Pi_0(X)$ will denote the set of path components in $X$.

If $X$ and $Y$ are spaces, we can also look at $\Pi_0(\mathrm{Map}(X,Y))$. If $X$ is locally compact and Hausdorff, $\Pi_0(\mathrm{Map}(X,Y))$ is called the set of <u>homotopy classes</u> of maps from $X$ to $Y$, also denoted by $[X,Y]$. In this case, $f$ and $g$ are in the same path component means that there exists $\alpha : I \to \mathrm{Map}(X,Y)$ with $\alpha(0) = f$, $\alpha(1) = g$. By the exponential law, $\alpha$ corresponds to $F : I \times X \to Y$, where $F(0,x) = f(x)$, $F(1,x) = g(x)$. This is known as a <u>homotopy</u> from $f$ to $g$. If $x_0 \in X$, a loop on $X$ based at $x_0$ is a path in $X$ starting and ending at $x_0$. We denote by $\Omega(X,x_0)$ or $\Omega X$ the <u>space of loops</u> on $X$ based at $x_0$, or simply <u>loopspace</u> of $X$. We now give $\Pi_0(\Omega X)$ a structure of a group, as follows.

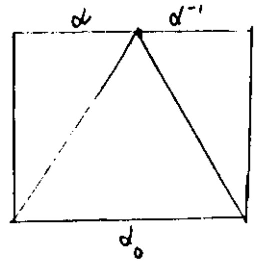


First we define a composition in $\Omega X$. If $\alpha, \beta$ are loops

$$\alpha\cdot\beta(t) = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

defines a loop, called the product of $\alpha$ and $\beta$.

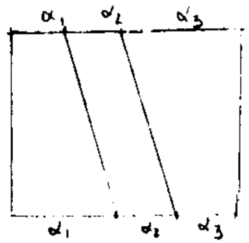
**Exercise.** Show that $[\alpha]\cdot[\beta] = [\alpha\beta]$ is a composition in $\Pi_0(\Omega X)$. Now $\alpha_0$ the constant loop at $x_0$ acts like the identity in this composition. Also defining $\alpha^{-1}(t) = \alpha(1-t)$, the "inverse loop", we have $\alpha\alpha^{-1}$ is homotopic to $\alpha_0$. Here is the diagram.

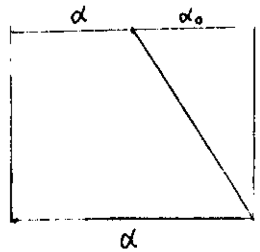



define

$$H(s,t) = \begin{cases} \alpha(0) & 0 \le s \le \frac{t}{2} \\ \alpha(2t-s) & \frac{t}{2} \le s \le \frac{1}{2} \\ \alpha(2-2t-s) & \frac{1}{2} \le s \le 1-\frac{t}{2} \\ \alpha(0) & 1-\frac{t}{2} \le s \le 1 \end{cases}$$

Also associativity $\alpha_1(\alpha_2\alpha_3) = (\alpha_1\alpha_2)\alpha_3$ and identity are pictured by:



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The group $\Pi_0(\Omega(X,x_0))$ is also known as the fundamental group of $X$ based at $x_0$ and denoted also by $\Pi_1(X,x_0)$. Again $\Pi_1(X,x_0)$ is a topological invariant, which is now a group.

More generally, one can take $\Omega^n(X,x_0)=\Omega(\Omega^{n-1}(X,x_0),x_0)$ where $x_0$ denote the constant loop and $\Pi_0(\Omega^n(X,x_0))=\Pi_n(X,x_0)$ is called the $n^{th}$ homotopy group of $X$. It turns out that $\Pi_n(X,x_0)$ is abelian for $n\geq 2$.

The set $\Pi_0(X)$ has the property that if $f:X\to Y$ then $f$ induces $f_*:\Pi_0X\to\Pi_0Y$. If $f$ is homotopic to $g$ then $f_*=g_*:\Pi_0X\to\Pi_0Y$.

What we are saying is that $\Pi_0X$ is a homotopy invariant. We say $X$ and $Y$ are homotopy equivalent if there exists maps $X\xrightarrow{f}Y$ and $Y\xrightarrow{g}X$ so that $g\circ f$ homotopic to $1_X$ and $f\circ g$ homotopic to $1_Y$. Then if $X$ and $Y$ are homotopy equivalent, $\Pi_0X$ and $\Pi_0Y$ are isomorphic, for $\Pi_0X\xrightarrow{f_*}\Pi_0Y$ $\Pi_0Y\xrightarrow{g_*}\Pi_0X$ are inverses of each other. It is easy to see that if $X$ and $Y$ are homotopy equivalent, then so are $\Omega^nX$ and $\Omega^nY$, if we are careful about base points. The group $\Pi_1(X,x_0)$ is not easy to compute. However, if $X$ is contractible, i.e. if there exists a homotopy $F:X\times I\to X$ with $F(x,0)=x$, $F(x,1)=x_0$ all $x$, then $\Pi_1(X,x_0)=1$. (Prove that in this case $X$ is homotopy equivalent to the space consisting of a single point.) In particular, $\Pi_1(R^n,x_0)=\Pi_1(I^n,x_0)=1$, where $R^n$ is euclidean n-space and $I^n$ is the n-cube.

We now try to establish some means of computing $\Pi_1 X$.



## §2. Covering spaces and the fundamental group.

A map $p:E\to X$ is called a covering space if every point $x\in X$ has an open neighborhood $U$ so that $p^{-1}(U)=\coprod W_\alpha$, disjoint union of open sets, and

$$p:W_\alpha\cong U$$

Then $W_\alpha$ are called the sheets, $U$ is said to be evenly covered. It follows that $p^{-1}(x)$ is a discrete space, that $p$ is a local homeomorphism, $p$ is onto and $X$ has the quotient topology.

Covering spaces have the unique lifting property (for connected spaces). Namely suppose $Y$ is connected, $e_0\in p^{-1}(x_0)$ and $f:Y\to X$ with $f(y_0)=x_0$. Then if there exists a lifting $g:Y\to E$ of $f$, i.e., $pg=f$, and $g(y_0)=e_0$, then $g$ is unique. In diagrams:

$$\begin{array}{ccc} & & (E,1e_0) \\ & {\scriptstyle g}\nearrow & \downarrow \\ (Y,y_0) & \xrightarrow[f]{} & (X,x_0) \end{array}$$

We show uniqueness as follows. Suppose $g'$ is another such lifting ($g'(y_0)=e_0$). Then decompose $Y=Y_0\coprod Y_1$ by

$$Y_0=\{y\in Y|g(y)=g'(y)\}$$
$$Y_1=\{y\in Y|g(y)\neq g'(y)\}$$

$y_0\in Y_0$. Now we show both $Y_0$ and $Y_1$ are open. If $y\in Y_0$, $g(y)=g'(y)$. Let $x=f(y)$. Then $x$ has a neighborhood $U$ so that $p:W_\alpha\approx U$, where $g(y)=g'(y)\in W_\alpha$. Then $g^{-1}(W_\alpha)\cap g'^{-1}(W_\alpha)$ is open in $Y$ and if $y'\in g^{-1}(W_\alpha)\cap g'^{-1}(W_\alpha)$, $pg'y'=pgy'$, but $p$ is $1-1$ in $W_\alpha$, thus $gy=gy'$. Hence $Y_0$ is open. Now suppose $y\in Y_1$, then $g(y)\neq g'(y)$, yet $x=pg(y)=pg'(y)$. So again by taking $U$ evenly covered, $g(y)\in W_\alpha$, $g'(y)\in W_\beta$ and $\alpha\neq\beta$. Take $g^{-1}(W_\alpha)\cap g^{1-1}(W_\beta)$ it is non-empty open in $Y$ and inside $Y_1$.

We show that for a covering space, we have the unique path lifting property:

$$\begin{array}{ccc} & & (E,e_0) \\ & {\scriptstyle \sigma'}\nearrow & \downarrow \\ (I,0) & \xrightarrow[\sigma]{} & (X,x_0) \end{array}$$

i.e. any path $\sigma$ in $X$ starting in $x_0$ lifts to a unique path $\sigma'$ starting at $e_0$. The uniqueness follows from the above. Now to show existence.

If $U$ is evenly covered and $\sigma : [t_0, t_1] \to U$ is a path $\sigma(t_0) = x_0$ and $e_0$ covers $x_0$. Then $e_0$ lies in a unique sheet $W_\alpha$ with $p_\alpha : W_\alpha \approxeq U$. Define $\sigma' : [t_0, t_1] \to E|U$ by $\sigma' = p_\alpha^{-1} \circ \sigma$. Now there is a covering of $X$ by evenly covered sets $\{U_\sigma\}$. Then $\{\sigma^{-1}(U_\alpha)\}$ covers $I$ which is compact, so we can find a finite partition of $I$, $0 < t_1 < t_2 < \cdots < t_n = 1$, so that $[t_i, t_{i+1}]$ maps into an evenly covered set $U_i$. Beginning at $[t_0, t_1]$, we have a unique path $\sigma^1 : [t_0, t_1] \to E$ lifting $\sigma$ and starting at $\sigma^1(t_0) = e_0$. Then $\sigma^1(t_1)$ is unique, so we can lift $\sigma'|[t_1, t_2]$ and so on, giving a path $\sigma^1$ in $E$ lifting $\sigma$.

Another important property is the <u>covering homotopy property</u>. Suppose given the diagram

$$\begin{array}{ccccc} & & & & (E, e_0) \\ & & & \overset{g}{\nearrow} \; \overset{G}{\nearrow} & \downarrow p \\ (Y \times 0, y_0 \times 0) & \xrightarrow{i_0} & (Y \times I, y_0 \times I) & \xrightarrow{F} & (X, x_0) \end{array}$$

Here $i_0$ is the imbedding at the bottom, and what is asserted is that $G$ <u>exists</u>.

To prove its existence, take any $y \in Y$, then one can find a neighborhood $N$ of $y$ and $G : N \times I \to E$ lifting $F$ and extending $g$, as follows. By compactness of $I$ and the fact that the evenly covered sets cover $X$, we can find a partition $0 < t_1 < \cdots < t_n = 1$ so that $N_i \times [t_i, t_{i+1}]$ maps into an evenly covered neighborhood of $F(y, t_i)$. Then by taking $N = \bigcap_i N_i$, we get a lifting $G : N \times I \to E$. Now the liftings $G : N \times I \to E$ are unique as extensions of $g$. Then at an intersection, $(N(y) \times I) \cap (N(y') \times I) = (N(y) \cap N(y')) \times I$; and if $y_1 \in N(y) \cap N(y')$, then $G_{(y)}(y_1, 0) = G_{(y')}(y_1, 0)$ so $G(y) = G(y')$ at $y_1 \times I$ and the lifting exists.

Note that because $g(y)$ lifts $F(y, 0)$, then the lifting $G$ of $y \times I$ is unique and hence $G$ is unique relative to being an extension of $g$ and lifting of $F$.

We now study the <u>circle</u> $S^1$. Think of $S^1$ as the complex numbers of norm 1. We have a homomorphism

$$\phi : R \to S^1$$



by $\phi(x) = e^{2\pi i x}$ which is open and onto $\operatorname{Ker}\phi = Z$ and

$$0 \to Z \to R \xrightarrow{\phi} S^1 \to 0$$

is a covering space, where we can take only 2 evenly covered sets, namely $\phi^{-1}(\phi(\frac{1}{2}, \frac{1}{2}))$ and $\phi^{-1}(\phi(0,1))$. We want now to see that $\pi_1(S^1, 1) = Z$. Let

$$\begin{array}{ccc} & & (R, 0) \\ & & \downarrow \phi \\ (I, (0,1)) & \xrightarrow{\sigma} & (S^1, 1) \end{array}$$

Then the path $\sigma$ can be uniquely lifted to $\sigma'$ and $\sigma'(1) \in Z$. Moreover if $\sigma \sim \overline{\sigma}$ is a homotopy, with end points fixed, then $\overline{\sigma}^1(1) = \sigma^1(1)$. We define $\theta : \Pi_1(S^1) \to Z$ by $\theta[\sigma] = \sigma'(1)$. Then $\theta$ is a homomorphism because $\sigma, \eta$ are two loops representing $[\sigma]$ and $[\eta]$, $\theta[\sigma \circ \eta]$ is obtained as follows. If $\sigma, \eta$ lift $\sigma$ and $\eta$ and $\sigma(1) = m, \eta(1) = n$ then putting $\eta''(s) = \eta'(s) + m$, gives $\phi(\eta'') = \eta$ and $\sigma'\eta''$ has initial point 0, end point $m + n$ and $\phi(\sigma'\eta'') = \sigma \circ \eta$.

$\theta$ <u>is onto</u>. Let $\sigma' : [0,1] \to R$, $\sigma'(s) = ns$, then $\sigma'(0) = 0$, $\sigma'(1) = n$, let $\sigma = \phi \circ \sigma'$, then $\theta[\sigma] = n$.

$\theta$ <u>is</u> $1-1$. Let $[\sigma], \theta[\sigma] = 0$. Hence $\sigma'$ the lift is a loop at 0. Now $R$ is <u>contractible</u>, so $\sigma'$ is homotopic to the trivial loop, hence $\phi(\sigma') = \sigma$ is also homotopic to the trivial loop.

We have thus

$$\Pi_1(S^1; 1) = Z.$$

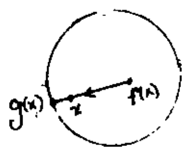
In fact, we have seen basically, suppose that $G$ is a connected topological group, $D$ a discrete subgroup, then

$$1 \to D \to G \to G/D \to 1$$

and $G \xrightarrow{p} G/D$ is a covering space and $\Pi_1(G, 1) = 1$, then $\Pi_1(G/D) \cong D$.

COROLLARY. *Brouwer fixed point theorem. Any map $f : D^2 \to D^2$ has a fixed point, namely $x \in D^2$ so that $f(x) = x$.*

PROOF: Suppose $x$ did not exist. Let $g(x)$ = point in $S^1$ obtained by extending




the vector from $f(x)$ to $x$ so as to intersect $S^1$ at $g(x)$. Note that if $x \in S^1$, $g(x) = x$. Then $g$ is continuous and we would have

$$S^1 \xrightarrow{i} D^2 \xrightarrow{g} S^1$$

$gi(x) = x$, hence $(gi)_* = id$ in $\Pi_1 S^1 \xrightarrow{i_*} \Pi_1 D^2 \xrightarrow{g_*} \Pi_1 S^1$, but $\Pi_1 D^2 = 1$.

This contradiction gives Brouwer theorem.



## §3. Singular Theory Definitions.

We will now construct an example of a functor from topological spaces to abelian groups, the singular homology groups, and prove they satisfy some properties (axioms) that makes them accessible for computation in particular cases. We considered for the homotopy groups, mappings from spheres to spaces, under homotopy relations. For singular homology, we take as domain the standard n-simplex $\Delta^n$.

Let $\Delta^n \subset R^{n+1}$ be defined as the set

$$\Delta^n = \left\{(t_0, \ldots, t_n) \mid 0 \le t_i \le 1, \sum t_i = 1\right\}$$

The n-simplex has $(n-1)$ faces,

$$\eta_i : \Delta^{n-1} \to \Delta^n \qquad i = 0, \ldots, n$$

$$\eta_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

A singular n-simplex in $X$ is a map $\sigma : \Delta^n \to X$. The faces of $\sigma$, $\partial_i \sigma : \Delta^{n-1} \to X$ are defined by $\partial_i \sigma = \sigma \circ \eta_i$. It is easy to verify that

$$\partial_i \partial_j = \partial_{j-1} \partial_i \qquad \text{for } i < j.$$

Let $R$ be a commutative ring. We let $\Delta_n(X)$ be the free R-module generated by the singular n-simplexes of $X$. It is called the R-module of n-chains in $X$. Then we define

$$\partial \sigma = \sum_{i=0}^{n} (-1)^i \partial_i \sigma$$

on generators and extend by R linearity to $\partial : \Delta_n(X) - 1\Delta_{n-1}(X)$. The above relations give $\partial\partial = 0$. We define submodules

$$B_n(X) \subset Z_n(X) \subset \Delta_n(X)$$

$$Z_n(X) = \left\{x \in \Delta_n \mid \partial x = 0\right\} : n - \underline{\text{cycles}}$$

$$B_n(X) = \left\{x \in \Delta_n(X) \mid x = \partial y\right\} : n - \underline{\text{boundaries}}$$

then $\quad \partial\partial = 0$ gives in fact that $B_n(X) \subset Z_n(X)$

and $\quad H_n(X) = Z_n(X)/B_n(X) : n - \underline{\text{homology}}$

We have now defined for any $X$, $H_n(X)$ an R-module. If $f : X \to Y$ is a map, $\sigma$ is a singular n-simplex in $X$, $f_\# \sigma : \Delta^n \to Y$ given by $f_\# \sigma = \sigma \circ f$ is an n-simplex in $Y$. Moreover $f_\# \partial_i \sigma = \partial_i f_\# \sigma$. Then $f$ induces a commutative diagram:

$$\begin{array}{ccc} \Delta_n(X) & \xrightarrow{f_\#} & \Delta_n(Y) \\ \downarrow \partial & & \downarrow \partial \\ \Delta_{n-1}(X) & \xrightarrow{f_\#} & \Delta_{n-1}(Y) \end{array}$$

Hence $f_\#$ sends cycles to cycles, boundaries to boundaries and homology to homology.

$$f_* : H_n(X) \to H_n(Y).$$

**Exercise.** Verify that $f = id : X \to X$ induces $f_* = id$ and $(f \circ g)_* = f_* \circ g_*$. Thus $H_n(X)$ <u>is a topological invariant of</u> $X$.

If we look at the notion of a chain complex, we see that it consists of a sequence $A = \{An\}_{n=0}^{\infty}$ of $R$-modules, with R-homomorphism $\partial : A_n \to A_{n-1}$ satisfying

$$\partial\partial = 0$$

Then as above, we may define $Z_n(A) = \ker(\partial : A_n \to A_{n-1})$, $B_n(A) = Im(\partial : A_{n+1} \to A_n)$ and $H_n(A) = Z_n(A)/B_n(A)$. The R-modules $B_n(A)$, $Z_n(A)$ and $H_n(A)$ are called the n-boundaries, n-cycles and n-homology of the chain complex $A$.

If $A, B$ are chain complexes, a <u>chain map</u> $f : A \to B$ is a family of maps $f : A_n \to B_n$, so that $\partial f = f \partial$ i.e.

$$\begin{array}{ccc} A_n & \xrightarrow{f} & B_n \\ \partial \downarrow & & \downarrow \partial \\ A_{n-1} & \xrightarrow{f} & B_{n-1} \end{array}$$

commutes. Then $f$ induces maps $Z_n(A) \to Z_n(B)$, $B_n(A) \to B_n(B)$ and $f_* : H_n(A) \to H_n(B)$, <u>induced map in homology</u>.

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an <u>exact sequence</u> of chain complexes, i.e., $f, g$ are chain maps and for every $n$, $0 \to A_n \to B_n \to C_n \to 0$ is an exact sequence.



Then $f, g$ induce $f_* : H_n(A) \to H_n(B)$, $g_* : H_n(B) \to H_n(C)$. We now describe $\partial_* : H_n(C) \to H_{n-1}(B)$, the <u>connecting homomorphism</u> as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \xrightarrow{f} & B_n & \xrightarrow{g} & C_n & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow{f} & B_{n-1} & \xrightarrow{g} & C_{n-1} & \longrightarrow & 0 \end{array}$$

Given $[x] \in H_n(C)$, $x \in C_n$ a cycle representing $[x]$, there exists $y \in B_n$ with $g(y) = x$, then $g(\partial y) = \partial g(y) = \partial x = 0$, so $\partial y = f(z)$, $z \in A_{n-1}$. Now $\partial z = 0$, for $f\partial z = \partial f z = \partial(\partial y) = 0$, and hence we want to define $\partial_*[x] = [z]$. In order to show it is well defined, we need to know (a) if $x = \partial x^1$, then $\partial_*[x] = 0$, (b) $\partial_*[x]$ is independent of $y$ and (c) $\partial_*$ is a homomorphism. The next result is fundamental.

**Theorem.** *If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence of chain complexes, it induces a long exact sequence in homology,*

$$\to H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\partial_*} H_{n-1}(A) \to \cdots$$

By a pair of spaces $(X, A)$, we mean a space $X$ and a subspace $A$ of $X$. A map $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ which sends $A$ into $B$. The inclusion $A \to X$ induces for every $n$ a monomorphism, $\Delta_n(A) \xrightarrow{i_\#} \Delta_n(X)$ and $i_\#$ is a chain map. We let $\Delta_n(X, A) \cong \Delta_n(X)/\Delta_n(A)$, and we call it the n-chains in $X$ mod $A$. The map $\Delta_n(X) \xrightarrow{j_\#} \Delta_n(X, A)$ induces a chain map $\partial : \Delta_n(X, A) \to \Delta_{n-1}(X, A)$ and we have an exact sequence of chain complexes

$$0 \to \Delta_*(A) \xrightarrow{i_\#} \Delta_*(X) \to \Delta_*(X, A) \to 0$$

which from the above induces a long exact sequence:

$$\to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1} \to \cdots$$

called the <u>exact sequence</u> of the pair $(X, A)$

If $f : (X, A) \to (Y, B)$, $f$ induces $f_* : H_n(X) \to H_n(Y)$, $f_* : H_n(A) \to H_n(B)$ and $\overline{f}_* : H_n(X, A) \to H_n(Y, B)$.

**Exercise.** Show that we have commutative diagram:

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{\partial_*} & H_{n-1}(A) \\ \downarrow \bar{f}_* & & \downarrow f_* \\ H_n(Y,B) & \xrightarrow{\partial_*} & H_{n-1}(B) \end{array}$$

Thus we have

THEOREM. *For every pair $(X,A)$ we have a "natural" exact sequence:*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \to \cdots$$

*so that if $f:(X,A) \to (Y,B)$ is a map of pairs, we have commutative squares*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \xrightarrow{\partial_*} & H_{n-1}(A) & \longrightarrow & \cdots \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\ \cdots & \longrightarrow & H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y,B) & \xrightarrow{\partial_*} & H_{n-1}(B) & \longrightarrow & \cdots \end{array}$$

Another important concept in the notion of chain homotopy:

Two chain maps $f,g : A \to B$ are chain homotopic if there exists a map

$$D : A_n \to B_{n+1}$$

so that $$D\partial + D\partial = f - g$$

PROPOSITION. *Chain homotopic maps induce the same map in homology.*

PROOF: If $[x] \in H_n(A)$, $x$ cycle representing $[x]$, then $f(x) - g(x) = D\partial x + \partial D = \partial D$, i.e., they differ by a boundary, thus $[f(x)] = [g(x)]$.

Two chain complexes $A$ and $B$ are called (chain) homotopically equivalent if there exist chain maps $f : A \to B$, $g : B \to A$ so that $f \circ g$ and $g \circ f$ are chain homotopic to the identity maps of $B$ and $A$ respectively. We say $f : A \to B$ or $g : B \to A$ is a chain equivalence.

PROPOSITION. *If $f : A \to B$ is a chain equivalence, then $f_* : H_n(A) \approx H_n(B)$.*



## §4. Singular homology – Homotopy invariance.

In this section we will see that $H_n(X)$ is a homotopy invariant. To do this it suffices to see that if $f$ and $g$ are homotopic maps from $X$ to $Y$, then the induced map $f_*$ and $g_*$ in homology are the same.

Suppose $F$ is a homotopy of $f$ and $g$, then we have

$$\begin{array}{ccc} X & & \\ \downarrow i_0 & & \\ X \times I & \xrightarrow{F} & Y \\ \uparrow i_1 & & \\ X & & \end{array}$$

where $i_j(x) = (x,j)$ $j = 0,1$ are the inclusions at the bottom and top. Hence $f = F \circ i_0$, $g = F \circ i_1$. Clearly, $i_0$ and $i_1$ are homotopic (id: $X \times I \to X \times I$ is the homotopy!) Thus if we can prove $i_{0*} = i_{1*} : H_*(X) \to H_*(X \times I)$, then

$$f_* = F_* i_{0*} = F_* i_{1*} = g_*.$$

It is easy to see from the definition of $H_n(X)$ that if $X$ is a one-point space, then $H_n(X) = 0$ for $n > 0, H_0(X) \cong R$. A space $X$ is called contractible if it is of the same homotopy type as a one-point space. We begin by showing

THEOREM. *If $X$ is contractible, then $H_n(X) = 0, n > 0, H_0(X) = R$.*

The space $X$ is contractible if there exists

$$F : X \times I \to X$$

with $F(x,0) = x, F(x,1) = x_0$.

If $X$ is a space, the cone of $X$, $CX$ is the quotient space of $X \times I$, where we identify $X \times 0$ to a point.

The cone of $X$ is contractible, for

$$F : CX \times I \to CX$$
$$F((x,s),t) = (x,st)$$

gives the required contraction.

Also, it is clear that there is a homeomorphism $\Delta^q \cong^{\varphi_q} C(Bdy\Delta^q)$. Choose one which fixes $Bdy(\Delta^q)$.

Given $\sigma : \Delta^q \to X$, we construct by induction a simplex $D(\sigma) : \Delta^{q+1} \to X$ as follows . $D(\sigma) : (Bdy\Delta^{q+1} \to X$ is described by

$$\partial_i D(\sigma) = D(\partial_{i-1}\sigma) \quad \text{for} \quad i \geq 1$$
$$\partial_0 D(\sigma) = \sigma$$

then we extend $D(\sigma)$ to $Bdy\Delta^{q+1} \times I \to X$ by $F \circ (D(\sigma) \times I)$ which factors through $C(Bdy\Delta^{q+1})$. Use $\varphi_{q+1}$ to define $D(\sigma) : \Delta^{q+1} \to X$. Check that (a) this is well-defined and (b) that we can start in $q = 1$, the induction.

We have thus extending linearly

$$D : \Delta_q(X) \to \Delta_{q+1}(X)$$

with
$$\partial D(\sigma) = \sum_{i=0}^{q+1} (-1)^i \partial_i D(\sigma)$$
$$= \sigma - \Sigma D(\sum_{i=0}^{q} (-1)^i \partial_i \sigma)$$
$$= \sigma - D(\partial\sigma)$$

hence
$$\partial D + D\partial = id$$

Thus if $z$ is a cycle, $\partial Dz + D\partial z = z$, but $\partial z = 0$, hence

$$\partial(Dz) = z.$$

Now we will construct inductively a "natural" chain homotopy

$$D : \Delta_n(X) \to \Delta_n(X \times I)$$

which satisfies $\partial D + D\partial = i_{1\#} - i_{0\#}$ and if $f : X \to Y$, then $f_\# D = Df_\#$. We do it on the singular simplices of $X$. Assume defined for $q < n$. Consider the canonical $n$-simplex $\xi_n : \Delta^n \to \Delta^n$, the identity map. Then $D\partial\xi_n$ is defined and



is an element of $\Delta_n(\Delta^n)$. Moreover the chain $h_n = -D\partial\xi_n + (i_{1\#}\xi_n - i_{0\#}\xi_n)$ is a cycle, for

$$\partial h_n = -\partial D\partial\xi_n + \partial(i_{1\#}\xi_n - i_{0\#}\xi_n) = +D\partial\partial\xi_n - (i_{1\#} - i_{0\#})\partial(\xi_n) + \partial(i_{1\#} - i_{0\#})(\xi_n) = 0$$

Now $\Delta^n$ is contractible, so $h_n$ must be a boundary of some chain $w, \partial w = h_n$. Define $Dh_n = w$. Then $\partial Dw = h_n$, ie.

$$(\partial D + D\partial)\xi_n = (i_{1\#} - i_{0\#})\xi_n$$

For any $n$-simplex $\sigma : \Delta^n \to X$, define $D\sigma = \sigma_\# D(\xi_n)$ Then $D$ will be natural and a chain homotopy between $i_{0\#}$ and $i_{1\#}$. Finally to start the induction if $\sigma_0 : \Delta^0 \to X$ is a o-simplex with $\sigma_0(1) = x_0$ then $D(\sigma_0) : \Delta^1 \to X \times I$ is given by $D(\sigma_0)(t) = (x_0, t)$.

We thus have:

THEOREM. *If $f$ and $g$ are homotopic maps from $X$ to $Y$, then $f_* = g_* : H_n X \to H_n Y$ for all $n$.*

**Corollary.** $H_n(X)$ is a homotopy invariant of $X$.

If $f, g : (X, A) \to (Y, B)$ are two maps, we say they are homotopic if there exists $F : X \times I \to Y$ a homotopy from $f$ to $g$ that sends $A \times I$ to $B$. We define the notion of homotopically equivalent pairs analogously.

THEOREM. *$H_n(X, A)$ is a homotopy invariant of the pair $(X, A)$.*

### §5. Singular Homology: Mayer-Vietoris Sequences.

Let $\mathfrak{A} = \{A_\alpha\}$ be a family of subspaces of $X$ so that

$$X = \cup \text{Int } A_\alpha .$$

We say that $\mathfrak{A}$ is a *cover* of $X$. We let $\Delta_n(\mathfrak{A})$ denote the submodule of $\Delta_n(X)$ generated by those simplices $\sigma : \Delta^n \to X$ so that $\sigma(\Delta^n) \subset A_\alpha$ for some $\alpha$. Then $\{\Delta_n(\mathfrak{A})\}$ is a subchain complex of $\{\Delta_n(X)\}$. We will prove:

THEOREM. *If $\mathfrak{A} = \{A_\alpha\}$ is a cover of $X$, the inclusion $\Delta_*(\mathfrak{A}) \xrightarrow{i_\#} \Delta_*(X)$ is a chain equivalence. In particular $i_* : H_n(\mathfrak{A}) \cong H_n(X)$ for all $n$.*

This theorem has as consequence the

THEOREM (MAYER-VIETORIS). *If $\{A, B\}$ is a cover of $X$, we have an exact sequence:*

$$\cdots \to H_n(A \cap B) \xrightarrow{j_*} H_n(A) \oplus H_n(B) \xrightarrow{k_*} H_n(X) \xrightarrow{\partial_*} H_{n-1}(A \cap B) \ldots$$

*where $j_* = (j_{1*}, -j_{2*})$ and $k_*$ are induced by the inclusions $A \cap B \subset A$, $A \cap B \subset B$, $A \subset X$, $B \subset X$ respectively.*

PROOF: Consider the sequence:

$$0 \to \Delta_n(A \cap B) \xrightarrow{j_\#} \Delta_n(A) \oplus \Delta_n(B) \xrightarrow{k_\#} \Delta_n(\{A, B\}) \to 0$$

where $j_\#(c) = (j_{1\#}c, -j_{2\#}c)$, $k_\#(\alpha, \beta) = k_{1\#}\alpha + k_{2\#}\beta$. Then it is an exact sequence and defines an exact sequence of chain complexes;

$$0 \to \Delta_*(A \cap B) \to \Delta_*(A) \oplus \Delta_*(B) \to \Delta_*(\{A, B\}) \to 0$$

which induces a long exact sequence, in which we may replace $H_n(\{A, B\})$ by $H_n(X)$ using the isomorphism $i_*$, giving the above exact sequence.

The Mayer-Vietoris sequence gives the excision property of singular homology.

THEOREM (EXCISION). *If $U$ is a subspace of $X$ so that $\overline{U} \subset$ Int $A$ then the inclusion $(X - U, A - U) \to (X, A)$ induces isomorphisms $H_n(X - U, A - U) \xrightarrow{\cong} H_n(X, A)$ for all $n$.*

PROOF: We look at the pair $\{A, X - U\}$. It is a cover of $X$. Then $\Delta_n(\{A, X - U\}) = \Delta_n(A) + \Delta_n(X - U)$ and

$$\begin{aligned} \Delta_n(A) + \Delta_n(X - U)/\Delta_n(A) &\approx \Delta_n(X - U)/\Delta_n(A) \cap \Delta_n(X - U) \\ &\approx \Delta_n(X - U)/\Delta_n(A - U) \end{aligned}$$

Thus we have $\Delta_*(X - U, A - U) \approx \Delta_*(\{A, X - U\})/\Delta_*(A)$. Now from

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta_*(A) & \longrightarrow & \Delta_*(X) & \longrightarrow & \Delta_*(X, A) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Delta_*(A) & \longrightarrow & \Delta_*(\{(\Delta, XU\}) & \longrightarrow & \Delta_*(\{A, X - U\})/\Delta_*(A) & \longrightarrow & 0 \end{array}$$

we obtain exact sequences

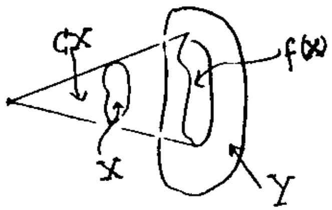
$$\begin{array}{ccccccccccc} \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) \\ & \uparrow\cong & & \uparrow\cong & & \theta\uparrow & & \cong\uparrow & & \cong\uparrow \\ \longrightarrow & H_n(A) & \longrightarrow & H_n(\{A, X - U\}) & \longrightarrow & H_n(\Delta_*\{A, X - U\})/\Delta_*(A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) \end{array}$$

and from the "five lemma," $\theta$ is an isomorphism. Thus $H_*(X - U, A - U) \simeq H_*(\Delta_*(\{A, X - U\})/\Delta_*(A)) \approx H_*(X, A)$ gives the result.

We make some applications.

Given spaces $X$ and $Y$, $f : X \to Y$ a map, then we can form $C_f$ the cone of $f$, obtained by glueing $CX$ with $Y$ via $f$, i.e. $CX = X \times I/\{(x, 0) \sim *\}$. Then

$$C_f = CX \cup Y/\{(x,1) \sim f(x)\}\,.$$




Recall that $CX$ is contractible, so $H_q(CX) = 0$ if $q > 0$. We let $C_f = C_+X \cup Y_+$, where $C_+(X) = X \times [0,3/4]/(x,0) \sim *$ and $Y_+ = Y \cup (X \times [1/2,1])/\{(x,1) \sim f(x)\}$ then $\{C_+X, Y_+\}$ is a cover of $C_f$, $C_+X \cap Y_+ = X \times [1/2, \frac{3}{4}]$ is of the same homotopy type as $X$. $C_+X$ is contractible and $Y_+$ is of the same homotopy type as $Y$ (prove it!)

We have the Mayer-Vietoris sequence:

$$\begin{array}{ccccccc} \longrightarrow H_n(C_+X \cap Y_+) & \longrightarrow & H_n(C_+X) \oplus H_n(Y_+) & \longrightarrow & H_n(C_f) \longrightarrow \cdots \\ \| & & \| & & \| \\ \cdots \longrightarrow H_n(X) & \xrightarrow{f_*} & H_n(Y) & \longrightarrow & H_n(C_f) \longrightarrow \cdots \end{array}$$

If we take $Y = CX$ and $X \xrightarrow{f} CX$ the inclusion, $C_f$ is called the suspension of $X$, denoted by $\Sigma X$ and the above sequence reduces to

$$\begin{array}{ccccccccc} \longrightarrow H_n(X) & \xrightarrow{f_*} & H_n(CX) & \longrightarrow & H_n(\Sigma X) & \xrightarrow{\partial_*} & H_{n-1}(X) & \longrightarrow & H_{n-1}(CX) \\ & & \| & & & & & & \| \\ & & 0 & & & & & & 0 \end{array}$$

and we obtain: there exists an isomorphism $\partial_* : H_n(\Sigma X) \cong H_{n-1}(X)$ for $n > 1$ and for $n = 1$, $H_1(\Sigma X) \simeq \tilde{H}_0(X)$, where $\tilde{H}_0(X)$ is the reduced homology of $X$, obtained by considering $\tilde{\Delta}_n(X) = \Delta_n(X)$ for $n > 0$ and $\tilde{\Delta}_0(X) = \ker\{\Delta_0(X) \xrightarrow{\alpha_0} R\}$, $\alpha_0(\sigma) = 1$ for 0-simplices $\sigma$.



The spheres $S^n$ are related to each other by suspension, $\Sigma S^n \approx S^{n+1}$. Now $S^0$ consist of two points. Thus $H_0(S^0) = R + R$, $H_q(S^0) = 0$ $q > 0$. Hence $\overline{H}_0(S^0) = R$, and $\overline{H}_q(S^1) = \begin{cases} 0 & q \neq 1 \\ R & q = 1\,. \end{cases}$

**THEOREM.** *The sphere $S^n$ has homology given by*

$$H_q(S^n) = \begin{cases} R & q = 0, n \\ 0 & q \neq 0, n\,. \end{cases}$$

**COROLLARY (BROUWER).** *Every map $f : D^{n+1} \to D^{n+1}$ has a fixed point.*

Let us see how to prove that $C_*(\mathfrak{A})$ is chain equivalent to $C_*(X)$. We use the notion of subdivision. We say that a simplex $\sigma : \Delta^q \to \Delta^n$ is *linear* if $\sigma(\Sigma t_i V_i) = \Sigma t_i \sigma(V_i)$. Here we are using the fact that if $v_i = (0, \ldots, \hat{1}, \ldots, 0)$, the points of $\Delta^q$ are uniquely expressible as $\Sigma t_i v_i$, $0 \leq t_i \leq 1$, $\Sigma t_i = 1$. If $\sigma$ is linear it only depends on $\{\sigma(v_i)\}$. We will denote it by $(x_0, \ldots, x_q)$ where $\sigma(v_i) = x_i$. Clearly if $\sigma = (x_0, \ldots, x_q)$, $\partial_i \sigma = (x_0, \ldots, \hat{x}_i, \ldots, x_q)$ and thus the linear simplexes generate a subchain complex of $\Delta_*(\Delta^n)$ $\{L_q(\Delta^n)\}_{q=0}^{\infty}$. Let $b_n$ be the barycenter of $\Delta^n$, i.e., $b_n = \sum_{i=0}^{n+1} \frac{v_i}{n+1}$. We use $b_n$ to describe a contraction of $L_*(\Delta^n)$. Define

$$\begin{array}{rl} & \beta_n : L_q(\Delta^n) \to L_{q+1}(\Delta^n) \\ \text{by} & \beta_n(x_0, \ldots, x_q) = (b_n, x_0, \ldots, x_q) \\ \text{then} & \partial\beta_n(x_0, \ldots, x_q) = (x_0, \ldots, x_q) + \Sigma(-1)^{i+1}(b_n, x_0, \ldots, \hat{x}_i, \ldots, x_q) \\ \text{while} & \beta_n(\partial(x_0, \ldots, x_q)) = \Sigma(-1)^i (b_n, x_0, \ldots, \hat{x}_i, \ldots, x_q) \\ \text{hence} & \partial\beta_n + \beta_n\partial = id\,. \end{array}$$

for $L_*(\Delta^n)$.

We use $\beta_n$ to construct a natural chain map

$$Sd : \Delta_*(X) \to \Delta_*(X)$$

and a natural chain homotopy

$$D : \Delta_q(X) \to \Delta_{q+1}(X)$$

with $\partial D + D\partial = Sd - id$ by induction, we use again $\xi_n \in L_n(\Delta^n)$ and set

$$Sd\xi_n = \beta_n(Sd\partial\xi_n)$$

Then one verifies $\partial Sd\xi_n = Sd\partial\xi_n$. Also one sets

$$D\xi_n = \beta_n(D\partial\xi_n - Sd\xi_n + \xi_n)$$

then one verifies $(\partial D + D\partial)(\xi_n) = Sd\xi_n - \xi_n$ so if $\sigma : \Delta^n \to X$ is an $n$-simplex, defining

$$Sd\sigma = \sigma_\#(Sd\xi_n)$$
$$D\sigma = \sigma_\#(D\xi_n)$$

one obtains $Sd$ and $D$ satisfying the above conditions. The importance of $Sd$ on a metric space is that it reduces the diameter of the simplices. More precisely if $X$ is a metric space $c$ is a singular $n$-chain in $X$, let mesh $(c) = \max(\text{diam}\ \sigma_i | c = \Sigma a_i\sigma_i, a_i \neq 0\}$. If we take $X = \Delta^n$, we obtain for $c$ a $q$-chain in $\Delta^n$,

$$\text{mesh}\ (Sdc) \leq \frac{a}{a+1}\ \text{mesh}\ (c)\,.$$

We are now ready to prove the main theorem of this section. Let $\mathfrak{A} = \{A_\alpha\}$ be a cover of $X$ and $\sigma : \Delta^q \to X$ be a $q$-simplex. $\sigma^{-1}\mathfrak{A} = \{\sigma^{-1}A_\alpha\}$ is a cover of $\Delta^q$. Let $\lambda$ be the Lebesgue number of this cover. Then there exists an $m$ so that $Sd^m\xi_q \in \Delta_*(\sigma^{-1}\mathfrak{A})$ namely so that $\left(\frac{q}{q+1}\right) m < \lambda$. Then $Sd^m\sigma \in \Delta_*(\mathfrak{A})$.

For any $\sigma \in \Delta^q \to X$ let $m(\sigma)$ be the minimal $m$ so that $sd^m\sigma \in \Delta_*(\mathfrak{A})$. Define

$$\theta\sigma = \sum_{i=0}^{m(\sigma)-1} Sd^i\sigma\,.$$



Define

$$\overline{D} : \Delta_q(X) \to \Delta_{q+1}(X)$$

by

$$\overline{D}\sigma = D\theta\sigma\,.$$

Now

$$\overline{D}(\sigma) = 0 \text{ iff } |\sigma \in \Delta_*(\mathfrak{A})\,.$$

Now

$$\begin{aligned}\partial\overline{D}\sigma = \partial D(\theta\sigma) &= -D\partial(\theta\sigma) + Sd\theta\sigma - \theta\sigma \\ &= -D\partial\theta\sigma + \sum_{i=0}^{m(\sigma)-1} Sd^{i+1}\sigma - \sum_{i=0}^{m(\sigma)} Sd^i\sigma \\ &= -D\partial\theta\sigma + Sd^{m(\sigma)}\sigma - \sigma \\ &= -D\left(\sum^{m(\sigma)-1} (-1)^j Sd^i\partial_j\sigma\right) + Sd^{m(\sigma)} - \sigma\,.\end{aligned}$$

Now

$$\begin{aligned}\overline{D}\partial\sigma &= D\theta(\Sigma(-1)^j\partial_j\sigma) \\ &= D\left(\sum_{i=0}^{m(\partial_j\sigma)-1} (-1)^j Sd^i\partial_j\sigma\right)\end{aligned}$$

$$\therefore\quad \partial\overline{D}\sigma + \overline{D}\partial\sigma = \left(Sd^{m(\sigma)}\sigma - \Sigma\sum_{i=m(\partial_j\sigma)}^{m(\sigma)-1}(-1)^j Sd^i\partial_j\sigma\right) - \sigma\,.$$

Notice that the chain inside ( ) is already in $\Delta_*(\mathfrak{A})$. Define

$$\tau\sigma = Sd^{m(\sigma)}\sigma - \Sigma\sum_{i=m(\partial_j\sigma)}^{m(\sigma)-1} (-1)^j Sd^i\partial_j\sigma\,.$$

Then

$$\partial\overline{D} = \overline{D}\partial = \tau - id$$

and clearly if $\sigma \in \Delta_*(\mathfrak{A})$, $(\tau i)(\sigma) = \sigma$. Thus $\overline{D}$ is a homotopy between $i \cdot \tau$ and $id$ of $\Delta_*(X)$ and $\tau i = id$ on $\Delta_*(\mathfrak{A})$. Hence $i* : H_n(\mathfrak{A}) \cong H_n(X)$ for all $n$.

## §6. $CW$-complexes.

We say that $X$ is obtained from $Y$ by attaching an $n$-cell if there exists a map $f : S^{n-1} \to Y$ so that $X = C_f = Y \bigcup_f e^n$. More generally $X$ is said to be obtained from $Y$ by attaching $n$-cells if

(a) $X = YU(Ue^n_\alpha)$ where $e^n_\alpha$ are subsets of $X$. If $\dot{e}^n_\alpha = e^n_\alpha \cap Y$, then

(b) $X - Y = \coprod e^n_\alpha - \dot{e}^n_\alpha$

(c) For every $\alpha$, there exists a map

$$f_\alpha : (D^n, S^{n-1}) - 1(e^n_\alpha, \dot{e}^n_\alpha)$$

so that $f_\alpha(D^n) = e^n_\alpha$, $f_\alpha : D^n - S^{n-1} \to e^n_\alpha - \dot{e}^n_\alpha$ is a homeomorphism and

(d) The topology of $e^n_\alpha$ is coinduced from $f_\alpha$ and the inclusion $\dot{e}^n_\alpha \xrightarrow{i_\alpha} e^n_\alpha$, i.e., $f : e^n_\alpha \to Z$ continuous iff $ff^{-1}_\alpha$ and $fi_\alpha$ continuous

(e) The topology of $X$ is coherent with the subspaces $\{Y, e^n_\alpha\}$, i.e. $W$ is closed in $X$ iff $W \cap Y$ and $W \cap e^n_\alpha$ closed for all $\alpha$.

Notice that $f_{\alpha *} : H_q(D^n, S^{n-1}) \cong H_q(e^n_\alpha, \dot{e}^n_\alpha)$ and excision, gives $H_q(X, Y) \approx \oplus H_q(e^n_\alpha, \dot{e}^n_\alpha)$ hence

$$H_q(X,Y) \cong \begin{cases} \bigoplus_\alpha R & q = n \\ 0 & \text{otherwise} \end{cases}$$

A $CW$-complex $X$ is a space that has a sequence of closed subspaces

$$X_0 \subset X_1 \subset \cdots X_n \subset \cdots$$

so that

(a) $X_0$ is a discrete set of points

(b) $X_n$ is obtained from $X_{n-1}$ by attaching $n$-cells

(c) $X = \cup X_n$

(d) The topology of $X$ is coherent with the $\{X_n\}$.

We have from the above that $H_q(X_{n+1}, X_n) = \begin{cases} \bigoplus_{\alpha \in J_n} R & q = n \\ 0 & \text{otherwise} \end{cases}$. Hence $H_q(X_{n-1}) \to H_q(X_n)$ is an isomorphism for $q \neq n$, $n+1$ and

$$0 \to H_{n+1}(X_{n+1}) \to H_{n+1}(X_{n+1}, X_n) \to H_n(X_n) \to H_n(X_{n+1}) \to 0$$

It thus follows that

$$H_n(X_n) \overset{\text{onto}}{\to} H_n(X_{n+1}) \xrightarrow{\cong} H_n(X_{n+1} \xrightarrow{\approx} \cdots \xrightarrow{\approx} H_n(X_{n+1}) \xrightarrow{\cong} \cdots$$

Now given an $n$-simplex $\sigma : \Delta^n \to X$, $\sigma(\Delta^n)$ is compact and the open cells $\{e^m_\alpha - \dot{e}^m_\alpha\}_{\alpha,n}$ cover $X$. If $\sigma(\Delta^n)$ were not contained in $X_m$ for some $m$, we would contradict the compactness of $\sigma(\Delta^n)$. Hence any chain $c \in \Delta_n(X)$ and any homology class $x \in H_n(X)$ lies in the image of a class in $H_n(X_m)$. Also if $y \in H_n(X_m)$ goes to zero in $H_n(X)$ it goes to zero in $H_n(X_{m'})$, for some $m'$. But then we obtain $H_n(X_{n+1}) \cong H_n(X)$.

Define

$$C_n(X) = H_n(X_n, X_{n-1}) \qquad \text{and}$$
$$d : C_n(X) \to C_{n-1}(X)$$
$$\text{by} \qquad H_n(X_n, X_{n-1}) \xrightarrow{\partial_*} H_{n-1}(X_{n-1}) \xrightarrow{j_*} H_{n-1}(X_{n-1}, X_{n-2})$$
$$\text{then} \qquad d^2 = 0$$

and we can define $\hat{H}_n(X)$ to be the $n^{th}$ homology group of $\dot{C}_*$ with respect to $d$.

**Theorem.** *We have isomorphisms* $H_n(X) \overset{\theta}{\cong} \hat{H}_n(X)$

**Proof:** To define $\theta$, take $x \in H_n(X)$, there exists $x' \in H_n(X_n)$ with $\bar{j}_* x' = x$. Take $j^* x' \in H_n(X_n, X_{n-1}) = C_n(X)$. Then $d(j^* x') = j_* \partial_* j_* x' = 0$, so we can take $[j_* x'] \in \hat{H}_n(X)$. If $x''$ is another class, $\bar{j}_* x'' = x$, then there exists $y \in H_{n+1}(X_{n+1}, X_n)$ $\partial_*(y) = x'' - x'$, then $j_* x'' - j_* x' = j_* \partial_*(y) = dy$, so $[j_* x'] = [j_* x'']$ and $\theta$ is well defined. It is additive. It is $1-1$ for if $\theta(X) = 0$ then $j_* x' = dw = j_* \partial_* w$, $w \in H_{n+1}(X_{n+1}, X_n)$ but $j_*$ is a momomorphism, hence $x' = \partial_* w$ and $x = \bar{j}_* \partial_* w = 0$. It is onto, for given $[z] \in \hat{H}_n(X)$, $z$ a representative cycle, $dz = j_* \partial z = 0$ implies $\partial_* z = 0$, so there exists u $\in H_n(X_n)$, $j_* u = z$ and we take $\bar{j}_* u \in H_n(X)$. Then $\theta(\bar{j}_* u) = [z]$. We want now to describe $d : C_n(X) \to C_{n-1}(X)$

$$H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$$

on a cell $e^n_\alpha \in H_n(X_n, X_{n-1})$. We have

$$\begin{array}{ccccc} S^{n-1} & \xrightarrow{f_\alpha} & X_{n-1} & \longrightarrow & X_{n-1} \cup e^n \\ {\scriptstyle id}\downarrow & & \downarrow & & \downarrow \\ S^{n-1} & \xrightarrow{f_\alpha} & X_{n-1}/X_{n-2} & \longrightarrow & X_{n-1}/X_{n-2} \cup e^n \end{array}$$

and

$$\begin{array}{ccccc}
H_n(X_n, X_{n-1} & \longrightarrow & H_{n-1}(X_{n-1}) & \longrightarrow & H_{n-1}(X_{n-1}, X_{n-2}) \\
\uparrow 1-1 & & \| & & \| \\
H_n(X_{n-1}\cup e^n, X_{n-1}) & \longrightarrow & H_{n-1}(X_{n-1}) & \longrightarrow & H_{n-1}(X_{n-1}, X_{n-2}) \\
\downarrow \cong & & \downarrow & \swarrow \cong & \\
H_n(X_{n-1}/X_{n-2}\cup e^n, X_{n-1}/X_{n-2}) & \xrightarrow{\partial_*} & H_{n-1}(X_{n-1}/X_{n-2}) & & \\
\uparrow \cong & \nearrow & & & \\
H_{n-1}(S^{n-1}) & & & &
\end{array}$$

All things are commutative. Hence we have

$$de^n_\alpha = \sum a^\alpha_\beta e^{n-1}_\beta, \text{ where } a^\alpha_\beta = \deg\left((f^\alpha_\beta)_*\right)$$

where $f^\alpha_\beta$ is the composite:

$$S^{n-1} \xrightarrow{f_\alpha} X^{n-1}/X^{n-2} \cong \vee S^{n-1} \underset{p_\beta}{\rightarrow} S^{n-1}$$

and $\deg(f^\alpha_\beta)_*$ is the unique integer $k$ so that

$$(f^\alpha_\beta)_*(1) = k \text{ in } H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1}).$$

We say $X$ is a finite $CW$-complex if it has a finite number of cells. If $X_0 = X_1, X_2 = X_3, \cdots, X_n = X_{n+1}, \cdots$ then

$$H_n(X) \cong H_n(X_n, X_{n-1})$$

For example if $X = S^n \cup e^{n+k}$, $k > 1$,

$$\overline{H}_q(X) = \begin{cases} Z & \text{if } k = n, n+k \\ 0 & \text{otherwise} \end{cases}.$$



## §7 Examples.

We begin by showing that we can obtain maps of arbitrary degree $k$, $f_k : S^n \to S^n$. We begin with a map

$$\alpha : S^1 \to S^1 \vee S^1$$

$$\alpha[t] = \begin{cases} ([2t],[0]) & 0 \le t \le \frac{1}{2} \\ ([0],[2t-1]) & \frac{1}{2}t \le 1 \end{cases}$$

where $S^1 = I/\{0,1\}$ and $I \to S^1 \quad t \to [t]$. Let $\rho_j$ be the projections onto the $j^{th}$ factor, $j = 1,2$. Then $\rho_1\alpha$ and $\rho_2\alpha$ are homotopic to the identity. Also if $fo : S^1 \vee S^1 \to S^1$ is given by

$$fo\begin{cases} ([t],[0]) & = [t] \\ ([0],[t]) & = [t] \end{cases}$$

and $i_j$ are the inclusion $S^1 \to S^1 \vee S^1 \quad j = 1,2$, then the $fo \circ ij$ are homotopic to the identity. It follows that if $u \in H_1(S^1)$ is a generator, $u_1 = (i_1)_*u$, $u_2 = (i_2)_*u$, then $\alpha_*(u) = u_1 + u_2$ and $fo_*u_1 = fo_*u_2 = u$. Hence

$$S^1 \xrightarrow{\alpha} S^1 \vee S^1 \xrightarrow{fo} S^1$$

satisfies $fo_*\alpha_*(u) = 2u$.

Suppose $f_k$ and $f_\ell$ are maps of degree $k, \ell$ respectively. Then

$$S^1 \xrightarrow{\alpha} S^1 \vee \xrightarrow{f_k \vee f_\ell} S^1 \vee S^1 \xrightarrow{fo} S^1$$

has the property that: $fo_*(f_k \vee f_\ell)_*\alpha_*u = (k+\ell)u$. This way, from the above, we get maps of degree $k \ge 0$. We now construct a map of degree $-1$. We use the usual representation of $S^n$. Consider the map $\theta(x_0, \dots, x_n) = (-x_0, x_1, \dots, x_n)$ where $S^n = \{(x_0,\dots,x_n) \in R^{n+1} | x_0^2 + \cdots + x_n^2 = 1\}$. For $S^0 = \{-1, 1\}, \theta(-1) = 1, \theta(1) = -1$. Consider the induced map in $\tilde{H}_0(S^0) \xrightarrow{\theta_*} \tilde{H}_0(S^0)$. We have $\tilde{\Delta}_0(S^0) \cong Z$ with generator $v_0 = \{1\} - \{0\}$. Then $\theta_\#(\{1\} - \{0\}) = \{0\} - \{1\} = -v$ i.e., $\theta_\#(v) = -v$. Hence $\theta_*(1) = -1$. Now, use the fact that the inclusion $S^n \to S^{n+1}$

$$(x_0, \cdots, x_n) \to (x_0, \cdots, x_{n+1})$$

commutes with $\theta$, hence using the suspension homorphism

$$\begin{array}{ccc} H_q(S^n) & \xrightarrow{\theta_*} & H_q(S^n) \\ \downarrow \cong & & \downarrow \cong \\ \overline{H}_{q-1}(S^{n-1}) & \xrightarrow{\theta_*} & H_{q-1}(S^n) \end{array}$$

we obtain:

$$\theta_* : H_n(S^n) \to H_n(S^n)$$

is of degree $-1$. In particular for $n = 1$, we obtain a map of degree $f_{-1} : S^1 \to S^1$. Its suspension is a map $f_{-1} : S^n \to S^n$ and this way by suspending we get maps of every degree. The antipodal map $a : S^n \to S^n$ is defined by $a(x_0, \cdots, x_n) = (-x_0, -x_1, \cdots, -x_n)$. Prove by induction that

COROLLARY. *The antipodal map* $a : S^n \to S^n$ *has degree* $(-1)^{n+1}$.

**The homology of projective spaces.** Let $F$ be the field of reals, complexes or quaternions. We let $F^*$ be the non- zero elements. In $F^{n+1} - 0$, $F^*$ acts diagonally $\alpha \bullet (a_0, \cdots, a_n) = (\alpha a_0, \cdots, \alpha a_n)$, and define:

$$FP^n = F^{n+1} - 0/F^*.$$

We denote $[a_0, \cdots, a_n]$ the equivalence class. We have natural embeddings $FP^n \subset FP^{n+1}$.

$$[a_0, \cdots, a_n] \to [a_0, \cdots, a_n; 0].$$

We now show that $FP^{n+1} - FP^n$ is homeomorphic to $F^{n+1}$ and use this to describe a cellular decomposition

$$* \subset FP^1 \subset FP^2 \subset \cdots \subset FP^n$$

where we attach one cell at a time

$$FP^{n+1} - FP^n = \{[a_0, \cdots, a_n, a_{n+1}] \text{ with } a_{n+1} \neq 0\}.$$

Dividing by $a_{n+1}$, we get $\left(\frac{a_0}{a_{n+1}}, \cdots, \frac{a_n}{a_{n+1}}, 1\right)$ is a unique representative of $[a_0, \cdots, a_n, a_{n+1}]$. The map

$$F^{n+1} \to FP^{n+1} - FP^n$$

is

$$(b_0, \cdots, b_n) \to [b_0, \cdots, b_n, 1].$$



Define

$$(D^{d(n+1)}, S^{d(n+1)}) \to (FP^{n+1}, FP^n)$$
$$b = (b_0, \cdots, b_n) \to [b_0, \cdots, b_n, 1 - \|b\|]$$

Now if $F = R$, $d = 1$, $F = C$, $d = 2$, $F = H$, $d = 4$, hence

$$RP^n = S^1 \cup e^2 \cup \cdots \cup e^n$$
$$CP^n = S^2 \cup e^4 \cup \cdots \cup e^{2n}$$
$$HP^n = S^4 \cup e^8 \cup \cdots \cup e^{4n}$$

Immediately, we see that

$$H_q(CP^n) = \begin{cases} Z & 0 \le q \le 2n, \quad q \text{ even} \\ 0 & \text{otherwise} \end{cases}.$$

$$H_q(QP^n) = \begin{cases} Z & 0 \le q \le 4n \quad q = 4k \\ 0 & \text{otherwise} \end{cases}.$$

We construct a $CW$-decomposition of $S^n$ which is compatible with the antipodal map, as follows. Let $S^k = \{(x_0, \cdots, x_k) | \Sigma x_i^2 = 1\}$ and $C_+^k = \{(x_0, \cdots, x_k) | x_k \ge 0\}$, $C_-^k = \{(x_0, \cdots, x_k) | x_k \le 0\}$ then if $\tau : S^n \to S^n$ is the antipodal map $\tau(x_0, \cdots, x_n) = (-x_0, \cdots, -x_n)$, we have $\tau C_\pm^k = C_\mp^k$. Thus we get a cell decomposition

$$S^0 \subset C_+^1 \cup C_-^1 \subset C_+^2 \cup C_-^2 \subset \cdots \subset C_+^n \cup C_-^n = S^n.$$

We compute the boundary in the chain complex as follows. We have $C_k(S^n)$ is free on two cells, $e_k$ and $\tau e_k$. Moreover, $\partial$ commutes with $\tau$. We claim:

$$\partial e_k = e_{k-1} + (-1)^k \tau e_{k-1} \qquad k = 1, \ldots, n$$

and $\partial(e_n + (-1)^{n-1} \tau e_n) = 0$.

Verify it by induction. Now $\Pi : S^n \to RP^n$ and $\overline{e}^k = \Pi e^k$ represents a cell decomposition of $RP^n$

$$e^0 \cup \overline{e}^1 \cup \cdots \cup \overline{e}^n = RP^n$$

consequently if $\overline{e}_k$ represents a generator of $C_k(RP^n)$ corresponding to $\overline{e}^k$, we have $\overline{\partial} \overline{e}_k = \overline{e}_{k-1} + (-1)^k \overline{e}_{k-1}$ which gives

$$\overline{\partial} \overline{e}_k = 2\overline{e}_{k-1}$$

and

$$\overline{\partial} \overline{e}_{2k+1} = 0.$$

Thus

$$\overline{H}_q(RP^{2n+1}) = \begin{cases} Z & q = 2n+1 \\ Z_2 & q = \text{ odd } \& < 2n+1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\overline{H}_q(RP^{2n}) = \begin{cases} Z_2 & 0 < q < 2n \quad \text{and } q \text{ odd} \\ 0 & \text{otherwise} \end{cases}.$$

**Euler characteristic in a finite $CW$-complex.**

Let $X$ be a finite $CW$-complex and let $n_k(X)$ be the number of $k$-cells in $X$. Suppose $\dim X = m$. Form

$$\chi(X) = \Sigma(-1)^k n_k(X)$$

on the other hand $H_q(X)$ is finitely generated abelian group. Let $b_q(X) =$ number of free cyclic groups in the decomposition of $H_q(X)$. Let

$$\chi'(X) = \Sigma(-1)^k b_k(X)$$

THEOREM. $\chi(X) = \chi'(X)$

PROOF: We have using cellular chains exact sequences of finite abelian groups,

$$0 \to Z_k(X) \to C_k(X) \to B_{k-1}(X) \to 0$$

and

$$0 \to B_k(X) \to Z_k(X) \to H_k(X) \to 0.$$

The first one is free. Thus $rk(C_k(X)) = rk(Z_k(X) + rk(B_{k-1}(X))$ and from (2), $rk(Z_k(X)) = rk(B_k(X)) + rk(H_k(X))$. Hence $n_k(X) = rk(Z_k(X)) + rk(B_{k-1}(X)); b_k(X) = rk(Z_k(X)) - rk(B_k(X))$ and the result follows from

$$\Sigma(-1)^k(n_k(X) - b_k(X)) = \Sigma(-1)^k(rk(B_{k-1}(X) + rk(B_k(X)) \\ = 0$$



COROLLARY. *$\chi(X)$ does not depend on the $CW$-decomposition*

$$\chi(X^n) = 1 + (-1)^n$$
$$X(CP^n) = n+1$$
$$\chi(RP^n)\begin{cases} 0 & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even} \end{cases}$$
$$\chi(X^n \times X) = (1 + (-1)^n)\chi(X) = \chi(S^n)\chi(X)$$

## §8 Cohomology.

Given a chain complex $A = \{A_n\}$ of $R$-modules and $G$ another $R$-module, we let $C^n(A;G) = \mathrm{Hom}_R(A_n, G)$. Then $d : A_{n+1} \to A_n$ induces $C^n(A;G) \xrightarrow{\delta} C^{n+1}(A;G)$, with $\delta\delta = 0$. Again we let $Z^n(A;G) = \mathrm{Ker}\,\delta : (C^n(A;G) \to C^{n+1}(A;G))$ and $B^n(A;G) = \mathrm{Im}\,\delta(C^{n-1}(A;G) \to C^n(A;G)$. Also

$$H^n(A;G) = Z^n(A;G)/B^n(A;G)$$

$\{C^n(A;G)\}$ is called a cochain complex, $\delta$ the coboundary, $Z^n(A;G)$ the $n$-cocycles of $A$ with coefficient in $G$, $B^n(A;G)$ the $n$-coboundaries of $A$ with coefficients in $G$, and $H^n(A;G)$ the $n^{th}$-cohomology of $A$ with coefficients in $G$. If $f : A \to A'$ is a chain map, it induces $f^\# : C^*(A';G) \to C^*(A;G)$ a cochain map, i.e. $\delta f^\# = f^\# \delta$, and hence an induced map in cohomology $f^* : H^n(A';G) \to H^n(A;G)$. In particular, if $X$ is a space, $\Delta_*(X)$ its singular chain complex with coefficients in $R$, $\Delta^*(X;G)$ will be the associated singular cochain complex of $X$ with coefficients in $G$, and $H^n(X;G)$ will be the $n^{th}$ cohomology group of $X$ with coefficients in $G$.

Again $H^n(X;G)$ is a homotopy invariant of $X$, in fact one may follow the proofs for homology to obtain:

(1) Naturality: If $f : X \to Y$, it induces

$$f^* : H^n(Y;G) \to H^n(X;G)$$

so that

a) if $f : X \to X$ is the identity, then $f^* =$ identity.

b) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a composition, then $(g \circ f)^* = f^* \circ g^*$.

(2) If $(X, A)$ is a pair, we have a natural long exact sequence

$$\cdots \to H^q(X,A) \to H^q(X) \to H^q(A) \xrightarrow{f^*} H^{q+1}(X,A) \to \cdots$$

(3) If $f$ and $g$ are homotopic maps from $X$ to $Y$, then $f^* = g^*$

(4) If $X$ is a one point space, then

$$H^q(X;G) = \begin{cases} G & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

(5) If $(X, A)$ is a pair and $U \subset \overline{U} \subset$ Int $A$, then

$$H^q(X, A) \cong H^q(X - U, A - U).$$

Again if $X$ is a $CW$-complex, we can define when $G = R$, $C^n(X) = H^n(X^n, X^{n-1})$ and $\delta : C^{n-1}(X) \to C^n(X)$ as the composition

$$H^{n-1}(X^{n-1}, X^{-2}) \to H^{n-1}(X^{n-1}) \xrightarrow{\delta^*} H^n(X^n, X^{n-1})$$

and one proves that $H^n(X)$ is isomorphic to the cohomology of $C^*(X)$.

We have thus a family of groups, $H^n(X)$ which is not so different than that of $H_n(X)$. However, we now show that in the cohomology we can introduce a product, making it into an algebra. Recall that an $n$-cochain $u \in \Delta^n(X)$ is defined on $n$-singular simplices $\sigma : \Delta^n \to X$. Given $\Delta^{m+n}$, we define the $m$-front face of $\Delta^{m+n}$ by $a_m : \Delta^m \to \Delta^{m+n}$, $a_m(t_0, \ldots, t_m) = (t_0, \ldots, t_m, 0, \ldots, 0)$ and $b_n : \Delta^n \to \Delta^{m+n}$, the $n$-back face by $b_n(t_0, \ldots, t_n) = (0, \cdots, 0, t_0, \cdots, t_n)$. Then for any $(n+m)$-singular simplex $\sigma : \Delta^{n+m} \to X$, we have $\partial_{a_m}\sigma : \Delta^m \to X$, $\partial_{b_n}\sigma : \Delta^n \to X$ defined by $\partial_{a_m}\sigma = \sigma \circ a_m$, $\partial_{b_n}\sigma = \sigma \circ b_n$. Now given cochains $u \in \Delta^n(X)$, $v \in \Delta^m(X)$, define $u \cdot v \in \Delta^{n+m}(X)$ by

$$u \cdot v(\sigma) = (-1)^{mn} u(\partial_{a_m}\sigma) v(\partial_{b_n}\sigma).$$

With these definitions, one may check

(a)

$$\delta(u \cdot v) = \delta u \cdot v + (-1)^m u \cdot \delta v$$

and thus $u \cdot v$ passes to cohomology, giving a product which we denote by $\cup$. Thus if $x \in H^m(X)$, $y \in H^n(X)$, $x \cup y \in H^{m+n}(X)$.

(b) $x \cup y$ is bilinear, i.e. $(x_1 + y_2) \cup y = x_1 \cup y + x_2 \cup y$ and similarly for $x \cup (y_1 \cup y_2)$.

(c) $x \cup y$ is graded commutative, i.e.

$$x \cup y = (-1)^{|x||y|} y \cup x$$

(d) The $\cup$ product is associative i.e. if $x, y, z$ are classes

$$(x \cup y) \cup z = x \cup (y \cup z)$$

(e) It is natural, if $f : Y \to X$, then $f^*(x \cup y) = f^*x \cup f^*y$.

More generally, if $(X, A)$ is a pair, $\Delta^m(X, A)$ are those cochains of $X$ which vanish in $A$. Given $u \in \Delta^m(X, A)$, $v \in \Delta^n(X)$, then $u \cdot v \in \Delta^{m+n}(X, A)$ and then defines $H^m(X, A) \otimes H^n(X) \to H^{m+n}(X, A)$.

If we let $X_1 \times X_2 \xrightarrow{p_i} X_i$, $i = 1, 2$. Define

$$u_1 \times u_2 = p_1^* u_1 \cup p_2^* u_2$$

the <u>cross-product</u>, which is then a homomorphism

$$H^m(X_1) \otimes H^n(X_2) \xrightarrow{\times} H^{m+n}(X_1 \times X_2)$$

THEOREM. *The cross-product map above is a monomorphism.*



## §9. Wang and Gysin sequences.

We want to obtain relations in the homology of a fiber bundle $Y$ when

(a) $$X \to Y \to S^n$$

or

(b) $$S^n \to Y \to X\,.$$

In case (a) we obtain the Wang sequence. In case (b) the Gysin sequence. More generally for (a), suppose

$$X \to Y \to \Sigma B$$

is a fiber bundle, where $\Sigma B = C_+B \cup C_-B$, $C_+B \cap C_-B = B$ and we assume that $Y_\pm = Y|C_\pm B$ is trivial, i.e. $Y_\pm \approx C_\pm B \times X$. Then also $Y_0 = Y_+ \cap Y_- \approx B \times X$. If we look at the cohomology of the pair $(Y, X)$, we have:

$$\begin{array}{ccccccccc}
\longrightarrow H^{q-1}(Y) \longrightarrow & H^{q-1}(X) & \xrightarrow{\delta} & H^q(Y,X) & \longrightarrow H^q(Y) \longrightarrow H^q(X) \longrightarrow \cdots \\
& \downarrow{\scriptstyle \mu^*} & & \uparrow{\scriptstyle \simeq} & \\
& H^{q-1}(B \times X) & & H^q(Y, Y_+) & \\
& \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \simeq} & \\
& H^q((C_-B, B)) & & \simeq H^q(Y_-, Y_0) &
\end{array}$$

where the top is exact and the diagram commutes. Where $\mu$ is defined as follows: let $\varphi_\pm : Y_\pm \to C_\pm B \times X$ be the homeomorphism, then $\varphi_- \varphi_+^{-1} : B \times X \to B \times X$ defines $\mu$ by $\varphi_- \varphi_+^{-1}(b, x) = (b, \mu(b, x))$. Suppose now that $\Sigma B = S^{n+1}$, then $B = S^n$. Then $H^q((C_-S^n, S^n) \times X) \simeq H^{q-n-1}(X)$ and we obtain:

$$\to H^{q-n-2}(X) \to H^{q-1}(Y) \to H^{q-1}(X) \to H^{q-n-1}(X) \to H^q(Y) \to H^q(X) \to \cdots$$

In particular, we obtain if $q < m + 1$, $H^q(Y) \xrightarrow{i^*} H^q(X)$ is an isomorphism. Let us apply this to describe the cohomology ring of $U(n)$.

THEOREM. $H^*(U(n)) = \Lambda(x_1, x_3, x_5, \ldots, x_{2n-1})$.

PROOF: We have a fiber bundle

$$U(n-1) \to U(n) \to S^{2n-1}.$$

Assume the theorem true for $U(n-1)$. Then we have $H^q(U(n)) \simeq H^q(U(n-1))$ for $q \leq 2n-2$. But $H^*(U(n-1))$ is generated by $x_1, x_3, x_5, \ldots, x_{2n-3}$ as an algebra. Hence $H^*(U(n)) \to H^*(U(n-1))$ is onto and we obtain an exact sequence:

$$0 \to H^{q-2n-1}(U(n-1)) \to H^q(U(n)) \to H^q(U(n-1)) \to 0.$$

Then additively it is true that $H^*(U(n)) \simeq \Lambda(x_1, \ldots, x_{2n-1})$. Now let $x_{2n-1} = p^* i_{2n-1}$. Because of the universal properties of the exterior algebra, we can define a ring homomorphism, $\Lambda(x_1, x_3, \ldots, x_{2n-1}) \to H^*(U(n))$, and with a little bit of care one shows it is an isomorphism.

Now I want to do the Gysin sequence.

We have already said that if $u \in H^n(D^n, S^{n-1})$ is a generator,

$$u\times : H^q(X) \simeq H^{q+n}((D^n, S^{n-1}) \times X).$$

We assume we are given a bundle pair $(D^n, S^{n-1}) \to (E, E_0) \to X$. We say it is orientable iff there exists a class $u \in H^n(E, E_0)$ which restricts to a generator of $H^n(E_x, (E_0)_x)$ for any $x \in X$.

THEOREM (THOM). *If $(D^n, S^{n-1}) \to (E, E_0) \to X$ is orientable, then*

$$\cup u : H^q(E) \to H^{q+n}(E, E_0)$$

*is an isomorphism.*

PROOF: We give the proof only when there is a finite cover of $X$, $V_1, \ldots, V_k$, so that $(E|V_i, E_0|V_i)$ is trivial. We use Mayer-Vietoris.



First, let $\alpha_i = \alpha|(E^i, E_0^i)$, then

$$H^q(V_i) \overset{\alpha_i}{\cong} H^{q+n}(E^i, E_0^i).$$

We assume that we have proved it for unions and intersections of the $V_i$ with less than $k$ factors. We want to prove it for $k$ factors, say on $V_1 \cup \cdots \cup V_k$. Then it is true for $U = V_1 \cup \cdots \cup V_{k-1}$, $V = U_k$, and $U \cap V = (V_1 \cap V_k) \cup \cdots \cup (V_{k-1} \cap V_k)$ and we have Mayer-Vietoris sequences for $(U \cup V)$ and for $((E, E_0)|V \cup V)$, linked together by homomorphism $\alpha_{U \cup V}$, etc. four of which are isomorphisms, so by the 5 lemma, the fifth one namely

$$\alpha_{U \cup V} : H^q(U \cup V) \Rightarrow H^{q+n}(\sigma(E, E_0)|V \cup V)$$

is also an isomorphism.

If $X$ is 1-connected, every bundle $(E, S^{n-1})$ over $X$ is orientable. In fact there is a line bundle, the determinant bundle of $E$, $\det E$ which is trivial iff $E$ is orientable.

<u>Gysin sequence:</u> If

$$S^{n-1} \to Y \to X$$

is an orientable fiber bundle, we have

$$\to H^q(X) \xrightarrow{\cup \chi} H^{q+n}(X) \to H^{q+n}(Y) \to H^{q+1}(X)$$

**Example.** $H^*(CP^n)$

$$S^1 \to S^{2n+1} \to CP^n$$
$$H^*(CP^n) = Z[w]/(w^{n+1})$$

Analogously

$$S^0 \to S^n \to RP^n$$
$$H^*(RP^n; Z_2) = Z_2[x]/(x^{n+1})$$

One of the most important results in the homology of fibre spaces is the Leray-Hirsh Theorem. Let $F \to X \to Y$, is $H^*(X) \to H^*(F)$ be onto. Then

$$H^*(X) \simeq H^*(Y) \otimes H^*(F)$$

as $H^*(Y)$-modules.

The above result on the Thom isomorphism is an important special case of this Leray-Hirsch Theorem.