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COLLEGE

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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HOMOLOGICAL ALGEBRA.

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1. Preliminaries

Let R be a commutative ring with unit, and let A, B, C be R-modules.

If

$$f: A \longrightarrow B \quad , \quad g: B \longrightarrow C$$

are R-homomorphisms, we say that the sequence

(1) $$A \xrightarrow{f} B \xrightarrow{g} C$$

is half-exact at B if $\operatorname{Im} f \subseteq \ker g$. If we have $\operatorname{Im} f = \ker g$, we say that (1) is exact at B.

Remark. The sequence (1) is half-exact if and only if $gf = 0$ (the zero-homomorphism).

A sequence

(2) $$\cdots \longrightarrow C_{n+1} \xrightarrow{\phi_{n+1}} C_n \xrightarrow{\phi_n} C_{n-1} \xrightarrow{\phi_{n-1}} \cdots$$

of R-modules and R-homomorphisms is a chain complex if it is half-exact at C_k, for each $k \in Z$ (i.e. $\operatorname{Im} \phi_{k+1} \subseteq \ker \phi_k \; \forall k$ or equivalently $\phi_k \phi_{k+1} = 0 \; \forall k$). Sometimes we write $Z_k = \ker \phi_k$, $B_k = \operatorname{Im} \phi_{k+1}$. The elements of Z_n are called n-cycles; the elements of B_n are called n-boundaries.

We will say that the sequence (2) is exact if it is exact at $C_k \; \forall k$.

Examples. (i) The sequence

(3) $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if f is 1-1 (exactness at A), $\operatorname{Im} f = \ker g$ (exactness at B) and g is onto (exactness at C). We say that (3) is a short exact sequence.

Notation. If A is a submodule of B, we write $i: A \hookrightarrow B$ for the inclusion of A in B; we write $f: A \rightarrowtail B$ if f is a monomorphism and $g: B \twoheadrightarrow C$ if g is an epimorphism.

If A is a submodule of B, then

$$0 \longrightarrow A \overset{i}{\hookrightarrow} B \overset{\psi}{\twoheadrightarrow} B/A \longrightarrow 0$$

is a short exact sequence. Here ψ indicates the canonical epimorphism.

If $\pi : B \twoheadrightarrow C$, then

$$0 \longrightarrow \ker \pi \overset{i}{\hookrightarrow} B \overset{\pi}{\twoheadrightarrow} C \longrightarrow 0$$

is a short exact sequence.

If A, C are two arbitrary R-modules, then

$$0 \longrightarrow A \overset{i_A}{\rightarrowtail} A \oplus C \overset{\pi_C}{\twoheadrightarrow} C \longrightarrow 0$$

is a short exact sequence. Here i_A, π_C are defined by setting $i_A(x) = (x,0)$, $\pi_C(x,y) = y$.

(ii) Consider the sequence

(4) $$0 \longrightarrow Z \overset{f}{\rightarrowtail} Z \overset{\psi}{\twoheadrightarrow} Z/2 \longrightarrow 0$$

where ψ is the canonical epimorphism and f is defined by setting $f(x) = 2kx$. The sequence (4) is a chain complex $\forall k \in Z$ and it is a short exact sequence if and only if $k = \pm 1$.

(iii) The sequence

$$\cdots \xrightarrow{0} Z \xrightarrow{id} Z \xrightarrow{0} Z \xrightarrow{id} Z \xrightarrow{0} Z \xrightarrow{id} \cdots$$

is exact.

(iv) The sequence

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

is a chain complex. It is exact if and only if f is an isomorphism

We end this section with the following technical lemma which is used in the proof of several theorems in algebraic topology.

<u>The five lemma</u>. Suppose we have a commutative diagram of R-modules and R-homomorphisms

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b", "\cong"'] & C \arrow[r, "h"] \arrow[d, "c"] & D \arrow[r, "\ell"] \arrow[d, "d", "\cong"'] & E \arrow[d, "e"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "\ell'"] & E' \end{tikzcd}$$

where the rows are exact. If b and d are isomorphisms, a is an epimorphism and e is a monomorphism, then c is an isomorphism.

<u>Proof</u>. This proof is a typical example of diagram chasing.

(i) c <u>is onto</u>. Let $z' \in C'$. As d is an isomorphism, $\exists!\ w \in D$ s.t. $d(w) = h'(z')$. We have

$$e\ell(w) = \ell' d(w) = \ell' h'(z') = 0$$

As e is 1-1, $\ell(w)=0$. So $w \in \ker \ell = \operatorname{Im} h$, i.e. $\exists\ z \in C$ s.t. $h(z)=w$. Now

$$h'(z' - c(z)) = h'(z') - h'c(z) = h'(z') - dh(z) = h'(z') - d(w) = 0$$

So $z' - c(z) \in \ker h' = \operatorname{Im} g'$ and $\exists\ y' \in B'$ s.t. $g'(y') = z' - c(z)$.
As b is an isomorphism, $\exists!\ y \in B$ s.t. $b(y) = y'$. Now we claim that $c(g(y) + z) = z'$. In fact

$$c(g(y) + z) = cg(y) + c(z) = g'b(y) + c(z) = g'(y') + c(z) = z'-c(z)+c(z)=z'$$

(ii) c <u>is 1-1</u>. Suppose $c(z) = 0$, for some $z \in C$. We have $dh(z)=h'c(z)=0$ and therefore $h(z)=0$ as d is an isomorphism. So $z \in \ker h=\operatorname{Im} g$ and $\exists\ y \in B$ s.t. $g(y) = z$. Now $g'b(y) = cg(y) = c(z) = 0$, hence $b(y) \in \ker g' = \operatorname{Im} f'$ and $\exists\ x' \in A'$ s.t. $f'(x') = b(y)$. As a is onto, $\exists\ x \in A$ s.t. $a(x) = x'$. Now $bf(x) = f'a(x) = f'(x') = b(y)$ and b is an isomorphism. Thus $f(x) = y$ and $z = g(y) = gf(y) = 0$.

2. <u>Chain complexes, chain maps and chain homotopies.</u>

Let $\mathcal{C} = \{C_n, \gamma_n\}$ be the sequence

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\gamma_{n+1}} C_n \xrightarrow{\gamma_n} C_{n-1} \longrightarrow \cdots$$

of R-modules and R-homomorphisms. Suppose $\mathcal{C}$ is a chain complex (i.e. $\gamma_n \gamma_{n+1}=0$).
If $\mathcal{D} = \{D_n, \delta_n\}$ is another chain complex, a chain map from $\mathcal{C}$ to $\mathcal{D}$ is a sequence

$$\phi = \{\phi_n\}_{n \in \mathbb{Z}}$$

where $\phi_n : C_n \longrightarrow D_n$ is an R-homomorphism and the diagram

$$\begin{tikzcd} C_n \arrow[r, "\phi_n"] \arrow[d, "\gamma_n"'] & D_n \arrow[d, "\delta_n"] \\ C_{n-1} \arrow[r, "\phi_{n-1}"] & D_{n-1} \end{tikzcd}$$

commutes $\forall n$. In this case we write $\phi : \mathcal{C} \to \mathcal{D}$.
If $\mathcal{E} = \{E_n, \varepsilon_n\}$ is yet another chain complex and $\psi : \mathcal{D} \to \mathcal{E}$ is a chain map, the composite $\psi\phi : \mathcal{C} \to \mathcal{E}$ is defined in the obvious way : $(\psi\phi)_n = \psi_n \phi_n$.
We write $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ for the sequence $\{\mathrm{id}_{C_n}\}$.
Suppose now that $\phi, \psi : \mathcal{C} \to \mathcal{D}$ are two chain maps. A chain homotopy h between ϕ and ψ is a sequence $h = \{h_n\}$ of R-homomorphisms $h_n : C_n \longrightarrow D_{n+1}$ such that $\delta_{n+1} h_n + h_{n-1} \gamma_n = \psi_n - \phi_n$.

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\gamma_{n+1}"] \arrow[d, "\phi_{n+1}"', shift right] \arrow[d, "\psi_{n+1}", shift left] & C_n \arrow[r, "\gamma_n"] \arrow[dl, "h_n"] \arrow[d, "\phi_n"', shift right] \arrow[d, "\psi_n", shift left] & C_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"] \arrow[d, "\phi_{n-1}"', shift right] \arrow[d, "\psi_{n-1}", shift left] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r, "\delta_{n+1}"'] & D_n \arrow[r, "\delta_n"'] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

If such h exists we say that ϕ is chain homotopic to ψ ($\phi \simeq \psi$).

<u>Proposition</u>. $\simeq$ is an equivalence relation in the set of chain maps of $\mathcal{C}$ into $\mathcal{D}$.

3. Homology groups of a chain complex.

Let $\mathcal{C}$ be a chain complex.

Definition. The k-th homology group of $\mathcal{C}$, $H_k(\mathcal{C})$ is defined by setting

$$H_k(\mathcal{C}) = \ker(\gamma_k)/\mathrm{Im}(\gamma_{k+1}) .$$

Remark. The complex $\mathcal{C}$ is an exact sequence if and only if $H_k(\mathcal{C}) = 0 \ \forall k$ (i.e. the homology groups measure the failure of $\mathcal{C}$ to be exact).

An element $\alpha \in H_k(\mathcal{C})$ is a class $[z]$ where $z \in \ker(\gamma_k)$, i.e. z is a k-cycle. $[z] = [z']$ if and only if $[z - z'] = 0$ i.e. if and only if $z - z'$ is a k-boundary, i.e. if and only if $\exists\ \bar{z} \in C_{k+1}$ s.t. $\gamma_{k+1}(\bar{z}) = z - z'$.

Let now $\phi : \mathcal{C} \to \mathcal{D}$ be a chain map. $\forall k$ we define a homomorphism

$$\phi_* : H_k(\mathcal{C}) \longrightarrow H_k(\mathcal{D})$$

called the induced homomorphism in homology, as follows. Let $\alpha \in H_k(\mathcal{C})$. Then $\alpha = [z]$ where the representative z of α is a k-cycle in $\mathcal{C}$.

We have $\delta_k \phi_k(z) = \phi_{k-1}\gamma_k(z) = 0$ (as z is a cycle and ϕ is a chain map).

Hence $\phi_k(z)$ is a k-cycle in $\mathcal{D}$ and defines a homology class $[\phi_k(z)] \in H_k(\mathcal{D})$.

We set $\phi_*[z] = [\phi_k(z)]$.

ϕ_* is well defined. In fact, if $[z'] = [z]$, we have $[z - z'] = 0$ in $H_k(\mathcal{C})$ i.e. $z - z' \in \mathrm{Im}(\gamma_{k+1})$ and $\exists\ \bar{z} \in C_{k+1}$ s.t. $\gamma_{k+1}(\bar{z}) = z - z'$. Thus we have

$$\phi_k(z) - \phi_k(z') = \phi_k(z - z') = \phi_k(\gamma_{k+1}(\bar{z})) = \delta_{k+1}\phi_{k+1}(\bar{z}) \in \mathrm{Im}(\delta_{k+1})$$

i.e. $[\phi_k(z)] = [\phi_k(z')]$.

ϕ_* is clearly a group homomorphism (as ϕ_k is a homomorphism).

Proposition. (i) $(\mathrm{id}_{\mathcal{C}})_* = \mathrm{id}_{H_k(\mathcal{C})} \quad \forall k$.

(ii) $(\psi\phi)_* = \psi_* \phi_*$.

(In other words the construction of the induced homomorphism is functorial).

Proposition. Let $\phi, \psi : \mathcal{C} \to \mathcal{D}$ be two chain maps. If $\phi \simeq \psi$, then $\phi_* = \psi_* : H_k(\mathcal{C}) \to H_k(\mathcal{D}) \quad \forall k$.

Proof. Let $[z] \in H_k(\mathcal{C})$. If h is a chain homotopy between ϕ and ψ, we have

$$\phi_k(z) - \psi_k(z) = \delta_{k+1}h_k(z) + h_{k-1}\gamma_k(z) = \delta_{k+1}(h_k(z)) \in \mathrm{Im}(\delta_{k+1})$$

(as $z \in \ker(\gamma_k)$).

Hence $[\phi_k(z)] = [\psi_k(z)]$ i.e. $\phi_*[z] = \psi_*[z]$.

Definition. A chain map $\phi : \mathcal{C} \to \mathcal{D}$ is a chain equivalence if $\exists\ \psi : \mathcal{D} \to \mathcal{C}$ s.t. $\psi\phi \simeq \mathrm{id}_{\mathcal{C}}$, $\phi\psi \simeq \mathrm{id}_{\mathcal{D}}$.

Proposition. If $\phi : \mathcal{C} \to \mathcal{D}$ is a chain equivalence, then the induced homomorphism

$$\phi_* : H_k(\mathcal{C}) \longrightarrow H_k(\mathcal{D})$$

is an isomorphism $\forall k$.

Proof. Using the functoriality of the induced map we see that

$$\psi_* \phi_* = (\psi\phi)_* = (\mathrm{id}_{\mathcal{C}})_* = \mathrm{id}_{H_k(\mathcal{C})}$$

(as $\psi\phi \simeq \mathrm{id}_{\mathcal{C}}$) and similarly $\phi_*\psi_* = \mathrm{id}_{H_k(\mathcal{D})}$. Hence ϕ_* is an isomorphism.

The above result tells us that the homology groups are chain homotopy invariants.

A contracting homotopy (or contraction) of the chain complex $\mathcal{C}$ is a chain homotopy h between $\mathrm{id}_{\mathcal{C}}$ and 0 (the zero chain-homomorphism). In other words we have a sequence $h = \{h_k\}$ of R-homomorphisms

$$h_k : C_k \longrightarrow C_{k+1}$$

s.t.

$$\gamma_{k+1}h_k + h_{k-1}\gamma_k = \mathrm{id}_{C_k} .$$

Proposition. If $\exists$ a contracting homotopy of $\mathcal{C}$, then $H_k(\mathcal{C}) = 0 \ \forall k$ (i.e. $\mathcal{C}$ is an exact sequence).

Proof. Let $[s] \in H_k(\mathcal{C})$. So $s \in \ker(\gamma_k)$ and we have

$$s = \gamma_{k+1} h_k(s) + h_{k-1}\gamma_k(s) = \gamma_{k+1}(h_k(s)) \in \operatorname{Im}(\gamma_{k+1}).$$

Therefore $[s] = 0$ in $H_k(\mathcal{C})$.

Example. A contracting homotopy for the chain complex

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r, "i_1"] & Z \oplus Z \arrow[r, "\pi_2"] \arrow[l, dashed, bend left, "h_2"] & Z \arrow[r] \arrow[l, dashed, bend left, "h_1"] & 0 \end{tikzcd}$$

is defined by setting

$$h_2 = \pi_1 \quad , \quad h_1 = i_2$$

Remark. It may happen that $\mathcal{C}$ is an exact sequence, but there is no contracting homotopy of $\mathcal{C}$.

Example. $\quad 0 \longrightarrow Z \xrightarrow{\times 2} Z \longrightarrow Z/2 \longrightarrow 0$.

4. The Fundamental Theorem of Homological Algebra.

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be three chain complexes and let

$$\phi : \mathcal{C} \longrightarrow \mathcal{D} \quad ; \quad \psi : \mathcal{D} \longrightarrow \mathcal{E}$$

be chain maps. We say that the sequence

$$(1) \qquad 0 \longrightarrow \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0$$

is a short exact sequence of chain complexes if

$$0 \longrightarrow C_k \xrightarrow{\phi_k} D_k \xrightarrow{\psi_k} E_k \longrightarrow 0$$

is a short exact sequence, $\forall k$. If (1) is a short exact sequence of chain complexes, we have the following commutative diagram, where each row is exact and each column is a chain complex.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_{k+1} \arrow[r, "\phi_{k+1}"] \arrow[d, "\gamma_{k+1}"] & D_{k+1} \arrow[r, "\psi_{k+1}"] \arrow[d, "\delta_{k+1}"] & E_{k+1} \arrow[r] \arrow[d, "\varepsilon_{k+1}"] & 0 \\
0 \arrow[r] & C_k \arrow[r, "\phi_k"] \arrow[d, "\gamma_k"] & D_k \arrow[r, "\psi_k"] \arrow[d, "\delta_k"] & E_k \arrow[r] \arrow[d, "\varepsilon_k"] & 0 \\
0 \arrow[r] & C_{k-1} \arrow[r, "\phi_{k-1}"] \arrow[d] & D_{k-1} \arrow[r, "\psi_{k-1}"] \arrow[d] & E_{k-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Remark. We get a sequence

$$H_k(\mathcal{C}) \xrightarrow{\phi_*} H_k(\mathcal{D}) \xrightarrow{\psi_*} H_k(\mathcal{E})$$

$\forall$ k. We want to put together all these sequences and construct a long exact sequence.

We start by defining a homomorphism

$$\Delta_k : H_k(\mathcal{E}) \longrightarrow H_{k-1}(\mathcal{C})$$

which is called the connecting homomorphism. Let $[s]\in H_k(\mathcal{E})$. This means that $s\in E_k$ and $\varepsilon_k(s)=0$. As ψ_k is onto, $\exists\ y\in D_k$ s.t. $\psi_k(y)=s$. We have

$$\psi_{k-1}\delta_k(y)=\varepsilon_k\psi_k(y)=\varepsilon_k(s)=0$$

i.e. $\delta_k(y)\in\ker\psi_{k-1}=\mathrm{Im}\,\phi_{k-1}$ and $\exists\ x\in C_{k-1}$ s.t. $\phi_{k-1}(x)=\delta_k(y)$. x is a cycle. In fact, in order to prove that $\gamma_{k-1}(x)=0$ it is enough to check that $\phi_{k-2}\gamma_{k-1}(x)=0$, as ϕ_{k-1} is 1-1. But

$$\phi_{k-2}\gamma_{k-1}(x)=\delta_{k-1}\phi_{k-1}(x)=\delta_{k-1}\delta_k(y)=0 \ .$$

So it makes sense to set

$$\Delta_k([s])=[x] \ .$$

Δ_k is well defined (exercise) and it is clearly a homomorphism.

<u>The Fundamental Theorem of Homological Algebra.</u>

The sequence

$$\cdots\xrightarrow{\phi_*} H_k(\mathcal{D})\xrightarrow{\psi_*} H_k(\mathcal{E})\xrightarrow{\Delta_*} H_{k-1}(\mathcal{C})\xrightarrow{\phi_*} H_{k-1}(\mathcal{D})\xrightarrow{\psi_*} H_{k-1}(\mathcal{E})\longrightarrow\cdots$$

is exact.

The above exact sequence is called the long exact sequence in homology, associated to the short exact sequence (1).

<u>Proof.</u> We have to check that

(a) $\mathrm{Im}(\psi_*)=\ker(\Delta_*)$.

(b) $\mathrm{Im}(\Delta_*)=\ker(\phi_*)$.

(c) $\mathrm{Im}(\phi_*)=\ker(\psi_*)$.

(a) <u>$\mathrm{Im}(\psi_*)\subseteq\ker(\Delta_*)$</u>. Let $[s]\in\mathrm{Im}(\psi_*)$. By definition $\exists\ y\in D_k$ s.t. $\delta_k(y)=0$ and $\psi_k(y)=s$. By looking at the construction of Δ_* we see that $\Delta_*[s]=[x]$, where $\phi_{k-1}(x)=\delta_k(y)$. But $\delta_k(y)=0$, and so $\phi_{k-1}(x)=0$, hence $x=0$ as ϕ_{k-1} is 1-1.

<u>$\mathrm{Im}(\psi_*)\supseteq\ker(\Delta_*)$</u>. Let $[s]\in\ker(\Delta_*)$. There exist $y\in D_k$, $x\in C_{k-1}$ s.t. $\phi_{k-1}(x)=\delta_k(y)$, $\gamma_{k-1}(x)=0$ and $\psi_k(y)=s$, and we have $\Delta_*[s]=[x]=0$. This means that x is a boundary, i.e. $\exists\ \bar{x}\in C_k$ s.t. $\gamma_k(\bar{x})=x$. Consider the element $y-\phi_k(\bar{x})\in D_k$. We have

$$\delta_k(y-\phi_k(\bar{x}))=\delta_k(y)-\delta_k\phi_k(\bar{x})=\delta_k(y)-\phi_{k-1}\gamma_k(\bar{x})=\delta_k(y)-\phi_{k-1}(x)$$
$$=\delta_k(y)-\delta_k(y)=0 \ .$$

So $y-\phi(\bar{x})$ is a cycle and it represents a class $[y-\phi_k(\bar{x})]\in H_k(\mathcal{D})$. Clearly

$$\psi_*[y-\phi_k(\bar{x})]=[\psi_k(y)-\psi_k\phi_k(\bar{x})]=[s] \ .$$

(b) <u>$\mathrm{Im}(\Delta_*)\subseteq\ker(\phi_*)$</u>. Let $\alpha\in H_{k-1}(\mathcal{C})$ and suppose that $\alpha\in\mathrm{Im}(\Delta_*)$. Then $\exists$ a representative x of α s.t. $\exists\ s\in E_k$ with $\varepsilon_k(s)=0$, $\exists\ y\in D_k$ s.t. $\psi_k(y)=s$ and $\delta_k(y)=\phi_{k-1}(x)$. We have

$$\phi_*(\alpha)=\phi_*[x]=[\phi_{k-1}(x)]=[\delta_k(y)]=0 \quad (\text{as } \delta_k(y) \text{ is a boundary})$$

and therefore $\alpha\in\ker(\phi_*)$.

<u>$\mathrm{Im}(\Delta_*)\supseteq\ker(\phi_*)$</u>. Let $[x]\in\ker(\phi_*)$. Then $[\phi_{k-1}(x)]=0$, i.e. $\exists\ y\in C_k$ s.t. $\phi_{k-1}(x)=\delta_k(y)$. Set $s=\psi_k(y)$. Then $\varepsilon_k(s)=\varepsilon_k(\psi_k(y))=$ $=\psi_{k-1}\delta_k(y)=\psi_{k-1}\phi_{k-1}(x)=0$, i.e. s is a cycle. Clearly $\Delta_*[s]=[x]$.

(c) <u>$\mathrm{Im}(\phi_*)\subseteq\ker(\psi_*)$</u>. Obvious: $\psi_*(\phi_*[x])=[\psi_k\phi_k(x)]=0$.

<u>$\mathrm{Im}(\phi_*)\supseteq\ker(\psi_*)$</u>. Let $[y]\in\ker(\psi_*)$. We have $y\in D_k$, $\delta_k(y)=0$ and $\psi_*[y]=[\psi_k(y)]=0$, i.e. $\psi_k(y)$ is a boundary. Hence $\exists\ s\in C_{k-1}$ s.t.

$\varepsilon_{k+1}(s) = \psi_k(y)$. As ψ_{k+1} is onto, $\exists\ \bar{y} \in D_{k+1}$ s.t. $\psi_{k+1}(\bar{y}) = s$. We have $\psi_k(y-\delta_{k+1}(\bar{y})) = \psi_k(y)-\psi_k\delta_{k+1}(\bar{y}) = \psi_k(y)-\varepsilon_{k+1}\psi_{k+1}(\bar{y}) = \psi_k(y)-\varepsilon_{k+1}(s) = \psi_k(y)-\psi_k(y) = 0$. So $y-\delta_{k+1}(\bar{y}) \in \ker(\psi_k) = \mathrm{Im}(\phi_k)$, i.e. $\exists\ x \in C_k$ s.t. $\phi_k(x) = y-\delta_{k+1}(\bar{y})$. We have $\phi_{k-1}\gamma_k(x) = \delta_k\phi_k(x) = \delta_k(y-\delta_{k+1}(\bar{y})) = \delta_k(y)-\delta_k\delta_{k+1}(\bar{y}) = 0$. Therefore $\gamma_k(x) = 0$, as ϕ_{k-1} is 1-1, and thus x is a cycle. Now we have

$$\phi_*[x] = [\phi_k(x)] = [y-\delta_{k+1}(\bar{y})] = [y]-[\delta_{k+1}(\bar{y})] = [y] .$$

In other words $[y] \in \mathrm{Im}(\phi_*)$.

5. Homology and cohomology with coefficients.

Let G be an abelian group and $\mathcal{C}$ be a chain complex of abelian groups. We set

$$\mathcal{C} \otimes G = \{ C_k \otimes G \ , \ \bar{\gamma}_k \}$$

where

$$\bar{\gamma}_k = \gamma_k \otimes \mathrm{id} : C_k \otimes G \longrightarrow C_{k-1} \otimes G .$$

Proposition. $\mathcal{C} \otimes G$ is a chain complex.

Proof. $\bar{\gamma}_k\bar{\gamma}_{k+1} = (\gamma_k \otimes \mathrm{id})(\gamma_{k+1} \otimes \mathrm{id}) = \gamma_k\gamma_{k+1} \otimes \mathrm{id} = 0$.

Definition. We set $H_k(\mathcal{C};G) = H_k(\mathcal{C} \otimes G)$.

$H_k(\mathcal{C};G)$ is the k-th homology group of $\mathcal{C}$ with coefficients in G.

Now we set

$$\mathcal{C}^*_G = \{ \mathrm{Hom}(C_n,G) \ , \ \gamma^*_n \}$$

where $\gamma^*_n : \mathrm{Hom}(C_{n-1},G) \longrightarrow \mathrm{Hom}(C_n,G)$

is defined by setting

$$\gamma^*_n(\phi) = \phi\gamma_n$$

Proposition. $\gamma^*_{n+1}\gamma^*_n = 0$.

Proof. $\gamma^*_{n+1}\gamma^*_n(\phi) = \gamma^*_{n+1}(\phi\gamma_n) = \phi\gamma_n\gamma_{n+1} = 0$.

We say that $\mathcal{C}^*_G$ is a cochain complex.

Definition. $H^n(\mathcal{C};G) = H^n(\mathcal{C}^*_G) = \ker(\gamma^*_{n+1})/\mathrm{Im}(\gamma^*_n)$.

$H^n(\mathcal{C};G)$ is the n-th cohomology group of $\mathcal{C}$ with coefficients in G.

We want to end with the statement of the Universal Coefficient Theorem for homology and cohomology, which explains the relation between the groups $H_k(\mathcal{C})$, $H_k(\mathcal{C};G)$, $H^k(\mathcal{C};G)$.

We need to introduce the notion of $\mathrm{Tor}(C,M)$, $\mathrm{Ext}(C,M)$ where C, M are two R-modules. We observe that it is possible to find two free modules A, B with $A \subseteq B$, and an epimorphism $\psi : B \twoheadrightarrow C$ s.t.

(1) $$0 \longrightarrow A \xhookrightarrow{i} B \xrightarrow{\psi} C \longrightarrow 0$$

is a short exact sequence.

Lemma. The sequence

$$A \otimes M \xrightarrow{i \otimes id} B \otimes M \xrightarrow{\psi \otimes id} C \otimes M \longrightarrow 0$$

is exact.

We point out explicitly that in general $i \otimes \mathrm{id}$ is not a monomorphism, and set

$$\mathrm{Tor}(C,M) = \ker[\, i \otimes \mathrm{id} : A \otimes M \longrightarrow B \otimes M \,] \ .$$

It can be easily checked that $\mathrm{Tor}(C,M)$ does not depend on the choice of A, B.

Now we start again with (1) and consider the sequence

(2) $$0 \longrightarrow \mathrm{Hom}(C,M) \xrightarrow{\psi^*} \mathrm{Hom}(B,M) \xrightarrow{i^*} \mathrm{Hom}(A,M)$$

Here $\psi^* : \mathrm{Hom}(C,M) \longrightarrow \mathrm{Hom}(B,M)$ is defined by setting $\psi^*(f) = f\psi$ $\forall\, f \in \mathrm{Hom}(C,M)$ and i^* is defined similarly.

Lemma. The sequence (2) is exact.

Again we remark that i^* is not onto. We set

$$\mathrm{Ext}(C,M) = \mathrm{coker}\, i^* = \mathrm{Hom}(A,M)/\mathrm{Im}(i^*) \ .$$

The definition of $\mathrm{Ext}(C,M)$ is independent of the choice of A,B.

We are ready for the Universal Coefficient Theorem for homology and cohomology.

Theorem. Let $\mathcal{C}$ be a free chain complex (i.e. C_n is free $\forall\, n$).

We have

(i) $H_n(\mathcal{C};G) \cong H_n(\mathcal{C}) \otimes G \oplus \mathrm{Tor}(H_{n-1}(\mathcal{C}),G)$

If either G is finitely generated or $H_k(\mathcal{C})$ is finitely generated $\forall\, k$, then

(ii) $H^n(\mathcal{C};G) = \mathrm{Hom}(H_n(\mathcal{C}),G) \oplus \mathrm{Ext}(H_{n-1}(\mathcal{C}),G)$.

Main reference

E.H. Spanier. Algebraic topology. Mc Graw Hill (1971) .