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COLLEGE
ON
GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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INTRODUCTION TO MORSE THEORY.

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Introduction to Morse Theory.

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Morse theory relates the topology of a manifold M to the behaviour of a function defined on M. As the name implies this was first done by Marston Morse in the 1920's and his work appears in [Morse]. During the 1950's there was a resurgence of interest and several interesting pieces of work were done, particularly by Samelson and Bott, Thom and Smale. Several expository books [Bott 1], [Milnor], [Eells] appeared at about this time. The first lecture will, more or less, follow the first part of Milnor's book. In the early 1980's Witten found a new approach that relates the behaviour of a function on M to its topology [Witten]. His ideas have been very influential and their full ramifications are probably still not realised. The second lecture will discuss Witten's work.

§1. Basic Definitions.

M^n will denote a compact, connected, smooth n-dimensional manifold without boundary.

$f : M \to \mathbb{R}$ will denote a smooth function.

A point $q \in M$ is a critical point of f if the differential of f vanishes at q. Since M is compact, if f is not constant, it will have at least two critical points, because f must attain its maximum and minimum values.

The Hessian of f at q (relative to a local coordinate system $x_1, \ldots, x_n$) is the $n \times n$ symmetric matrix $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]$ evaluated at q. It is easily checked that at a critical point the nonsingularity of the Hessian is independent of the choice of coordinates.

A critical point of f at which the Hessian is non-singular is called non-degenerate. It follows from the compactness of M that f cannot have infinitely many non-degenerate critical points. The index of the critical point q is the dimension of a largest linear subspace of T_qM on which the quadratic form defined by the Hessian matrix is negative definite; that is, the index is the number of negative eigenvalues of the Hessian matrix.

A function all of whose critical points are non-degenerate is called a Morse function. Morse functions exist on every manifold M, indeed the set of Morse functions form a dense open subset of the space of all smooth functions on M. This fact is a standard consequence of Sard's theorem.

Morse's Lemma. Let $f : M \to \mathbb{R}$ have a critical point of index k at q. Then one can choose a local coordinate system $x_1, \ldots, x_n$ on some neighbourhood U of q such that $f|U$ is given by

$$f(q) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2 .$$

This means that by suitable choice of coordinates, all the higher terms in Taylor's expansion of f can be eliminated. This lemma is proved in all the references.

Examples.

1. Let $T = \mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ consists of all integral multiples of 2π, be a 2-dimensional torus. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x,y) = \sin x + \sin y$. Prove that f defines a Morse function on T. (f has exactly 4 critical points, one (the minimum) of index 0, one (the maximum) of index 2 and two (saddlepoints) of index 1.)

2. Let $K = \mathbb{R}^2/\Gamma$, where Γ is the group generated by the two Euclidean transformations

$$(x,y) \to (x+\pi, -y)$$

and

$$(x,y) \to (x, y+2\pi) .$$

Prove that K is the Klein bottle and that $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \sin x \sin y$ defines a Morse function on K with exactly 4 critical points, with indices as in the previous example.

3. Let M be $\mathbb{RP}^{n-1}$ and A be an $n \times n$ symmetric matrix. Prove that the function $f : S^{n-1} \to \mathbb{R}$ defined by $f(\underline{x}) = \underline{x}^t A \underline{x}$ defines a function on $\mathbb{RP}^{n-1}$. Prove that $\underline{x}$ is a critical point for f if and only if $\underline{x}$ is an eigenvector of A. If A has n distinct eigenvalues $\lambda_1 < \lambda_2 < \dots < \lambda_n$ with corresponding critical points $\underline{x}_1, \dots, \underline{x}_n \in \mathbb{RP}^{n-1}$ prove that f is a Morse function and that the index of $\underline{x}_k$ is $k-1$.

§2. The Topological Behaviour.

We will now assume that f is a Morse function on M such that the values of f at its critical points are distinct. An arbitrarily small change in f will ensure that the extra condition is satisfied.

Let $f_\alpha = \{x \in M \mid f(x) = \alpha\}$ be a level set of f and let

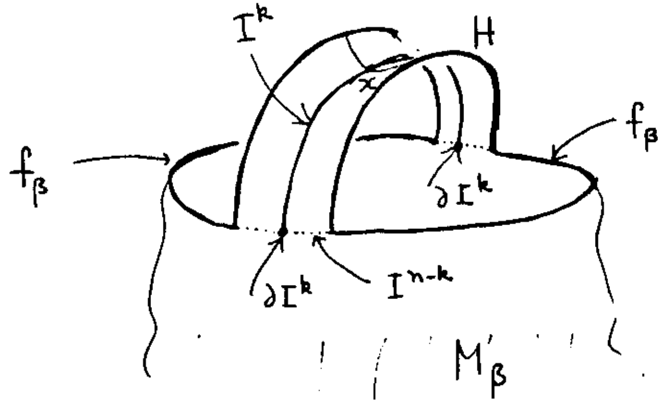
$$M_\alpha = \{x \in M \mid f(x) \leq \alpha\} .$$

If α is not a critical value (i.e. there are no critical points in f_α) , then, by the implicit function theorem, f_α is an $(n-1)$-dimensional manifold and M_α is an n-dimensional manifold with boundary f_α. (We will only consider α such that M_α is non-empty.)

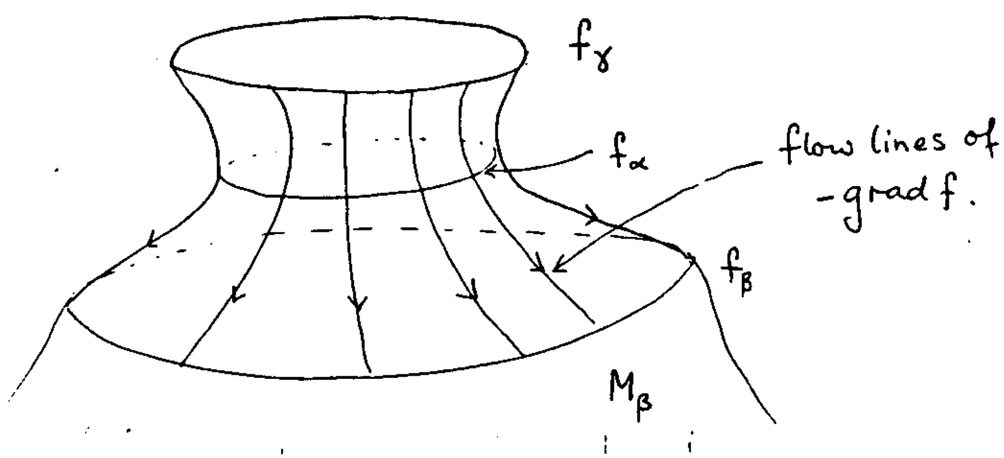
The following basic results describe the topology of M in terms of the critical points of f.

<u>Theorem</u> (i) If there are no critical values of f in the interval $[\beta,\gamma]$ (where $\beta < \gamma$) then M_β and M_γ are diffeomorphic.

(ii) If there is exactly one critical value, say α, of f in the interval $[\beta,\gamma]$ (where $\beta < \alpha < \gamma$) then M_γ is diffeomorphic to the manifold $M_\beta \cup H$. The 'handle' H is diffeomorphic to the unit cube $I^n \cong I^k \times I^{n-k}$ glued onto M_β by an inclusion mapping $\partial I^k \times I^{n-k} \subset f_\beta$, where k is the index of the critical point q such that $f(q) = \alpha$.

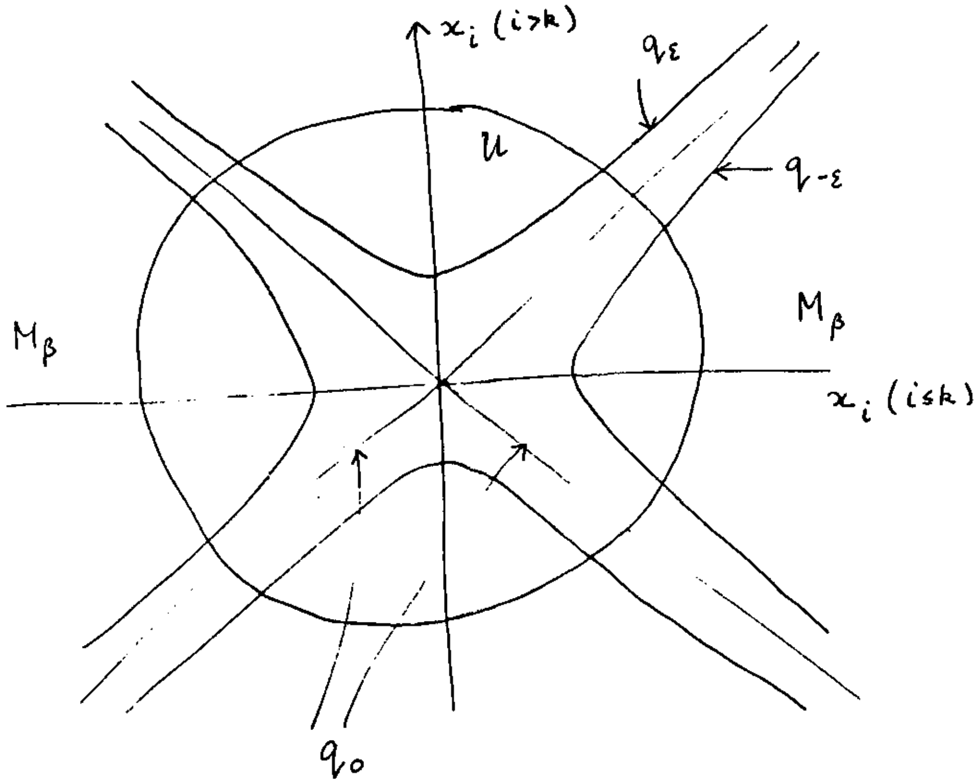


<u>Proof</u> (i) Introduce a Riemannian metric on M and consider the vector field $\operatorname{grad} f$ induced by f on M. It is orthogonal to the level sets f_α. Consider the flow induced by $\operatorname{grad} f$, it maps f_α diffeomorphically to $f_{\alpha+\varepsilon}$ as long as there is no critical point in the region $[f_\alpha, f_{\alpha+\varepsilon}]$ and can be used to induce a diffeomorphism of M_α with $M_{\alpha+\varepsilon}$. The result follows.

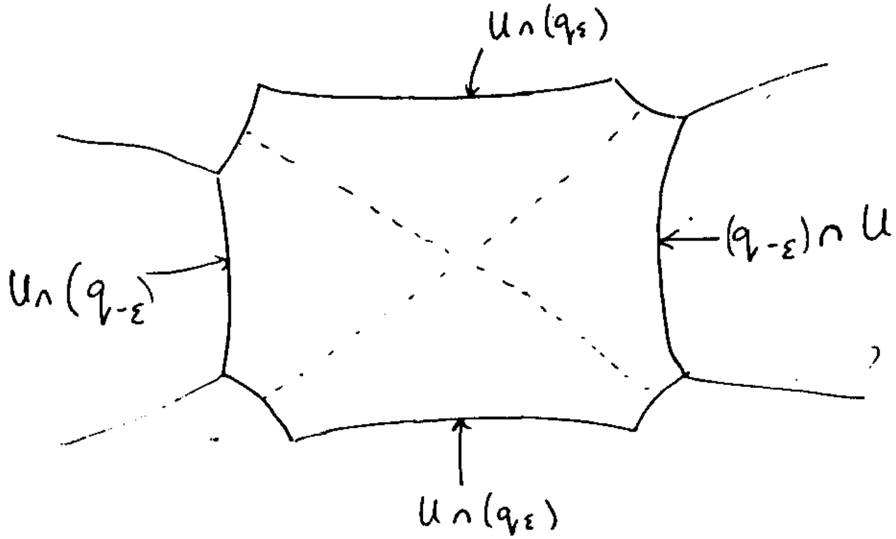


(ii) Let q be the unique critical point in $M_\gamma \backslash M_\beta$, so $f(q) = \alpha$. Choose coordinates in a neighbourhood U of q using Morse's lemma. We consider what happens to $U \cap M_{\alpha-\varepsilon}$ as ε increases through the value 0, by the argument of (i), up to diffeomorphism, M does not change outside U. It is enough to consider $\mathbb{R}^n$ with the function

$$q(x) = -x_1^2 - x_2^2 - \dots - x_k^2 + x_{k+1}^2 \dots + x_n^2 .$$



Changing up to diffeomorphism one has the following diagram



from which one can see the result.

<u>Corollary</u> Let $f : M \to \mathbb{R}$ be a Morse function with critical points $x_1, \dots, x_r$ all at different levels $\alpha_1 < \alpha_2 < \dots < \alpha_r$ and indices $k_1 = 0$, $k_2, \dots, k_r = n$. Then M is homotopically equivalent to a cell complex K with r cells of dimensions $k_1, k_2, \dots, k_r$.

Note that it is not strictly necessary to assume that the critical levels are distinct. If there are ℓ critical points at level α then $M_{\alpha+\varepsilon}$ is obtained from $M_{\alpha-\varepsilon}$ by attaching ℓ handles simultaneously. The ℓ cells of the cell complex K are also attached simultaneously.

Examples.

1. Follow the proof of the above theorem to describe the topology of the torus, Klein bottle and real projective spaces using the functions defined previously.

2. Suppose α is a critical level of $f : M \to \mathbb{R}$ and that the interval $(\alpha, \alpha+\varepsilon]$ has no critical values. Show that $M_{\alpha+\varepsilon}$ is homotopy equivalent to M_α (although M_α is not a manifold).

§3. The homology of a cell complex.

A good reference for the details of this material is [Cooke and Finney].

Let K be a finite cell complex obtained inductively (on n) by attaching an r-cell $(D^r, S^{r-1}) \to (K_n, K_{n-1})$ so that $K_n = K_{n-1} \cup_f D^r$ where $f : S^{r-1} \to K_{n-1}$ is the attaching map.

Define a chain complex $\{C_r\}$ for K (over a field) by letting C_r be the vector space whose basis is the set of r-cells of K. The differential $d : C_r \to C_{r-1}$ is defined on an r-cell as the sum of the incidence numbers of its boundary S^{r-1} with each $(r-1)$-cell.

Note that for $f : M \to \mathbb{R}$, the chain complex of the cell complex obtained using f can be obtained by taking C_r to be the vector space whose basis is the set of critical points on index r. The differential can be computed directly from the gradient flow of f, this is outlined in [Witten], although I know of no detailed references.

The homology $H_*(C)$ of this chain complex gives the homology of M.

Example Calculate the chain complex associated to the cell complexes obtained for the three examples already considered and hence calculate their homology.

§4. The Morse inequalities.

These now follow easily because of the following algebraic result.

Lemma Let $\{C_r\}$ be a finite dimensional chain complex over a field, with $\gamma_r = \dim C_r$ and $\beta_r = \dim H_r$ the Betti numbers. Let $\gamma(t) = \sum \gamma_r t^r$ and $\beta(t) = \sum \beta_r t^r$ be the corresponding polynomials, then there is a polynomial $\alpha(t)$ with non-negative (integer) coefficients such that

$$\gamma(t) = \beta(t) + (1+t)\alpha(t) .$$

Proof As usual, let $B_r = \text{Image } d : C_{r+1} \to C_r$

$Z_r = \text{Kernel } d : C_r \to C_{r-1}$.

Since $d^2 = 0$ one has $B_r \subset Z_r$ and $H_r = Z_r/B_r$ hence $Z_r \cong B_r \oplus H_r$; also $C_r \cong Z_r \oplus B_{r-1}$, so $C_r \cong B_r \oplus H_r \oplus B_{r-1}$. Under these isomorphisms d is zero on $B_r \oplus H_r$ and is the identity on B_{r-1}.

Let $\alpha_r = \dim B_r$, then

$$\gamma_r = \alpha_r + \beta_r + \alpha_{r-1}$$

which, when put into polynomial form, is the stated result.

Setting $t = -1$ gives

Corollary (i) $\sum (-1)^r \gamma_r = \sum (-1)^r \beta_r$

(ii) $\sum_{i=0}^{k} (-1)^r \gamma_r \geq \sum_{i=0}^{k} (-1)^r \beta_r$ for each $k \geq 0$.

In the special case where $\{C_r\}$ is the chain complex obtained from a Morse function one obtains inequalities between the Betti numbers of M and the numbers of critical points of f of various indices ($\beta_r = r^{\text{th}}$ Betti number of M, γ_r = number of critical points of f of index r). These are the Morse inequalities.

§5. de Rham and Hodge Theory.

Let $\Omega^*(M)$ be the de Rham complex of the smooth manifold M, $\Omega^r(M)$ is the space of smooth differential r-forms on M and $d : \Omega^r(M) \to \Omega^{r+1}(M)$ is the exterior differential, defined in local coordinates $x_1, \ldots, x_n$ by

$$d(f dx_{i_1} \wedge \ldots \wedge dx_{i_r}) = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r}$$
$$= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r} .$$

The de Rham cohomology of M is defined to be the homology of the de Rham complex. It is a theorem [de Rham] that this is isomorphic to the singular cohomology of M with real coefficients and hence to the cellular cohomology of M with real coefficients.

Now consider a compact Riemannian manifold M with metric $\langle\ ,\ \rangle$ defined in each tangent space T_xM. This defines an inner product, also denoted $\langle\ ,\ \rangle$ on each exterior product $\Lambda^r T_x^* M$ and hence, by integration over M, on $\Omega^r(M)$; making each $\Omega^r(M)$ into a pre-Hilbert space.

By conjugating with the Hodge star operator

$$* : \Omega^r(M) \to \Omega^{m-r}(M)$$

and introducing appropriate signs one defines an operator

$$d^* : \Omega^{r+1}(M) \to \Omega^r(M)$$

such that $\langle d\omega, \eta\rangle = \langle \omega, d^*\eta\rangle$ for all $\omega \in \Omega^r$, $\eta \in \Omega^{r+1}$.

Define $L = dd^* + d^*d : \Omega^r(M) \to \Omega^r(M)$ the Hodge Laplacian. Then one can show that $\operatorname{Ker} L = \operatorname{Ker} d \cap \ker d^*$ the space of harmonic forms; it is Hodge's theorem [Hodge] that $\operatorname{Ker} L$ in dimension r is isomorphic to the de Rham cohomology $H^r(M)$.

A good reference for all this material is [Warner].

§6. Witten's differential.

If $f : M \to \mathbb{R}$ is smooth, Witten defined a modified exterior differential which depends on f and on a parameter $t \in \mathbb{R}$.

Definition $d_t = e^{-tf} \circ d \circ e^{tf} : \Omega^r(M) \to \Omega^{r+1}(M)$ where e^{tf} is the operator obtained by multiplying a form by the function e^{tf}.

Since d_t is conjugate to d, it satisfies $d_t^2 = 0$ and the homology of the chain complex $(\Omega^*(M), d_t)$ is isomorphic to the de Rham cohomology of M.

One can also define a Laplacian

$$L_t = d_t d_t^* + d_t^* d_t$$

whose kernel is isomorphic to de Rham cohomology as is the case for the Hodge Laplacian.

In the case where f is a Morse function on M, one can consider the "low lying eigenvalues" of the operator L_t, that is those eigenvalues λ satisfying $\lambda/t \to 0$ as $t \to \infty$. The corresponding eigenforms span a finite dimensional subcomplex C^r of the de Rham complex such that the dimension of C^r is equal to the number of critical points of index r and whose homology is isomorphic to the space of harmonic forms. The algebraic lemma of §4 then gives the Morse inequalities directly. This is the argument of [Witten] and the following sections will elaborate some of the details.

First, we will give an alternative description of Witten's operator and study the Laplacian L_t. If ω is a k-form on M then one can define an operator

$$E_\omega : \Omega^r(M) \to \Omega^{r+k}(M)$$

by $E_\omega(\eta) = \omega \wedge \eta$. Then $d_t = d + tE_{df}$ because

$$e^{-t} \circ d \circ e^{tf}(\eta) = e^{-tf} \circ (te^{tf} df \wedge \eta + e^{tf} d\eta) = t df \wedge \eta + d\eta .$$

To study the Laplacian we need a simple fact from linear algebra. Let V be an inner product space, then each $\Lambda^k V$ has an induced inner product. For $\omega \in \Lambda^1 V$ we have $E_\omega : \Lambda^r V \to \Lambda^{r+1} V$ and let $I_\omega : \Lambda^{r+1} V \to \Lambda^r V$ be its adjoint.

Lemma $E_\omega I_\eta + I_\eta E_\omega = \langle\omega,\eta\rangle$ for all $\omega, \eta \in \Lambda^1 V$.

Proof Choose an orthonormal basis $e_1, \ldots, e_n$ for V then $\{e_{i_1}\wedge\ldots\wedge e_{i_r}\}$ is an orthonormal basis for $\Lambda^r V$. By linearity, it is enough to check the formula for $\omega = e_i$ and $\eta = e_j$.

Now

$$E_{e_i}(e_{i_1}\wedge\ldots\wedge e_{i_r}) = e_i \wedge e_{i_1} \wedge \ldots \wedge e_{i_r} = 0 \quad \text{if} \quad i \in I = \{i_1,\ldots,i_r\} .$$

Hence

$$I_{e_i}(e_{i_1}\wedge\ldots\wedge e_{i_{r+1}}) = 0 \quad \text{if} \quad i \notin I$$

$$= \pm e_{i_1} \wedge \ldots \wedge e_{i_{r+1}} \quad \text{with } e_i \text{ omitted if } i \in I .$$

So by a straightforward calculation

$$(E_{e_i}I_{e_j}+I_{e_j}E_{e_i})(e_{i_1}\wedge\ldots\wedge e_{i_r}) = 0 \quad \text{if} \quad i \neq j$$

$$= e_{i_1} \wedge \ldots \wedge e_{i_r} \quad \text{if} \quad i = j .$$

Similarly, if ω, η are 1-forms one has that

$$E_\omega I_\eta + I_\eta E_\omega = \langle\omega,\eta\rangle$$

as mappings of $\Omega^r(M)$.

Witten's Laplacian L_t equals

$$(d+tE_{df})(d^*+tI_{df}) + (d^*+tI_{df})(d+tE_{df})$$

$$= dd^* + d^*d + t(E_{df}d^*+dI_{df}+d^*E_{df}+I_{df}d) + t^2(E_{df}I_{df}+I_{df}E_{df})$$

$$= L + tA + t^2\|df\|^2$$

where, in local coordinates $x_1, x_2, \ldots, x_n$; A is the operator

$$\sum \frac{D^2 f}{Dx_i Dx_j}(E_{dx_i}I_{dx_j}-I_{dx_i}E_{dx_j})$$

and $\frac{D^2 f}{Dx_i Dx_j}$ is the second covariant derivative of f . In Euclidean coordinates this is just $\frac{\partial^2 f}{\partial x_i \partial x_j}$. Near points where $df \neq 0$ the term $t^2\|df\|^2$ dominates the behaviour of L_t for large t . Hence for large t the eigenforms of L_t must be small near points where $df \neq 0$. In this case everything is dominated by the behaviour of the eigenforms near the critical points. To understand the local behaviour (carried out in §8) one must study the harmonic oscillator equation.

§7. Eigenvalues of the harmonic oscillator.

The one dimensional harmonic oscillator operator is

$$H = -\frac{d^2}{dx^2} + t^2x^2 = -D^2 + t^2x^2 .$$

It is a standard problem studied in texts on quantum mechanics to consider the eigenvalue problem for H . (See [Miller] for example.)

Let $J^\pm = \pm D - tx$ be the "raising" and "lowering" operators (with $t > 0$) . Then

$$J^-J^+ = H + t , \quad J^+J^- = H - t \quad \text{so} \quad J^-J^+ - J^+J^- = 2t$$

and

$$HJ^\pm = J^\pm H = \pm 2tJ^\pm .$$

If φ, ψ are functions such that they and their derivatives vanish as $x \to \pm\infty$ then

$$\langle J^+\varphi,\psi\rangle = \langle\varphi,J^-\psi\rangle$$

and

$$\langle H\varphi,\psi\rangle = \langle\varphi,H\psi\rangle$$

where $\langle\ ,\ \rangle$ is the usual inner product.

If φ is an eigenfunction for H such that $\|\varphi\| = 1$ so that $H\varphi = \lambda\varphi$, the relations above show that

$$H(J^\pm\varphi) = J^\pm(H\pm 2t)\varphi = (\lambda\pm 2t)J^\pm\varphi$$

and

$$\langle J^+\varphi, J^+\varphi\rangle = \langle J^-J^+\varphi, \varphi\rangle$$

$$= \langle H\varphi, \varphi\rangle + t\langle\varphi, \varphi\rangle$$

$$= \lambda + t .$$

Also $\langle J^-\varphi, J^-\varphi\rangle = \lambda - t$ so $\lambda \leq t$. Therefore there is a 'ladder' of eigenvalues $\lambda + 2kt$, $k \in \mathbb{N}$. For $\lambda = t$ one has the single eigenfunction $\exp(-tx^2/2)$ and one now easily deduces that the only eigenfunctions are $\psi_k(x) = (J^+)^k\exp(-tx^2/2)$ with eigenvalues $t(2k+1)$ for $k \in \mathbb{N}$

§8. The local model for Witten's Laplacian.

We consider the operator L_t in the situation where $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$f(x_1,\dots,x_n) = \tfrac{1}{2}(-x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2)$$

and $\mathbb{R}^n$ has its standard metric.

Then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

and $\|df\|^2 = x_1^2 + x_2^2 + \dots + x_n^2$.

So
$$L_t = \sum L_i + t \sum \varepsilon_i K_i = L + tK$$

where
$$L_i = -\frac{\partial^2}{\partial x_i^2} + t^2 x_i^2 ,$$
$$\varepsilon_i = -1 \quad \text{for} \quad 1 \le i \le k$$
$$= +1 \quad \text{for} \quad k+1 \le i \le n ,$$

and
$$K_i = E_{dx_i} I_{dx_i} - I_{dx_i} E_{dx_i} .$$

Consider the behaviour of L_t on an r-form

$$\phi(x) dx_{i_1} \wedge \dots \wedge dx_{i_r} ;$$

L_i acts as the identity on $dx_{i_1} \wedge \dots \wedge dx_{i_r}$ and K_i acts as the identity on $\phi(x)$. Hence, it follows that all the K_i and L_i commute. Now the eigenfunctions for L_i are $\psi_{r_i}(x_i) = (J_i^+)^{r_i} \exp(-tx_i^2/2)$ with eigenvalue $t(1+2r_i)$. So the eigenfunctions for $L = \sum L_i$ are products of the $\psi_{r_i}(x_i)$ with eigenvalue the sum of the corresponding eigenvalues. Hence the operator L on forms has an $\begin{bmatrix} n \\ r \end{bmatrix}$ dimensional space of eigenforms in $\Omega^r(M)$ corresponding to each eigenfunction $\psi_{r_1}(x_1)\psi_{r_2}(x_2) \dots \psi_{r_n}(x_n)$ with eigenvalue $(n+2(r_1+\dots+r_n))t$.

Now we consider K_i acting on $dx_{j_1} \wedge \dots \wedge dx_{j_r}$. It is straightforward to check that if $i \in J = \{j_1, \dots, j_r\}$ then K_i acts as $+1$ and if $i \notin J$ then it acts as -1 .

<u>Lemma</u> Let $K(dx^J) = n_J dx^J$ where dx^J denotes $dx_{j_1} \wedge \dots \wedge dx_{j_r}$, then

$$n_J = 2k + 2r - n - 4s$$

where s is the cardinality of $J \cap \{1, 2, \dots, k\}$.

<u>Proof</u> $K_i(dx^J) = +1 \iff i \in J$

so $\quad \varepsilon_i K_i(dx^J) = +1 \iff i \in J$ and $i \notin \{1, \dots, k\}$

or $i \notin J$ and $i \in \{1, \dots, k\}$

i.e. $K(dx^J)$ has $r - s + k - s$ terms with $+1$

and $n - r - k + 2s$ terms with -1 .

Hence $n_J = 2(r+k-2s) - n$ as required.

Note that $s \le r$ and $s \le k$ so $n_J \ge -n$.

<u>Theroem</u> The kernel of $L_t : \Omega^r(\mathbb{R}^n) \to \Omega^r(\mathbb{R}^n)$ is one dimensional if $r = k$ and is zero otherwise.

<u>Proof</u> The eigenvalues of L_t are $\lambda = t(n+2(r_1+\dots+r_n)) + n_J t$ with $r_i \in \mathbb{N}$ and $n_J \ge -n$ so $\lambda \ge 0$. To ensure $\lambda = 0$ one must have each $r_i = 0$ and $n_J = -n$ so $r + k = 2s$ hence $k = s$ and $r = s$ i.e. one must have $k = r$ and $J = \{1, 2, \dots, k\}$. So the only form in the kernel of L_t is

$$\psi_0(x_1)\psi_0(x_2) \dots \psi_0(x_n) dx_1 \wedge dx_2 \wedge \dots \wedge dx_k .$$

§9. The localisation theorem.

To compare the behaviour of Witten's Laplacian on M with its behaviour near the critical points of f , one can utilise the fact that the eigenforms of L_t are concentrated near the critical points of f . This has been made precise by B. Simon and others [Simon], [Cycon] and [Helffer]. We state it in the form given by Simon.

Theorem Let $\{q_i\}$ $1 \leq i \leq Q$ be the set of critical points of the Morse function $f : M \to \mathbb{R}$ and let L_t be Witten's Laplacian. Let $\lambda_1(t) \leq \lambda_2(t) \leq \dots \leq \lambda_r(t) \leq \dots$ be the eigenvalues of L_t and let $\mu_1 \leq \mu_2 \leq \dots \leq \mu_r \leq \dots$ be the eigenvalues of the operator

$$K = \bigoplus_{i=1}^{Q} K_i(t)$$

where $K_i(t) : \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^n)$ is Witten's Laplacian on $\mathbb{R}^n$ with $f = -x_1^2 - \dots - x_{k_i}^2 + x_{k_{i+1}}^2 + \dots + x_n^2$ as in §8 (k_i is the index of the critical point q_i). Then $\lambda_r(t)/t \to \mu_r$ as $t \to \infty$.

Since the kernel of each $K_i(t)$ is one dimensional and lies in $\Omega^{k_i}(\mathbb{R}^n)$, it follows that the eigenspaces of L_t corresponding to the lowlying eigenvalues (i.e. those $\lambda(t)$ satisfying $\lambda(t)/t \to 0$) have total dimension Q and lie in $\Omega^r(M)$ for r corresponding to the indices of the critical points.

The proof of the localisation theorem can be found in [Cycon] and variants in [Helffer] and [Bismut 1]. Another account of Witten's work can be found in [Henniart].

§10. Further developments.

Finally I mention a few other developments related to the Morse theory.

Morse theory was originally introduced to study the calculus of variations and to prove the existence of solutions. The finite dimensional theory was extended by R. Bott to the case where there are critical submanifolds but the function is non-degenerate in the normal directions. In another direction R. Thom showed how some theorems about the topology of smooth algebraic varieties could be proved using Morse theory, see [Milnor] §7. This has now been extended to the study of singular algebraic varieties [Goresky-MacPherson]. The study of degenerate critical points is explained well in [Fomenko] and an application of this to differential geometry is given in [Duan-Rees]. The results of Morse theory were extended from gradient flows to general flows by S. Smale in 1960 [Smale].

Witten's ideas have been used by him and others to give new proof of the Atiyah-Singer theorem [Witten], [Bismut 2]. The extension of this work to infinite dimensions has been developed by A. Floer to give interesting new results about 3-dimensional topology [Floer].

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