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COLLEGE

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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SOME BACKGROUND FOR GLOBAL ANALYSIS.

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Some background for global analysis.

James Eells November 1988.

I. Differentiation.

1. Let V and W be finite dimensional real vector spaces and

$$F : U \to W$$

a map of a domain $U \subset V$ into W. Its directional derivative at $x \in U$ in the direction $v \in V$ is

$$\left.\frac{d\,F(x+hv)}{dh}\right|_{h=0}.$$

Suppose these exist for all $x \in U$ and $v \in V$. Then the differential of F at $x \in U$ is a map

$$DF(x) : V \longrightarrow W.$$

2. If $DF(x)v$ is continuous in x for each $v \in V$, then $DF(x)$ is a linear map.

Thus $DF : U \to \mathcal{L}(V, W) =$ the vector space of all linear transformations $V \to W$.

3. If $U \xrightarrow{F} W \xrightarrow{G} X$, *then*

$$D(G \circ F) = DG \circ DF. \quad (\text{\underline{Chain rule}})$$

Example. If $V = \mathbb{R}^m$, $W = \mathbb{R}^n$, then the directional derivative of F at x in the direction $v = (0, \dots, 0, 1, 0, \dots, 0)$ is

$$\left(\frac{\partial F^1(x)}{\partial x^i}, \dots, \frac{\partial F^n(x)}{\partial x^i}\right).$$

And the right member in the chain rule is the matrix product

$$\left(\frac{\partial G^a}{\partial y^\alpha}\right)\left(\frac{\partial F^\alpha}{\partial x^i}\right).$$

4. Suppose that $\dim V = \dim W$ and that each $DF(x)$ is an isomorphism. Then *for each point* $x \in U$ *there is an open set* $U_x \subset U$ *which is mapped homeomorphically onto an open subset* $V_y \subset V$ *containing* $F(x) = y$. *Furthermore,*

$$DF^{-1} = (DF)^{-1}.$$

(*Inverse mapping theorem*. That is equivalent to the implicit function theorem.)

II. *Integration.*

5. We want to define the notion of p-dimensional integration in an m-dimensional vector space:

$$\int_C \omega,$$

a bilinear form with real values. In particular, the *integration domain* C should have an *orientation*, so

$$\int_{-C} \omega = -\int_C \omega.$$

An especially important integration domain is associated with the boundary bC of an oriented domain C. Then

$$bbC = 0.$$

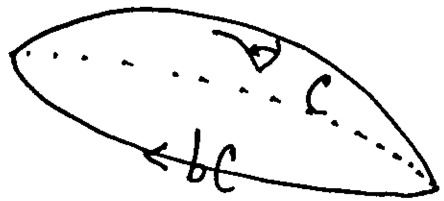
6. The nature of the *integrand* ω is essentially forced upon us by the requirements in (5):

Example. Take $V = \mathbb{R}^m$. For

$p = 1$: 

an oriented curve. The integrand ω is a linear combination of $dx^1, \dots dx^m$, where $dx^i \in \mathcal{L}(\mathbb{R}^m; \mathbb{R})$ is defined as 0 on all vectors of $\mathbb{R}^m$ except for real multiples of $(0, \dots, 0, \underset{(i^{\text{th}}\text{ place})}{1}, 0, \dots, 0)$.

For $p=2$:



The integrand ω is now a linear combination of $dx^i dx^j$ (formal multiplication subject to the anti-symmetry $dx^j dx^i = -dx^i dx^j$.

For generally, on an open $U \subset \mathbb{R}^m$ we form the integrands

$$\omega = \sum_{1 \le i_1 < \dots < i_p \le m} w_{i_1 \dots i_p}\, dx^{i_1} \dots dx^{i_p}$$

where each $w_{i_1 \dots i_p} : U \to \mathbb{R}$ is a function. (That construction has an intrinsic description as a map of U into the p^{th} exterior power of V.)

7. Its <u>differential</u> (or <u>exterior differential</u>)

$$d\omega = \sum (Dw_{i_1 \dots i_p})\, dx^{i_1} \dots dx^{i_p}.$$

Then from $\dfrac{\partial^2 w_{i_1 \dots i_p}}{\partial x^j \partial x^k} = \dfrac{\partial^2 w_{i_1 \dots i_p}}{\partial x^k \partial x^j}$ we conclude that

$$dd\omega = 0.$$

8. $$\int_C d\omega = \int_{bC} \omega$$ (<u>Stokes' theorem</u>).

9. If $F : \underset{\substack{\cap \\ \mathbb{R}^m}}{U} \longrightarrow \mathbb{R}^n$ is a map, we can define $F^*(dy^\alpha) = \sum_{i=1}^{m} \dfrac{\partial F^\alpha}{\partial x^i} dx^i$.

That induces a linear map F^* on integrands.

$$\int_{FC} \omega = \int_C F^* \omega$$ (<u>Transformation of integral formula</u>.)

III Analysis on manifolds.

10. Most of the concepts to be studied in this College are of a global nature. They require the main ideas of analysis, not just in domains of Euclidean spaces, but on manifolds — for instance, on spheres and projective spaces. The formulas in (I) and (II) above are in suitable form for immediate transfer to manifolds.

Example. The m-sphere

$$S^m = \{x \in \mathbb{R}^{m+1} : \sum_{i=1}^{m} (x^i)^2 = 1\}$$

can be covered by two open sets $O_+ = S^m -$ south pole, $O_- = S^m -$ north pole. Stereographic projection carries them to open sets U_+, U_- in $\mathbb{R}^m$. To see that 1-dimensional integration $\int_C \omega$ is well defined for curves on S^m, we can calculate pieces in U_+ and in U_-, and compare them using the transformation of integral formula.

Stokes theorem is valid on any manifold.

11. Let M be any manifold (e.g., an open $U \subset V$), and let $\mathcal{F}(M) =$ the vector space of all integrands of all dimensions. In fact, multiplication of integrands is defined, and makes $\mathcal{F}(M)$ an associative algebra.

The differential

$$d : \mathcal{F}(M) \to \mathcal{F}(M)$$

satisfies $dd \equiv 0$, so

$$\operatorname{Kernel} d \supset d\,\mathcal{F}(M).$$

The quotient $\operatorname{Kernel} d / d\,\mathcal{F}(M)$ is a topological invariant of M (de Rham's theorem), called the real cohomology algebra of M.

IV. The tangent bundle.

12. Let M be a manifold. Then each point $x \in M$ has neighbourhoods U

mapped homeomorphically to $\mathbb{R}^m$; write $\theta : U \to \mathbb{R}^m$. There are many such *charts*. If $\theta_i : U_i \to \mathbb{R}^m$ and $\theta_j : U_j \to \mathbb{R}^m$ are two such, then $\theta_j^{-1}\circ\theta_i$ is a map from one open set of M to another. Its differential at x

$$\lambda_{ji} = D(\theta_j^{-1}\circ\theta_i).$$

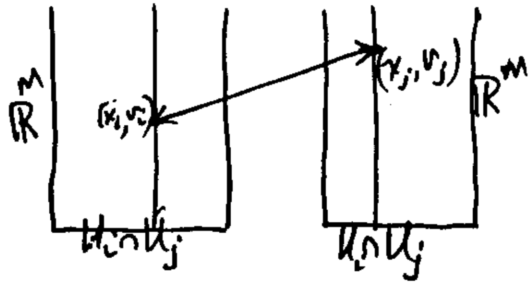
If $x \in U_i \cap U_j \cap U_k$, then

$$\lambda_{ij}(x) = \lambda_{ji}^{-1}(x);$$

$$\lambda_{ki}(x) = \lambda_{kj}(x)\,\lambda_{ji}(x).$$

$$\lambda_{ii}(x) = \text{identity map}.$$

These can be used to form an identification space of great importance: Let (U_i) be a covering of M by charts $\theta_i : U_i \to \mathbb{R}^m$. Form the topological union $\bigcup\{U_i \times \mathbb{R}^m\}$, and perform the identifications whenever $U_i \cap U_j \neq \emptyset$:



Say $(x_i, v_i) \sim (x_j, v_j)$ if $x_i = x = x_j$ and $\lambda_{ij}(x)\,v_j = v_i$. Then $\sim$ is an equivalence relation, and $T(M) = \bigcup\{U_i \times \mathbb{R}^m\}/(\sim)$ is a well defined space, called the *tangent vector bundle* of M. There is a natural map $\pi : T(M) \to M$ of $T(M)$ onto M, given by $\pi(x_i, v_i) = x$.

A key feature is that for every $x \in M$ the space $\pi^{-1}(x) = T_x(M)$ *is naturally an* m-*dimensional vector space*.