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COLLEGE

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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AN INTRODUCTION TO FIBRE BUNDLES.

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An introduction to fibre bundles

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§1 Real vector bundles

A *real vector bundle* ξ over a space B consists of a continuous map $\pi : E \to B$ such that for each $b \in B$, $\pi^{-1}(b)$ has the structure of a vector space. Moreover, the following *local triviality condition* must be satisfied: For each $b \in B$, $\exists$ a neighbourhood U, an integer $n \geq 0$ and a homeomorphism

$$U \times \mathbb{R}^n \xrightarrow{h} \pi^{-1} U$$

so that for each $b \in U$, $x \longmapsto h(b,x)$ is an isomorphism of vector spaces between $\mathbb{R}^n$ and $\pi^{-1}(b)$.

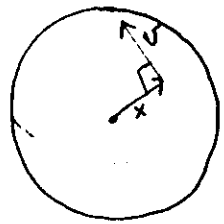
B is called the *base space*
E - - - *total space*
π - - - *projection map*
$F_b(\xi) = \pi^{-1}(b)$ - - - *fibre over b*
(U,h) - - a *local coordinate system* for ξ about b.
n is a locally constant function (which in most cases of interest is indeed constant).
One then speaks of an *n-dimensional bundle* or *an $\mathbb{R}^n$-bundle*

The simplest $\mathbb{R}^n$-bundle over B is the *product bundle* $B \times \mathbb{R}^n$.

Let us describe carefully two interesting examples.

Example 1. The tangent bundle of S^2.

S^2 is the unit sphere in $\mathbb{R}^3$, that is, $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2$ and a tangent vector v of S^2 at a point x satisfies $v \cdot x = 0$



So, the tangent bundle of S^2 can be described as a subspace of $S^2 \times \mathbb{R}^3$, namely,

$$\tau(S^2) = \{(x,v) \in S^2 \times \mathbb{R}^3 \mid x \cdot v = 0\}$$

In order to verify the local triviality conditions we need charts on S^2. Let $U = \{(x_1,x_2,x_3) \in S^2 \mid x_3 > 0\}$ be the north hemisphere and let $\varphi : U \to \mathbb{R}^2$ be the projection on the first two coordinates, $\varphi(x_1,x_2,x_3) = (x_1,x_2)$. Then (U,φ) is a chart. $\varphi(U) = \{(x_1,x_2) \mid x_1^2 + x_2^2 < 1\}$ is the open unit disc in $\mathbb{R}^2$. The inverse of φ, $\varphi^{-1} : \varphi(U) \to \mathbb{R}^3$, is given by $\varphi^{-1}(x_1,x_2) = (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$. φ^{-1} is a smooth map and its derivative is given by the matrix of partial derivatives

$$D\varphi^{-1}_{(x_1,x_2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x_1}{\sqrt{1-x_1^2-x_2^2}} & \frac{-x_2}{\sqrt{1-x_1^2-x_2^2}} \end{pmatrix} \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$$

which is of rank 2, and so its image is a plane in $\mathbb{R}^3$.
Moreover, this plane is infact orthogonal to $(x_1, x_2, \sqrt{1-x_1^2-x_2^2})$:

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \frac{-x_1}{\sqrt{1-x_1^2-x_2^2}} & \frac{-x_2}{\sqrt{1-x_1^2-x_2^2}} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\right]\cdot\left(x_1, x_2, \sqrt{1-x_1^2-x_2^2}\right) =$$

$$= \begin{pmatrix} u \\ v \\ \frac{-x_1 u}{\sqrt{1-x_1^2-x_2^2}} - \frac{x_2 v}{\sqrt{1-x_1^2-x_2^2}} \end{pmatrix}\cdot\left(x_1, x_2, \sqrt{1-x_1^2-x_2^2}\right)$$

$$= u x_1 + v x_2 - x_1 u - x_2 v = 0$$

So $h: U\times\mathbb{R}^2 \longrightarrow \pi^{-1}(U)$ defined by

$$h(x,w) = \left(\varphi^{-1}(x), D\varphi^{-1}_{(x)}(w)\right)$$

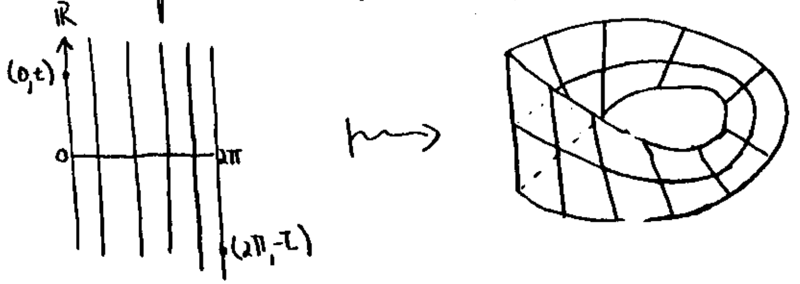
is a homeomorphism.

By choosing 5 more hemispheres we can actually see that $\tau(S^2)$ is locally trivial. More generally, the tangent bundle of a smooth manifold can be shown to be a vector bundle.

Example 2. The Moebius band.

The unit circle in $\mathbb{R}^2$ can be parametrized by $(\cos\theta, \sin\theta)$, where $0 \le \theta \le 2\pi$.

Let E be the quotient space of $[0,2\pi]\times\mathbb{R}$ obtained by identifying the left boundary $0\times\mathbb{R}$ with the right boundary $2\pi\times\mathbb{R}$ under the correspondence $(0,t)\longmapsto(2\pi,-t)$



Thus E is an open Moebius band and is the total space of an $\mathbb{R}$-bundle over S^1, the projection given by
$p[\theta,t] = (\cos\theta, \sin\theta)$.

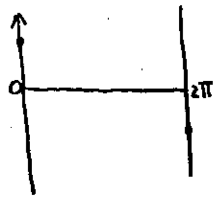
Two bundles ξ and η are *isomorphic*, $\xi\cong\eta$ if there exists a homeomorphism $f: E(\xi)\to E(\eta)$ such that for all $b\in B$ f maps $F_b(\xi)$ isomorphically onto $F_b(\eta)$. An n-bundle is *trivial* if it is isomorphic to a product bundle.

If the tangent bundle of a manifold is trivial, then the manifold is called parallelizable.

Sections

Let $\pi: E \to B$ be a vector bundle. A continuous map $s: B \to E$ with $\pi \circ s = 1$ is called a <u>section</u>. Every vector bundle possesses a canonical section, the <u>zero section.</u> This map is defined locally by using local coordinates systems $h: U \times \mathbb{R}^n \to \pi^{-1}(U)$, $(u, s(u) = h(u,0))$. Observe that since the 0 vector of any vector space is unique this is a well defined and continuous map.

A section $s: B \to E$ is <u>non-vanishing</u> if $s(b) \neq 0$ for all b. Not all vector bundles admit non-vanishing sections, for instance, a section of the Moebius band $s: S^1 \to B$ gives rise to a map $\bar{s}: [0,2\pi] \to \mathbb{R}$ with $\bar{s}(0) = \bar{s}(2\pi)$.



But then, the intermediate value theorem shows that for some θ_0, $\bar{s}(\theta_0) = 0$. Thus every section of the Moebius band vanishes at some point.

Sections of the tangent bundle of a manifold are called <u>vector fields</u>.

<u>Example</u>. S^n admits a vector field if n is odd.

$$\tau(S^n) = \{(x,v) \mid x \in S^n,\ v \in \mathbb{R}^{n+1} \text{ and } x \cdot v = 0\}$$

$$s(x_1, x_2, \ldots, x_{n+1}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{n+1}, -x_n).$$

The following result expresses trivial bundles in terms of sectio

<u>Theorem</u> An $\mathbb{R}^n$-vector bundle ξ is trivial $\Leftrightarrow$ ξ admits n every linearly independent functions.

Proof. $\Leftarrow$)

$$B \times \mathbb{R}^n \longrightarrow E(\xi)$$
$$(b, x) \longmapsto (x_1 s_1(b) + \cdots + x_n s_n(b))$$

where $x = (x_1, \ldots, x_n)$ and $s_1, \ldots s_n$ are linearly independent sections.

$\Rightarrow$) If ξ is trivial, then there is a local coordinate system $h: B \times \mathbb{R}^n \to E(\xi)$ and if $e_1, \ldots, e_n$ is the usual basis of then the everywhere linearly independent sections are given by $h(b, e_i)$, $i = 1, \ldots, n$. ■

<u>Example</u>. S^3 is parallelizable. It is enough to produce 3 vector fields everywhere line independent.

$$s_1(x) = (x_2, -x_1, x_3, -x_4)$$
$$s_2(x) = (x_3, -x_4, -x_1, x_2)$$
$$s_3(x) = (x_4, x_3, -x_2, -x_1).$$

Operations on vector bundles

Induced bundles

Let $\pi: E(\xi) \to B$ be an $\mathbb{R}^n$-bundle and let $f: X \to B$ be any continuous map. One can construct the *induced bundle* or *pull-back* $f^*(\xi)$ over X as follows:

Total space $\{(x, v) \in X \times E(\xi) \mid f(x) = \pi(v)\}$

projection $\pi'(x,v) = x$. One then has a commutative diagram

$$\begin{tikzcd} E(f^*(\xi)) \arrow[r, "\hat{f}"] \arrow[d, "\pi'"'] & E(\xi) \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

where $\hat{f}(x,v) = v$. The vector space structure on $(\pi')^{-1}(x)$ is defined by $\alpha_1(b,e_1) + \alpha_2(b,e_2) = (b, \alpha_1 e_1 + \alpha_2 e_2)$

$\hat{f}$ carries each fibre of $f^*\xi$ isomorphically onto some fibre of ξ.

Finally, if $U \times \mathbb{R}^n \xrightarrow{h} \pi^{-1}(U)$ is a local coordinate system of ξ, then

$$f^{-1}(U) \times \mathbb{R}^n \longrightarrow (\pi')^{-1}(f^{-1}(U))$$
$$(x, v) \longmapsto (x, h(f(x), v))$$ is a local coordinate system of $f^*(\xi)$.

It follows then that the pull-back of a trivial bundle is also trivial. An important theorem asserts that if $f \overset{\text{homotopic}}{\simeq} g$, then the pull backs are $f^*\xi$ and $g^*\xi$ are isomorphic. In particular, if f is homotopic to a constant map, then $f^*\xi$ is trivial.

A bundle map from η to ξ is a continuous map $g: E(\eta) \to E(\xi)$ which carries each fibre of η isomorphically onto one fibre of ξ. Such a g induces a map $\bar{g}$ of the base spaces

$$\begin{tikzcd} E(\eta) \arrow[r, "g"] \arrow[d] & E(\xi) \arrow[d] \\ B' \arrow[r, "\bar{g}"'] & B \end{tikzcd}$$

and it is not difficult to verify that $\bar{g}^*(\xi) \cong E(\eta)$.

Cartesian Products

If $\pi_1: E_1 \to B_1$ is an $\mathbb{R}^n$-bundle and $\pi_2: E_2 \to B_2$ is an $\mathbb{R}^m$-bundle, then $\pi_1 \times \pi_2 : E_1 \times E_2 \to B_1 \times B_2$ is an $\mathbb{R}^{n+m}$-bundle. The fibre $(\pi_1 \times \pi_2)^{-1}(b_1, b_2)$ is simply $\pi_1^{-1}(b_1) \times \pi_2^{-1}(b_2)$.

Example. If M, N are manifolds, then $\tau(M \times N) \cong \tau M \times \tau N$.

Whitney sums

If $\pi: E \to B$ is a bundle say ξ, then $\xi \oplus \xi$ is defined to be $\Delta^*(\xi \times \xi)$, where $\Delta: B \to B \times B$ is the diagonal map $\Delta(b) = (b,b)$. Observe that $F_b(\xi \oplus \xi) = F_b(\xi) \times F_b(\xi)$ which can be identified with $F_b(\xi) \oplus F_b(\xi)$

Given a vector bundle $\pi: E(\xi) \longrightarrow B$ and two local coordinate systems $h_\alpha: U_\alpha \times \mathbb{R}^n \longrightarrow \pi^{-1}(U_\alpha)$, $h_\beta: U_\beta \times \mathbb{R}^n \longrightarrow \pi^{-1}(U_\beta)$ with $U_\alpha \cap U_\beta \neq \emptyset$, then for each $b \in U_\alpha \cap U_\beta$ one has an isomorphism $g_{\beta\alpha}(b)$ of $\mathbb{R}^n$ defined by

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^n \xrightarrow{h_\alpha} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{h_\beta} (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

$$(b, x) \longmapsto (b, g_{\beta\alpha}(b)(x))$$

this assignment $b \longmapsto g_{\beta\alpha}(b)$ turns out to be a continuous map on b, ie $g_{\beta\alpha}: U_\alpha \cap U_\beta \longrightarrow GL(n, \mathbb{R})$, called *transition map.* If $\{U_\alpha\}$ is an open covering of B and $h_\alpha: U_\alpha \times \mathbb{R}^n \to \pi^{-1}(U_\alpha)$ are local coordinate systems for each α, then we obtain a set $\{g_{\beta,\alpha}: U_\alpha \cap U_\beta \longrightarrow GL(n, \mathbb{R})\}$ of transition functions.

These satisfy $g_{\beta\alpha} \cdot g_{\alpha\gamma} = g_{\beta\gamma}$ whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

In fact, one can recover $E(\xi)$ from the $\{g_{\beta\alpha}\}$ since $E(\xi)$ is homeomorphic to the space obtained by identifying in $\bigcup_\alpha (U_\alpha \times \mathbb{R}^n)$ all elements (b, x) in $U_\alpha \times \mathbb{R}^n$ with $(b, g_{\beta\alpha}(x))$ in $U_\beta \times \mathbb{R}^n$.

The Whitney sum is just one example for constructing new bundles out of old, but in differential geometry many other constructions play an important role. These are based on operations on vector spaces (functors). For example, to any pair of vector spaces one can assign:

1) the vector space $V \otimes W$
2) the vector space $\mathrm{Bil}^s(V \times V, W)$ of all symmetric bilinear transformations from $V \times V$ to W
3) the vector space $\mathrm{Hom}(V, W)$ of linear transformations from V to W

To a single vector space V one can assign

4) The dual $V^* = \mathrm{Hom}(V, \mathbb{R})$
5) the space $T^p(V)$ of all multilinear transformations $V \times \cdots \times V \longrightarrow \mathbb{R}$
6) The k-th exterior power $\Lambda^k(V)$ of all alternating multilinear transformations $V \times \cdots \times V \longrightarrow \mathbb{R}$

l these examples of <u>continuous functors on the category of vector</u> <u>ces and isomorphisms</u>.

instance, to each V one associates a vector space V^*, nd to each $V \xrightarrow{A} W$, one associates a transformation $^*: W^* \longrightarrow V^*$. The association $A \longmapsto A^*$ is continuous: once ses are chosen on V and W, if A is the matrix corresponding to the linear transformation, then the matrix * in the dual bases is simply the transpose of A. Therefore, e obtain a continuous map $(\ )^t: GL(\mathbb{R}, n) \longrightarrow GL(\mathbb{R}, n)$.

Given the n-bundle $\pi: E(\xi) \to B$ one can construct the dual dle $\pi^*: E(\xi^*) \to B$. The fibre over b is the dual space of $\pi^{-1}(b)$, $\pi^{-1}(b)^*$, nd the topology is given by considering local coordinate systems n $E(\xi)$, say $U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$, which induce a local coordinate system $U \times \mathbb{R}^n \longrightarrow (\pi^*)^{-1}(U)$. The transition maps are now given by considering the composites

$$U_\alpha \cap U_\beta \xrightarrow{g_{\beta\alpha}} GL(\mathbb{R}, n) \xrightarrow{(\ )^t} GL(\mathbb{R}, n).$$

In this manner one can construct bundles $\xi \otimes \eta$, $Bil^s(\xi \times \xi, \eta)$, $Hom(\xi, \eta)$, ξ^*, $T^p(\xi)$, $\Lambda^k(\xi)$, and so on.

Let us consider for instance $T^2(\xi)$. The fibre over each b is the space of real valued bilinear products. A section $\mu: B \longrightarrow T^2(\xi)$ satisfying $\mu(b)(\alpha,\beta) = \mu(b)(\beta,\alpha)$ for all $\alpha, \beta \in F_b(\xi)$ and $\mu(b)(\alpha,\alpha) > 0$ if $\alpha \neq 0$ corresponds to an inner product on each $F_b(\xi)$ which varies continuously on b. Such a section is called a <u>Euclidean metric on ξ</u>.

In the case of $\tau(M)$, a Euclid metric μ is called a <u>Riemannian metric</u>.

An important theorem asserts that <u>any v.b over a paracompact base admits Euclidean metrics</u>.

Consider two bundles ξ and η over the same base B with $E(\eta) \subset E(\xi)$, then <u>η is a subbundle of ξ</u> ($\eta \subset \xi$) if for each $b \in B$, $F_b(\eta)$ is a subspace (as vector spaces) of $F_b(\xi)$.

If $\eta \subset \xi$ and ξ is a Euclidean bundle, then we can construct the *orthogonal complement* of η in ξ, $\eta^\perp$, which satisfies $\eta \oplus \eta^\perp = \xi$. For example, if $f: M \to N$ is an embedding, and N is a Riemannian manifold, then $\tau(M) \subset f^*(\tau(N))$ and $\tau(M)^\perp$ is called the normal bundle of f, ν_f.

Let us now consider a single bundle ξ. A section of $\Lambda^k(\xi)$ is a map $s: B \to \Lambda^k(\xi)$ which associates to each b an alternating multilinear map $F_b(\xi) \times \cdots \times F_b(\xi) \to \mathbb{R}$. These sections are called *k-forms* and play an important role in integration theory on manifolds ($\xi = \tau(M)$)

§3 Universal bundles

Let $M^k \subset \mathbb{R}^{k+1}$ be an hypersurface. Then the *Gauss map*: $M \to S^k$ is defined by associating to each $p \in M$ the unit outward (inward) normal vector at p, $n(p)$

If $M^k \subset \mathbb{R}^{n+k}$, a *generalized Gauss map* can be defined by associating to each $p \in M$ the k-plane in $\mathbb{R}^{n+k}$ through the origin parallel to $\tau_p(M)$. One must then think of k-planes through the origin as points of a new space.

The *Grassmann manifold* $G_k(\mathbb{R}^{n+k})$ is the set of all k-planes through the origin in $\mathbb{R}^{n+k}$. It is topologized as a quotient space.

A k-frame of $\mathbb{R}^{n+k}$ is k-tuple of orthonormal vectors of $\mathbb{R}^{n+k}$. The set of all k-frames $V_k(\mathbb{R}^{n+k})$ can be identified with some subspace $\mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$. $V_k(\mathbb{R}^{n+k})$ is called the *Stiefel manifold of k frames of* $\mathbb{R}^{n+k}$. There is a natural function $V_k(\mathbb{R}^{n+k}) \to G_k(\mathbb{R}^{n+k})$

...o give $G_k(\mathbb{R}^{n+k})$ the quotient topology: U is open in $G \iff$ ...(U) is open in $V_k(\mathbb{R}^{n+k})$

...orem. (i) $G_k(\mathbb{R}^{n+k})$ is a compact $(n+k)$-dimensional manifold.

(ii) $G_k(\mathbb{R}^{n+k}) \cong G_n(\mathbb{R}^{n+k})$.

...etch proof. (See below)

...canonical vector bundle $\gamma^k(\mathbb{R}^{n+k})$ over $G_k(\mathbb{R}^{n+k})$ is ...nstructed as follows.

$$E(\gamma^k(\mathbb{R}^{n+k})) = \{(k\text{-plane in } \mathbb{R}^{n+k}, \text{ a vector in that } k\text{-plane})\}$$

$$E(\gamma^k(\mathbb{R}^{n+k})) \subset G_k(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$$

In particular, $G_1(\mathbb{R}^{n+1})$ is the set of lines of $\mathbb{R}^{n+1}$ through the origin and is called n-projective space $\mathbb{R}P^n$. And $E\gamma^1(\mathbb{R}^2)$ is the Moebius band.

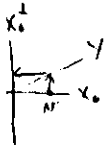
...etch proof

...et $X_0 \in G_k(\mathbb{R}^{n+k})$ and regard $\mathbb{R}^{n+k} \cong X_0 \oplus X_0^\perp$, $\begin{cases} P_2: X_0\oplus X_0^\perp \to X_0^\perp \\ P_1: X_0 \oplus X_0^\perp \to X_0 \end{cases}$ the orthogonal proj.

...et $U = \{Y \in G_k(\mathbb{R}^{n+k}) \mid P_1(Y) = X_0\} = \{Y \in G_k(\mathbb{R}^{n+k}) \mid Y \cap X_0^\perp = 0\}$

$Y \in U$ can be considered as the graph of the linear transformation

$$T(Y): X_0 \xrightarrow[(P_1)^{-1}]{\cong} Y \xrightarrow[P_2]{} X_0^\perp \Rightarrow T: U \cong \mathrm{Hom}(X_0, X_0^\perp) \cong \mathbb{R}^{nk}$$



Compact – $V_k(\mathbb{R}^{n+k}) \subset S^{n+k-1} \times \cdots \times S^{n+k-1}$ is already compact.

To verify that $\gamma^k(\mathbb{R}^{n+k})$ is locally trivial one considers a neighbourhood U of X_0 as before. Then a local coordinate system can be defined by

$$h: U \times X_0 \longrightarrow \pi^{-1}(U)$$

where $h(Y, x) = (Y, y)$ where $y \in Y$ is the unique vector with $P_1(y) = x$ $\quad (P_1|: Y \xrightarrow{\cong} X_0$, in fact $y = x \oplus T(Y)x)$

We will show now that Gauss maps can be further generalized:

Proposition For any n-plane bundle ξ over a compact space B there exists a bundle map $\xi \to \gamma^n(\mathbb{R}^{n+k})$ provided k is sufficiently large.

Proof. It is enough to produce a map $\bar{f}: E(\xi) \to \mathbb{R}^N$ which is linear and injective on each $F_b(\xi)$, since the required fibre map $f: E(\xi) \to \gamma^n(\mathbb{R}^N)$ is then defined by

$$f(e) = (\bar{f}(F_b(\xi)), \bar{f}(e))$$

To produce $\bar{F}$ choose open sets $U_1, \dots, U_r$ covering B so that $\xi|U_i$ is trivial. Since B is normal there exist open sets V_i, W_i with $W_i \subset \bar{W}_i \subset V_i \subset \bar{V}_i \subset U_i$ such that $\{W_i\}$ still cover B. There are maps $\lambda_i : B \to \mathbb{R}$ so that

$$\lambda(x) = \begin{cases} 1 & \text{if } x \in \bar{W}_i \\ 0 & \text{if } x \notin V_i \end{cases}$$

Let $g_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the projection.

Define $g_i' : E(\xi) \to \mathbb{R}^n$ by $g_i'(e) = \begin{cases} 0 & \text{if } e \notin \pi^{-1}(U_i) \\ \lambda_i(\pi(e)) \cdot g_i(e) & \text{if } e \in \pi^{-1}(U_i) \end{cases}$

Then g_i' is linear on each fibre. Now define $\bar{F}$ by

$$\bar{F}(e) = (g_1'(e), \dots, g_r'(e)) \in \mathbb{R}^n \oplus \dots \oplus \mathbb{R}^n \cong \mathbb{R}^{rn}.$$

In order to take care bundles over more exotic spaces it is necessary to consider infinite Grassmann manifolds $G_n(\mathbb{R}^\infty)$

$$G_n(\mathbb{R}^n) \subset G_n(\mathbb{R}^{n+1}) \subset G_n(\mathbb{R}^{n+2}) \subset \dots$$

$$G_n(\mathbb{R}^\infty) = \bigcup_{k \geq 0} G_n(\mathbb{R}^{n+k})$$ with the limit topology.

U open in $G_n(\mathbb{R}^\infty) \iff U \cap G_n(\mathbb{R}^{n+k})$ open in $G_n(\mathbb{R}^{n+k})$ for all k.

$\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$, $\quad \mathbb{R} \subset \mathbb{R}^2 \subset \dots$ is a vector space.

$\mathbb{R}^\infty$ is also given the limit topology. $G_n(\mathbb{R}^\infty)$ is then the space of all n-planes through the origin in $\mathbb{R}^\infty$.

As a special case we obtain the *infinite projective space* $\mathbb{R}P^\infty = G_1(\mathbb{R}^\infty)$.

A *canonical n-plane bundle γ^n over $G_n(\mathbb{R}^\infty)$* is obtained as in the finite dimensional case: $E(\gamma_n)$ is a subspace of $G_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty$, namely, $\{(n\text{-plane in } \mathbb{R}^\infty, \text{vector in that plane})\}$.

The previous propn. can be generalized as follows:

Theorem: Any n-bundle ξ over a paracompact space admits a bundle map $\xi \to \gamma^n$.

Recall that a bundle map is

$$\begin{array}{ccc} E(\xi) & \longrightarrow & E(\gamma^n) \\ \downarrow & & \downarrow \\ B & \xrightarrow[f_\xi]{} & G_n(\mathbb{R}^\infty) \end{array}$$

and that $E(\xi) \cong f_\xi^*(\gamma^n)$.

So the theorem asserts that any n-bundle over a paracompact space is the pull-back of the *"universal" bundle* γ_n. $G_n(\mathbb{R}^\infty)$ is called *the classifying space for n-bundles* and f_ξ is called a *classifying map for ξ*. In fact,

Theorem Two n-bundles ξ, η over a paracompact space B are isomorphic if and only if $f_\xi \simeq f_\eta$.

The two theorems imply that there is a bijection between the set of all isomorphism classes of v.b over B and the set of homotopy classes of maps $[B, G_n(\mathbb{R}^\infty)]$