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COLLEGE

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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LIE GROUPS AND LIE ALGEBRAS.

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These notes are meant as a supplement to the four lectures on Lie groups and Lie algebras mini-course. No attempt of originality has been made. The basic reference is [Adams]. The subject of representations hasn't been touched except for the adjoint and the low dimensional identifications in Chapter I.

I am sure these notes are full of mistakes and misprints, which the reader is kindly asked to forgive and try to correct.

Campinas, November 19, 1988

GENERALITIES, DEFINITIONS AND EXAMPLES

1. Definition: A real Lie group G is a real C^∞ manifold with a group structure, such that the map $(x,y) \to xy^{-1}$ from $G \times G \to G$ is C^∞.

2. Exercise: (1) It follows then that both maps $x \to x^{-1}$ and $(x,y) \to xy$ are C^∞.

(2) The connected component of the identity element of G is also a Lie group and all components are mutually diffeomorphic.

3. Definition: Two Lie groups are isomorphic iff there exists a C^∞ group isomorphism between them.

4. Examples: (i) $(\mathbb{R}^n, +)$

(ii) $\mathbb{C} - \{0\}$ under multiplication.

(iii) Let $S^1 = \{z \text{ in } \mathbb{C} \text{ with } |z| = 1\}$. Then S^1 is a subgroup of (ii).

(iv) The algebra of quaternions $\mathbb{H}$ is isomorphic to $\mathbb{C} \oplus \mathbb{C}$ as a vector space and is given the following associative, non-commutative product: $(z,w) \equiv z + jw$ with $wj = j\bar{w}$, $j^2 = -1$,

So $(z + jw)(z_1 + jw_1) = (zz_1 - \bar{w}w_1) + j(\bar{z}w_1 + wz_1)$.

We can also consider it as a product in R^4 with basis 1, i, j and $ij \equiv k$ and the following multiplication table: 1 commutes with everything, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$.

The conjugate of $x \equiv x_o + x_1 i + x_2 j + x_3 k$ is $\bar{x} = x_o - x_1 i - x_2 j - x_3 k$ and it is easy to see that $x\bar{x} = |x|^2$.

So $x^{-1} = \frac{\bar{x}}{|x|^2}$, for all x in $\mathbb{H} - \{0\}$, $\mathbb{H} - \{0\}$ is a Lie group and S^3, the unit quaternions, is a compact subgroup.

Show that $|xy| = |x||y|$.

(v) $Gl(n, \mathbb{R}) = \{A \text{ real } n \times n \text{ matrix with } \det(A) \neq 0\}$

(vi) All upper-triangular, non-singular, matrices.

(vii) $0(n) = \{A \text{ in } Gl(n,\mathbb{R}) \mid A^{tr} = A^{-1}\}$, A^{tr} means the transpose of the matrix A.
Show that 0(n) is a compact lie group with 2 connected components.
$S0(n) = \{A \text{ in } 0(n) \text{ with } \det(A) = 1\}$ is the component of I in 0(n).

(viii) $Gl(n, \mathbb{C}) = \{B \text{ complex } n \times n \text{ matrix with } \det(B) \neq 0\}$

(ix) $U(n) = \{B \text{ in } Gl(n,\mathbb{C}) \mid B^* = B^{-1}\}$
Where $B^* = \bar{B}^{tr}$, ie, the entries of B^* are the conjugates of the entries of the transpose matrix of B.
$SU(n) = \{B \text{ in } U(n) \mid \det(B) = 1\}$

Show that U(n) is diffeomorphic as a manifold with $S^1 \times SU(n)$. We will see later that they are not isomorphic as Lie groups).

Let scalars act on the right on column vectors in $\mathbb{H}^n$ now:

(x) $Sp(n) = \{C \text{ in } Gl(n, \mathbb{H}) \mid CC^* = C^*C = I\}$

(**Exercise**: Show that S^3 above is isomorphic to $SU(2)$ and to $Sp(1)$)

(xi) Given two Lie groups G_1 and G_2 the product $G_1 \times G_2$ can be given the product structure $(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2)$ and this turns $G_1 \times G_2$ in to a Lie group.

There are, however, other possible Lie group structures on a product manifold:

5. **Exercise**: Define the Lie group of affine motions of $\mathbb{R}^n$.

6. **Definition**: A Lie subgroup H of G is a subgroup which is also a submanifold of G. Let now $\mathbb{C}$ be included in $M_2(\mathbb{R})$, the algebra of real 2 x 2 matrices, and $\mathbb{H}$ in $M_2(\mathbb{C})$ as follows: [Curtis]

$$(x_0 + x_1 i) \longmapsto \begin{pmatrix} x_0 & -x_1 \\ x_1 & x_0 \end{pmatrix}, \quad x_0, x_1 \text{ in } \mathbb{R}$$

and

$$(z + jw) \longmapsto \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}, \quad z, w \text{ in } \mathbb{C}$$

7. **Exercise**:

1) Show that these are ring inclusions and use them to define ring inclusions ϕ of $M_{2n}(\mathbb{C})$ into $M_{2n}(\mathbb{R})$ and of $M_n(\mathbb{H})$ into $M_{2n}(\mathbb{C})$, where $M_n(K)$ denotes the n x n matrices with entries in K. Show that ϕ restricts to the respective $Gl(n, K)$ ' s.

Show $\phi(A^*) = \phi(A)^*$ and that $\phi(U(n)) \subsetneq SO(2n)$, $\phi(Sp(n)) \subsetneq U(2n)$.

2) Show that S^1 is Lie group isomorphic to $SO(2)$.

An example of an infinite dimensional Lie group is the following:

Let G be a (finite dimensional) Lie group and consider the set of all C^∞ functions $f: S^1 \to G$ that satisfy $f(1) = e$ the unit element of G. Introduce the obvious product of two such functions: $f.g\,(x) = f(x)g(x)$ where we consider the product of G on the right. This group is usually denoted by $\Omega(G)$ and is called the "loop group of G" ([Bott], [Pressler + Segal], [Eells-Lemaire]).

On K^n (K being $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) we define the following scalar product:

$$\langle x,y \rangle := \sum_i \bar{x}_i y_i \quad \text{in } K.$$

Then we observe that $\langle , \rangle$ is bi-additive with $\langle x\lambda, y\rangle = \bar{\lambda}\langle x,y\rangle$, $\langle x, y\lambda\rangle = \langle x,y\rangle\lambda$ and $\overline{\langle x,y\rangle} = \langle y, x\rangle$, for all x,y in K^n and all λ in K.

Conjugation here is the usual (anti) automorphism of K : the identity for reals and $\overline{x_0 + x_1 i + x_2 j + x_3 k} = x_0 - x_1 i - x_2 j - x_3 k$ for $\mathbb{C}$ and $\mathbb{H}$.

Observe that the above product is non degenerate and that it defines a norm on K^n

by $\quad |x| = \langle x, x\rangle^{1/2}$.

For A, B in $M_n(K)$, $(AB)^* = B^*A^*$,

$\langle Ax, y\rangle = \langle x, A^* y\rangle$ and A in $M_n(K)$ preserves the scalar product in K^n in the sense $\langle Ax, Ay\rangle = \langle x,y\rangle$ for all x,y in K^n iff $A^*A = I$.

In the case $K \equiv \mathbb{H}$, using the inclusion of $Gl(n, \mathbb{H})$ in $Gl(2n, \mathbb{C})$ we see that $A^*A = I$ implies $AA^* = I$ too. For $K = \mathbb{C}$ or $\mathbb{R}$ this is immediate.

The above identifies $O(n)$, $U(n)$ and $Sp(n)$ with the groups of (invertible) linear operators which preserve the respective scalar products on $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$.

8. Exercise: i. A in $O(n)$, $U(n)$ or $Sp(n)$ iff the columns (and rows) of A form an orthonormal basis of the respective K^n, iff $|Ax| = |x|$ for all x in K^n.

ii. Show that A in $O(n)$ is the product of, at most, n reflections (Hint: Use induction).

Recall that the vector space $\mathfrak{X}(M)$ of C^∞ vector fields X on a manifold M (derivations of the ring of C^∞ functions on M) under the bracket operation $[x,y](f) := x(yf) - y(xf)$ satisfies:

(i) $[x,y] = -[y,x]$ (anticommutativity)

(ii) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ (Jacobi identity)

9. Definition: An algebra over $\mathbb{R}$ or $\mathbb{C}$ is a "Lie algebra" iff it satisfies (i) and (ii) above. Observe that (ii) measures in a way, the deviation of a Lie algebra from being associative. The example above is an infinite dimensional Lie algebra. Here are some more

10. Examples:

i. Let V be a real , resp. complex, vector space with the trivial bracket: $[x,y] = 0$, for all x,y in V. Such a V is called a commutative Lie Algebra.

ii. R^3 with the vector product: $[x, y] \equiv x \times y$.

iii. The algebra of real or complex $n \times n$ matrices gives rise to a Lie algebra with bracket $[A, B] := AB - BA$ where AB is the usual product of matrices.

iv. Given any associate algebra Ω over $\mathbb{R}$ on $\mathbb{C}$ we can form a Lie algebra by defining $[x,y] = xy - yx$ for all x, y in Ω.

v. Exercise: A derivation of an algebra Ω over $\mathbb{R}$ or $\mathbb{C}$, is a linear map

$$D: \Omega \to \Omega \quad \text{satisfying}$$

$$D(xy) = (Dx)y + x(Dy).$$

Show that the set of all derivations of Ω, $Der(\Omega)$, is a Lie algebra with product $[D, D'] = DD' - D'D$.

11. Definition: A morphismof Lie algebras is a linear map that commutes with the bracket.

12. Exercise: Show that for a Lie algebra Ω we have a linear map $ad: \Omega \to Hom(\Omega, Der\ \Omega)$ defined by $x \to (:y \to [x,y])$,

i.e., $ad_x(y) := [x,y]$,

where the vector space on the right ccnsists of all Lie algebra morphisms.

Some notable subalgebras of the above are:

i. The skew symmetric matrices A in $M_n(\mathbb{R})$ with $A+A^{tr}=0$. This subalgebra is denoted by $\hat{SO}(n)$.

ii. $\hat{U}(n) = \{B \text{ in } M_n(\mathbb{C}) \mid B + B^* = 0\}$

iii. $\hat{SU}(n) = \{B \text{ in } \hat{U}(n) \mid tr(B) = 0\}$, where (tr(B) is the trace of the matrix B).

iv. $\hat{S}_p(n) = \{C \text{ in } M_n(H) \mid C + C^* = 0\}$

v. An example of an infinite dimensional Lie algebra is the following: Let $\hat{G}$ be any finite dimensional Lie algebra (over $\mathbb{R}$ or $\mathbb{C}$) and consider all smooth (C^∞) maps $f : S^1 \to \hat{G}$ with $f(1) = 0$.
Define $[f,g](x) := [f(x), g(x)]$ for all x in S^1 where the bracket on the right is the one in $\hat{G}$.
This Lie algebra is denoted by $\Omega(\hat{G})$.

Back to Lie groups now: If G is a Lie group for any g in G we define "left translation by g" to be the diffeomorphism of G

$$L_g(h) := gh.$$

Similarly, right translation $R_g(h) := hg$.

12. <u>Definition</u>: A vector field X in G is called "left invariant" iff $dL_g(X_h) = X_{gh}$, for short, $dL(X) = X$. We define "right invariant" vector fields, similarly.

13. <u>Exercise</u>: (i) A left invariant vector field X is always smooth.

(ii) The set of all left invariant vector fields on G form a vector space with dimension equal to dim G. Call this space L(G).

(iii) L(G) in $\mathfrak{X}(G)$ is closed under the bracket of vector fields, forming a Lie subalgebra of the algebra $\mathfrak{X}(G)$.

(iv) L(G) can be identified with T_eG, the tangent space of G at the identity, and it provides a global section of the tangent bundle TG, implying that G is parallelizable. Sometimes we denote T_eG by $\hat{G}$.

(Note: The connected, simply connected finite dimensional Lie groups are determined, up to isomorphism, by their Lie algebras. We will comment more on this theorem of E. Cartan later).

14. <u>Exercise</u>: i. Let $\{X_1, \ldots, X_n\}$ be a basis for the Lie algebra L(G) and let the constants C_{ij}^k be defined by

$$[X_i, X_j] = \sum_{k=1}^{n} C_{ij}^k X_k$$

Show that $C_{ij}^k + C_{ji}^k = 0$, corresponding to anticommutativity of the bracket, and find the relation imposed on C_{ij}^k'S by the Jacobi identity.

The C_{ij}^k'S are called the structural constants of G.

ii. A form w in $\Lambda(G)$ is called "Left invariant iff $\delta L_g(w) = w$ for all g in G.

Show that left invariant forms are C^∞ and that they form a subspace of $\Lambda(G)$, which is a real subalgebra.

Calculate

$$\dim \Lambda^m_{\text{left inv.}}(G), \quad \text{all } m.$$

Is this a differential subalgebra, i.e., closed under d?

If w in $\Lambda^1_{\text{left inv.}}(G)$, x, y in $L(G)$,

Show that $dw(x,y) = -w[x,y]$.

Show that δ is a contravariant functor on left invariant forms.

If $\{w^1, \ldots, w^n\}$ is the dual basis of $\{X_1, \ldots, X_n\}$ above, then show that

$$dw^i = \sum_{j<k} C^i_{jk}\, w^k \wedge w^j$$

(Maurer - Cartan equations).

15. Definition: A homomorphism $\phi: G \to Aut(V)$, where V is a vector space and Aut(V) is the space of the automorphisms of V, is called a "representation of G on V". In particular Aut(V) may be $Gl(n, \mathbb{C})$ or $Gl(n, \mathbb{R})$.

A Lie algebra morphism $\psi: \hat{G} \to \hat{H}$ is called a Lie algebra representation iff $\hat{H} = End(V)$: The Lie algebra of endomorphisms of the vector space V.

16. Proposition: If $\phi: G \to H$ is a Lie group morphism, then $d\phi: L(G) \to L(H)$ is a Lie algebra morphism.

Proof: Since $\phi(gg') = \phi(g)\,\phi(g')$ implies $dL_{\phi(g)} \circ d\phi(x) = d\phi \circ dL_g(x)$ for g, g' in G, X in $\mathfrak{X}(G)$, one has

that to each (left invariant) X in L(G) corresponds a $\tilde{X} \equiv d\phi(X)$ defined only on Im (G) $\subseteq$ H and extendable on the whole of H by left invariance.

Now X and $\tilde{X}$ are ϕ - related, i.e. $d\phi(X) = \tilde{X}$, and since the brackets of ϕ - related vector fields are ϕ - related we get

$$[x, y]^\sim = [\tilde{x}, \tilde{y}]$$

Q.E.D.

17. Definition: A one-parameter subgroup of G is the image of a C^∞ homomorphism

$$\phi: (\mathbb{R}, +) \to G.$$

18. Example: Let M be a C^∞ manifold and D(M) the group of diffeomorphisms of M under composition. To each one parameter subgroup F of D(M) associate a C^∞ vector field X in $\mathfrak{X}(M)$ by

$$(Xf)_m := \frac{d}{dt}\Big|_{t=0} (f(F_t(m)) - f(m))/t$$

for any real smooth function f on M. We say that X generates F.

Conversely, if X in $\mathfrak{X}(M)$ vanishes outside a compact set in M, then it generates a unique one parameter subgroup of D(M), by

$$\dot{F}_t(m) = X(F_t(m))$$

$$F_o(m) = m.$$

The restriction on X is imposed to guarantee that F is defined on the whole real line. We can define local one parameter subgroups of D(M) as well.

An interesting fact is the following: If X in $\mathfrak{X}(M)$ generates F and if Y is in $\mathfrak{X}(M)$, then

$$[X, Y]_m = \frac{d}{dt}\Big|_{t=0} (dF_{-t})(Y(F_t(m)).$$

18a. Exercise: Express the one parameter subgroup of D(G) generated by a left invariant vector field X in terms of translations.

19. Proposition: There is a 1-1 correspondence between one-parameter subgroups of G and integral curves of left invariant vector fields through e.

Proof: Given X in L(G) let $\gamma_X(t) \equiv \gamma(t)$, $-\varepsilon < t < \varepsilon$ be defined by the solution of the 1^{st} order D.E. : $X_{\gamma(y)} = \dot{\gamma}(t)$ and $\gamma(0) = e$.

Observe that $\frac{d}{dt}\Big|_{t=0} \gamma(s).\gamma(t) = dL_{\gamma(s)}\dot{\gamma}(0) = dL_{\gamma(s)} X(e) =$

$= X(\gamma(s))$ and also

$$\frac{d}{dt}\Big|_{t=0} = (\gamma(s+t) = \dot{\gamma}(s) = X(\gamma(s))$$

and that both curves $\gamma(s).\gamma(t)$ and $\gamma(s+t)$ go through $\gamma(s)$ for $t = 0$. This shows they are equal for all s, t, in $(-\varepsilon, \varepsilon)$ and that γ is defined for all t in $\mathbb{R}$ and is a one-parameter subgroup.

Given now a one-parameter subgroup $\gamma(s)$ let $X(\gamma(s)) \equiv \dot{\gamma}(s)$ for all s in $\mathbb{R}$. To show that X extends to a left invariant vector field on G it is enough to show $X(\gamma(s)\gamma(t)) = dL_{\gamma(s)} (X(\gamma(t)))$.

But

$$X(\gamma(s)\gamma(t)) = X(\gamma(s+t)) = \dot{\gamma}(s+t) =$$

$$= \frac{d}{d\tau}\Big|_{\tau=0} \gamma(s+t+\tau) = \frac{d}{d\tau}\Big|_{\tau=0} L_{\gamma(s)} (\gamma(t+\tau)) =$$

$$= dL_{\gamma(s)} (\dot{\gamma}(t)) = dL_{\gamma(s)} (X(\gamma(t))).$$

Q E D.

20. Definition: (The exponential map) Define

$\exp: T_eG \to G$, by

$\exp(tv) := \phi_v(t)$

Where ϕ_v is the one-parameter subgroup that corresponds to the left invariant vector field V defined by v. Equivallently we can write $e^v = \phi_v(1)$.

21. Theorem: exp is a C^∞ map.

Proof: $\phi_v(t)$ is the solution to the differential equation

$$\phi_v'(t) = v(\phi_v(t)) = dL_{\phi_v(t)}(V_e).$$

But $L_g(v)$ is C^∞ in g and v and consequently, so is the solution .

Q E D.

22. Corollary: (1) $d(\exp)_o = id : T_e G \to T_e G$,

Which implies that exp is a diffeomorphism from a neighborhood $U_o \subseteq T_e G$ on to a neighborhood $W_e \subseteq G$.

(2) If $f: G \to H$ is a Lie group morphism

then, $f \circ \exp_G = \exp_H \circ df$.

(Observe that both $\phi(e^{tx})$ and $e^{td\phi(x)}$ are one-parameter subgroups with initial tangent $d\phi(x)$.)

23. Examples of exponential maps.

i. $\mathbb{R} \to S^1$, where $\mathbb{R} \equiv$ Imaginary complex numbers, defined by $\theta \mapsto e^{2\pi i\theta}$

ii. ($\mathbb{R}^3$, vector product) $\to S^3$, the same map as in i:
Choose cylindrical coordinates in $\mathbb{R}^3 - \{0\}$: (θ, J) with θ in $\mathbb{R}$ and $J^2 = -1$ in S^2, let $(\theta, J) \mapsto e^{\theta J} :=$ $:= \cos\theta + \sin\theta J$.
See what happens along each ray, θJ_0, and compare with example i.

iii. $V = \mathbb{R}^n$ or $\mathbb{C}^n$, $G = \mathrm{Aut}(V)$ open in $\mathrm{Hom}(V,V)$.
Identify T_eG with $\mathrm{Hom}(V,V)$ and for A in $\mathrm{Hom}(V,V)$ let

$$e^{tA} = I + tA + \frac{t^2A^2}{2!} + \dots + \frac{t^nA^n}{n!} + \dots$$

Observe that this is a C^∞ homomorphism from $(\mathbb{R}, +)$ to $\mathrm{Aut}(V)$, i.e., a one-parameter subgroup with tangent vector at $t = 0$ equal to A. So in all matrix groups we have $e^A = \sum_0^\infty \frac{A^n}{n!}$, where $A^0 = I$, i.e., the usual matrix exponential. (Show convergence).
Similarly, for any finite dimensional vector space V, $\exp : \mathrm{End}(V) \to \mathrm{Aut}(V)$ is given by the usual exponential, where the product in $\mathrm{End}(V)$ is composition.

24. Exercise:

i. For matrices A and B in $\hat{Gl}(n,\mathbb{C}) = M_n(\mathbb{C})$ we have $[A, B] = 0 \Rightarrow e^{A+B} = e^A e^B$.

ii. If $\mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n/\mathbb{Z}^n \equiv T^n$ is the covering homomorphism then show π is the exponential map of the torus T^n.

iii. Show that the image of exp "determines" G_0:
The connected component of e in a Lie group G, in the sense that w in G_0 is a product of elements in $\exp(U_0)$.

iv. If G is connected, a homomorphism of Lie groups $\phi: G \to H$ is determined by $(d\phi)_e$.

v. Show that there is no A in $M_2(\mathbb{R})$ with $\exp(A) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$.

25. Exercise:

i. Classify all real two-dimensional Lie algebras. (: The trivial one and the one with product $[e_1, e_2] = e_1$ for a basis $\{e_1, e_2\}$).

ii. What is the 1-connected Lie group that corresponds to the second Lie algebra?

iii. Let $G = \{(x,y) \text{ in } \mathbb{R}^2,\ y > 0\}$ and the following composition law $(x,y) \mapsto g : \mathbb{R} \to \mathbb{R}$ given by $g(t) = yt + x$.
Now compose $g_1 \cdot g_2$ as functions $g_1 \cdot g_2(t) =$

$= y_1(y_2t + x_2) + x_1 = y_1y_2t + y_1x_2 + x_1$.

Show it is a connected simply connected Lie group and find its Lie algebra.

iv Classify all 3-dimensional real Lie algebras and find their corresponding simply connected Lie groups. (See [Kirillov]).

26. Remark: A theorem of Elie Cartan asserts that for any real finite dimensional Lie algebra Ω there is a simply connected Lie group G with L(G) isomorphic to Ω .

This theorem is no longer true, in this generality, if we consider infinite dimensional Lie algebras. For a relatively short proof of Cartan's theorem based on the vanishing of the first and second de Rham cohomology groups of a finite dimensional simply connected G, see [Gorbatsevich].

27. Theorem: If ϕ is a continuous one-parameter subgroup of a Lie group G, then ϕ is C^∞.

(For the proof, show that $\phi(t) = \exp(tX)$ for some X in $T_e G$ and small t. Then translate using "ϕ is a homomorphism").

28. Corollary: (1) A continuous homomorphism between Lie groups is C^∞ .

(2) A locally euclidean (locally homeomorphic to some euclidean space) topological group can have at most one differential structure, making it into a Lie group.

Hilbert's problem: (Montgomery and Zippin; Gleason 1952). Every locally euclidean topological group has a differentiable structure which makes it into a Lie group.

(Actually, a C^∞ structure on a Lie group contains an analytic structure, a fact which implies uniqueness of the C^∞ structure). The following two theorems determine the relations between Lie subgroups of G and subalgebras of L(G) and their proofs may be found in [Warner].

29. Theorem: Let $\phi : H \to G$ define a Lie subgroup of G. Then ϕ is an embedding (homeomorphism of H with $\phi(H)$ in the relative topology) iff $\phi(H)$ is closed in G.

30. Exercise: Give an example of a non closed subgroup of a Lie group.

31. Theorem: Let $\hat{H}'$ be a subalgebra of $\hat{G}$: The Lie algebra of G. Then there exists exactly one connected subgroup (H, ϕ) of G, such that, $d\phi(\hat{H}) = \hat{H}'$.

32. Corollary: There is a one to one correspondence between connected Lie subgroups of a Lie group G and subalgebras of its Lie algebra.

We recall from topology that if $\tilde{G} \xrightarrow{\gamma} G$ is a covering of a topological group, then $\tilde{G}$ can be given a topological group structure, so that γ in a group morphism. So, if G in a Lie group, its fundamental covering space $\tilde{G}$ is also a Lie group, locally isomorphic to G through γ.

33. Exercise:

i. Show that the product group $\mathbb{R} \times SU(n)$ is the fundamental covering group of U(n) for all n.

ii. Show that the product group $S^1 \times SU(n)$ is the n-fold covering group of U(n).

iii. Is U(n) an (n-1)-fold covering group of itself?

iv. Is the m - fold covering group of U(n) unique for all $m = 2, 3, \ldots$?

34. Example: $\pi_1 SO(n) \cong \mathbb{Z}_2, n \geq 3$

In order to investigate this example and to facilitate our understanding of the construction of the fundamental covering groups (double coverings in this case) of the SO(n)'s, we first introduce the concept of group actions on manifolds.

35. Definition: A C^∞ map $\mu: G \times M \to M$ where G is a Lie group and M is a C^∞ manifold is called a left action of G on M, iff

(i) it is smooth,

(ii) $\mu(e,x) = x$, for all x in M, e the unit of G,

(iii) $\mu(h, \mu(g,x)) = \mu(hg, x)$, for all g, h in G and x in M.

A C^∞ map $\nu: G \times M \to M$ with $\nu(e, x) = x$ and $\nu(h, \nu(g,x)) = \nu(gh, x)$ is called a right action of G on M.

Observe that a right action becomes a left action by passing to the inverse ($: g \mapsto g^{-1}$) and vice-versa.

An action is called effective iff $gx = x$ for all x in M implies $g = e$ (Observe that we have simplified $\mu(gx)$ to gx and will do so from now on when there is no danger of confusion).

We can turn non-effective actions into on effective ones by dividing G by its normal subgroup that fixes everything.

So we assume all actions effective from now on.

An action is called (i) FREE iff $gx = x$ for some x in M implies $g = e$,

(ii) TRANSITIVE iff there is x_o in M with

$$\{ gx_o \mid g \text{ in } G \} = M.$$

If μ is a left action of G on M we have that to each g in G corresponds a diffeomorfism L_g of M defined by $L_g(x) = gx$. This gives us a group monomorphism $G \to \text{Diff}(M)$ provided μ is effective.

Let $Gx := \{ gx \mid g \text{ in } G \}$: orbit of x in M.

$G_x := \{ g \text{ in } G \mid gx = x \}$: Isotropy subgroup of x.

If G/G_x denotes the quotient manifold of G by its subgroup G_x then Gx is diffeomorphic to G/G_x. All elements of the same orbit have isomorphic isotropy subgroups.

If the action is transitive, i.e., there is just one orbit, them $M \cong G/_H$, i.e., M is a homogenious space of the group G, where H is the isotropy subgroup of some x_o in M.

36. Examples:

(i) $G \times G \to G$ by $(g, h) \mapsto gh$

This is a free and transitive action.

ii. $G \times G \to G$

$(g, h) \mapsto ghg^{-1}$: conjugation.

This action is a rich source of examples.

iii. $G \times G/_H \to G/_H$

$(g, g_1H) \to gg_1H$

This is the usual way G acts on anyone of its homogenious spaces.

iv. $Gl(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$

$(A, \xi) \mapsto A\xi$ Linear action of the matrix group.

v. Restrict the above to $SO(n)$ and find the orbits.

vi. Restrict action v. to S^{n-1} and show it is of the type described in example iii. Find H.

vii. If μ is an action of G on M, then it restricts to each subgroup H of G.

viii. If V is a $\mathbb{C}$ - or $\mathbb{R}$ - vector space with $\dim V = n$, there is no natural action of $Gl(n, \mathbb{C})$ on V (i.e., without having to choose a basis). There is, however, an action of $Gl(n, \mathbb{R} \text{ or } \mathbb{C})$ on the manifold of bases of V

by
$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{pmatrix} = \begin{pmatrix} a_{11}\vec{v}_1 + \cdots + a_{1n}\vec{v}_n \\ \vdots \\ a_{n1}\vec{v}_1 + \cdots + a_{nn}\vec{v}_n \end{pmatrix}$$

This is a free and transitive action.

ix. A principal G - bundle amounts, essentially, to a free G - action on a manifold P.

37. Exercise: Show that in the case of a Lie group action with just one orbit type (i.e., all possible isotropy subgroups are conjugate in G) the quotient space can be given the structure of a manifold in a natural way.

We assume principal actions from the right for convenience.

38. Examples: of principal bundles:

(i) All $H \ldots G \to G/_H$, H subgroup of G.

(ii) $SO(n) \ldots SO(n+1) \to S^n$ (special case of (i)).

(iii) $SO(n) \ldots O(n) \to Z_2$

(iv) $O(n) \ldots Gl(n) \to R^{(n^2+n)/2}$

(v) $Gl(n) \ldots B(M) \to M$

Where M is any n - manifold, B(M) is the manifold of bases of M, i.e., b in B(M) iff $b = (X_1(m), \ldots, X_n(m))$ a basis of $T_x(M)$, the tangent space of M at m.

(vi) Let CP^1 be the manifold of complex lines in $\mathbb{C}^2$. In homogenious Coordinates, $[{x \atop y}]$ belongs to CP^1, where $({x \atop y})$ in $\mathbb{C}^2 - \{0\}$ and $[{x \atop y}] \equiv [{\lambda x \atop \lambda y}]$ for any λ in $\mathbb{C} - \{0\}$.

Let $SU(2)$ act on CP^1 by matrix multiplication from the left. Show that the action is transitive and prove that this way we obtain the principal bundle

$$S^1 \ \dots\ S^3 \to S^2 .$$

where S^2 is diffeomorphic to $\mathbb{C}P^1$.

39. <u>Exercise</u>:

(i) If $M \times G \to M$ is a right action and $G \times F \to F$ is a left action, then $G \times (M \times F) \to M \times F$ by $g,\ (m, f) \longrightarrow (mg, g^{-1}f)$ is a right action which is free if any one of the above two is free.

Take $B(M)$ in place of M, $Gl(n, \mathbb{R})$ in place of G and $\mathbb{R}^n$ in place of F. Now $Gl(n, \mathbb{R}) \times (B(M) \times \mathbb{R}^n) \to$ $\to B(M) \times R^n$ is a free action with quontient TM : the tangent bundle of M.

I.e., we have:

$$\begin{array}{ccccc} & & & & \mathbb{R}^n \\ & & & & \vdots \\ Gl(n, \mathbb{R}) \ \dots\ B(M) \times \mathbb{R}^n & \longrightarrow & \dfrac{B(M) \times \mathbb{R}^n}{Gl(n, \mathbb{R})} & = & TM \\ & & & & \downarrow \\ & & & & M \end{array}$$

(ii) Show that this is a general phenomenon: If $P \times G \to P$ is a free action with quontient M, which must, therefore, be a manifold, and if $G \times F \to F$ is any action, the following diagram is commutative

$$\begin{array}{ccccc} & & F & & F \\ & & \vdots & & \vdots \\ G \ \dots & & P \times F & \longrightarrow & P \underset{G}{\times} F \\ & & \downarrow & & \downarrow \\ G \ \dots & & P & \longrightarrow & M \end{array}$$

With rows principal G - bundles and columns fibre bundles with group G and fibre F.

This procedure defines the "associated bundle with fibre F" (right column) to the G - principal bundle defined by the bottom row.

(iii) If $f : G \times M \longrightarrow M$ is a (left) action and m_o is a fixed point of M, i.e., $gm_o = m_o$ for all g, then the map $\Psi : G \to \mathrm{Aut}\,(T_{m_o}M)$ defined by $\Psi\,(g) = (dL_g)_{m_o}$ is a representation of G.

<u>The adjoint representation</u>

Consider the conjugation $\alpha : G \times G \to G$ (action of G on itself by inner automorphisms) $(g, h) \to ghg^{-1} \equiv \alpha_g\,(h)$.

The neutral element is fixed by α and so $g \to d\alpha_g$ defines a representation $G \to \mathrm{Aut}(T_e\, G)$ called the adjoint representation and denoted by $Ad: G \to \mathrm{Aut}(\hat{G})$, $Ad(g)$ is usually written as Ad_g, (where $\hat{G}$ is identified with T_eG).

Recall that if ϕ is a Lie group morphism then the following diagram commutes:

$$\begin{array}{ccc} \hat{G} & \xrightarrow{d\phi} & \hat{H} \\ {\scriptstyle exp_G}\downarrow & & \downarrow{\scriptstyle exp_H} \\ G & \xrightarrow[\phi]{} & H \end{array}$$

40. <u>Corollary</u>: If $ad = d(Ad)_e$ and Exp is the matrix expo- nential $\mathrm{Exp}(s) = I + s + \dots + \dfrac{s \circ \dots \circ s}{n!} + \dots$ then the following is commutative

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\ ad\ } & End(\hat{G}) \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle Exp} \\ G & \xrightarrow[Ad]{} & Aut(G) \end{array}\quad \text{, i.e., } Exp(ad_X) = Ad_{e^X}.$$

Notation: $ad(x) \equiv ad_x$, for all x in $\hat{G}$.

The following diagram also commutes:

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\ Ad_\sigma\ } & \hat{G} \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle xp} \\ G & \xrightarrow[\alpha_\sigma]{} & G \end{array}$$

and this implies $e^{t\,Ad_\sigma(X)} = \sigma e^{tX} \sigma^{-1}$, for all X in $\hat{G}$, t in R and all σ in G. Taking the derivative at $t = 0$ we get:

$$Ad_\sigma(X) = \left(\frac{d}{dt}\right)_{t=0} \sigma e^{tX} \sigma^{-1},$$

Which is quite useful when operating with the adjoint. In the case $G = Aut(V)$ then $\hat{G} = End(V)$ and $Exp: \hat{G} \longrightarrow G$ is the usual "matrix" exp. For B in $Aut(V)$ and C in $End(V)$ the formula will become

$$Ad_B(C) = \frac{d}{dt}\Big|_{t=0} B\,Exp(tC)B^{-1} =$$

$$= \frac{d}{dt}\Big|_{t=0} B\left(I + \frac{tC}{1!} + \frac{t^2C^2}{2!} + \dots\right)B^{-1}$$

$$= \frac{d}{dt}\Big|_{t=0} \left(I + \frac{t\,BCB^{-1}}{1!} + \dots + \frac{t^n(BCB^{-1})^n}{n!} + \dots\right) = BCB^{-1}$$

which shows that, in the case of matrices, conjugation is precicely the adjoint.

41. Proposition: For a Lie group G and X, Y in $\hat{G}$

$$ad_X(Y) = [X, Y].$$

Proof:
$$ad_X(Y) = \frac{d}{dt}\Big|_{t=0} Ad_{e^{tX}}(Y) =$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} (e^{tX} e^{sY} e^{-tX}).$$

This means the following: For a fixed t, $e^{tX} e^{sY} e^{-tX}$ is a curve in G, going through e for $s = 0$. Its tangent at $s = 0$ is an element Y_t of $\hat{G}$.
Varying t now, we get a curve Y_t in $\hat{G}$, passing through Y at $t = 0$. Its tangent vector at $t = 0$ translated to the origin of $\hat{G} \equiv T_e G$ is $ad_X(Y)$.

Recall now from Example 18 that if F is the one parameter group of diffeomorphisms of G generated by X, then

$$[X, Y]_g = \frac{d}{dt}\Big|_{t=0} d(F_{-t})\, Y(F_t(g)).$$

As we saw at the beginning of the proof

$$ad_X(Y) = \frac{d}{dt}\Big|_{t=0} dR_{e^{-tX}}(Y\, e^{tX}).$$

But the integral curve of a left invariant vector field X through g in G is

$$ge^{tX} = R_{e^{tX}}(g).$$

So, $F_{-t}(g) = R_{e^{-tX}}(g)$, which implies

$$\frac{d}{dt}\Big|_{t=0} dR_{e^{-tX}}(Y_{e^{tX}}) = \frac{d}{dt}\Big|_{t=0} dF_{-t}(Y_{e^{tX}}) =$$

$$= [X, Y].$$

Q E D.

Clifford algebras and Spin

We go back now to $SO(n)$ and its double covering for $n \geq 3$.

Observe that the conjugate action of S^3 on itself

$\alpha : S^3 \times S^3 \to S^3$ by

$\alpha(q)(r) := q\, r\, \bar{q}$

is not effective and $\mathbb{Z}_2 = \{1, -1\}$ is the normal subgroup of S^3 that fixes everything. The corresponding effective action is then $\alpha : S^3/\mathbb{Z}_2 \times S^3 \to S^3$

which extends to an action of $S^3/\mathbb{Z}_2$ on $\mathbb{R}^4 \equiv \mathbb{H}$, by

isometries of the usual euclidean metric. We have, therefore the representation $\alpha : S^3 \to O(4)$ which is given by $\alpha(x)(y) = xy\bar{x}$, when x in S^3 and y in $\mathbb{R}^4$. Since $\alpha(x)1 = 1$, $\alpha(S^3) \subseteq O(3)$ the isometries of $\mathbb{R}^3 = \operatorname{Im}(\mathbb{H})$. As S^3 is a connected 3 - dimensional manifold, image of α is $SO(3)$, the identity component of $O(3)$ and we have the double covering

$$\mathbb{Z}_2 \ldots S^3 \xrightarrow{\alpha} SO(3).$$

As S^3 is simply connected, $\pi_1 SO(3) \cong \mathbb{Z}_2$ from the theory of covering spaces (or from the homotopy sequence of the above fibration).

42. Exercise: Consider the map $\mathbb{Z} \to \mathbb{Z}_2$ induced by the inclusion $SO(2) \to SO(3)$ on the π_1 - level. Exhibit a homotopy, in $SO(3)$, between the image of

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}$$

and the identity map.

(Hint: Consider the map $\alpha \circ \beta$ where $\beta : S^1 \to S^3$ is the restrition of the inclusion $\mathbb{C} \subseteq \mathbb{H}$.)

Now the homotopy sequence of the principal bundle $SO(n) \ldots SO(n+1) \to S^n$ implies that the inclusion $SO(n) \subseteq SO(n+1)$ induces isomorphisms of $\pi_1 SO(n) \cong \pi_1 SO(n+1)$ for all $n \geq 3$, so that $\pi_1 SO(n) \cong \mathbb{Z}_2$ for all $n \geq 3$.

43. Exercise: Show that

i. The Euclidean scalar product in $\mathbb{R}^4 \equiv \mathbb{H}$ is given in terms of conjugation by $2\langle u, v\rangle = u\bar{v} + v\bar{u}$

ii. $\operatorname{Im}\mathbb{H}$ consists of all u in $\mathbb{H}$ with $u^2 = -|u|^2$

iii. If v, u in $\operatorname{Im}\mathbb{H}$ then $\langle u, v\rangle = 0$ iff $uv = -vu$. In this case uv is in $\operatorname{Im}\mathbb{H}$.

iv. If q in $\operatorname{Im}(\mathbb{H})$, $-\alpha(q) = R_q$; the reflection of

$R^3 \equiv \mathrm{Im}\,\mathbb{H}$, in the hyperplane perpendicular to q. I.e., $\alpha(q)$ is the reflection in the straight line defined by q.

v. For each A in SO(3) there exist v_1, v_2 in $S^2 = S^3 \cap \mathrm{Im}\,\mathbb{H}$, with $A = R_{v_1} \circ R_{v_2}$.

vi. Give a direct proof that $\alpha : S^3 \to SO(3)$ is surjective.

The universal covering group of SO(n) is denoted by Spin(n) for $n \geq 3$ and the above shows that Spin (3) $\cong S^3$. The construction of Spin (n), $n > 3$, is basically an imitation of the process described in the above example [Atiyah, Bott and Shapiro]: To construct Spin (n), we include $\mathbb{R}^n$ in an associative algebra with unit, denoted by C_n, which is the analog of $\mathbb{H}$ in the case $n = 3$, so that for all v in $\mathbb{R}^n \subset C_n$ we have $v^2 = -|v|^2$. This implies

$2\langle x, y\rangle = \langle x, x\rangle + \langle y, y\rangle - \langle x-y, x-y\rangle = -x^2 - y^2 - (x-y)^2 =$

$= -(xy + yx)$. So, $xy = -yx$ for x and y mutually orthogonal in $\mathbb{R}^n$. This property guarantees that the reflection $x \mapsto R_v(x)$ is given by $x \to vxv$. The algebra C_n should not be "too small" in order to cover all of SO(n) or "too large" so that it doesn't contain too many elements.

To define C_n, let $\{e_1, \ldots, e_n\}$ be the "standard" basis of $\mathbb{R}^n$ (or any orthonormal basis for that matter) and let

$$\{1, e_1, \ldots, e_n, e_1 e_2, \ldots, e_{n-1}e_n, \ldots, e_1 \ldots e_n\}$$

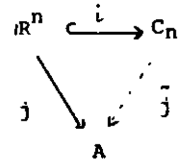
be the "standard" basis for C_n, so dim $C_n = 2^n$.

The requirements above impose

$$e_i^2 = -1, \; e_i e_j = -e_j e_i, \; i \neq j$$

which in turn give us a multiplication table for C_n.

44. <u>Exercise</u>: C_n is "universal" in the following sense: If $j : \mathbb{R}^n \to A$ is a linear map, A any associative algebra with unit 1_A, satisfying $j(x)^2 = -|x|^2 . 1_A$, then there exists exactly one algebra morphism $\tilde{j} : C_n \to A$ with $\tilde{j} \circ i = j$:



(The inclusion i is given by the fact that $e_1, \ldots, e_n$ are basis elements of C_n).

If $C_n^0 = \mathrm{span}\,\{1, e_{i_1} \ldots e_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n, \; k \text{ even}\}$

$C_n^1 = \mathrm{span}\,\{e_{i_1} \ldots e_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n, \; k \text{ odd}\}$

then $C_n = C_n^0 \oplus C_n^1$ where C_n^0 is a subalgebra of C_n and the above direct sum provides a $\mathbb{Z}_2$ - graduation of C_n. I.e., for x in C_n^i, y in C_n^j, xy in $C_n^{[i+j]}$ where $[i+j]$ is considered modulo 2.

45. <u>Examples</u>: [Curtis] (i) $C_1 \cong \mathbb{C}$ with basis $\{1, e_1\}$ and $e_1^2 = -1$. $C_1^0 \cong \mathbb{R}$, $C_1^1 \cong \mathrm{Im}\,\mathbb{C}$.

(ii) $C_2 = \mathrm{Span}\,\{1, e_1, e_2, e_1 e_2\}$ with

$C_2^0 = \mathrm{Span}\,\{1, e_1 e_2\} \cong C_1$

and $C_2 \cong \mathbb{H}$.

(iii) Let $\phi : C_{k-1} \to C_k$ be defined by $\phi(e_i) = e_i e_k$, $i = 1, \ldots, k-1$.

Show that ϕ extends to a well defined algebra isomor phism of C_{k-1} on to C_k^o.

(iv) Consider $(i, i) \equiv e_1$; $(j, j) \equiv e_2$ and $(k, -k) \equiv e_3$ in $\mathbb{H} \oplus \mathbb{H}$, observe that these are anticommuting complex structures, i.e., $(i, i)^2 = -(1, 1)$, etc., such that their products $\{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}$ are a basis of $\mathbb{H} \oplus \mathbb{H}$ over the reals. So, $C_3 \cong \mathbb{H} \oplus \mathbb{H}$.

(v) Consider the following elements of $M_2(\mathbb{H})$:

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \equiv E_1, \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \equiv E_2, \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} \equiv E_3 \text{ and } \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \equiv E_4.$$

Show that they are anticommuting complex structures, i.e., $E_i E_j + E_j E_i = -2\delta_{ij}$ and that the set

$\{I, E_1, \ldots, E_4, E_1E_2, \ldots, \ldots, E_1 \ldots E_4\}$ is a basis of $M_2(\mathbb{H})$ over the reals. Conclude that $\Psi: M_2(\mathbb{H}) \cong C_4$ by $E_i \to e_i$.

Observe now that the identifications in Examples (i) (ii) and (iii) can be obtained by successive inclusions $\mathbb{C} \subsetneq \mathbb{H} \subsetneq \mathbb{H} \oplus \mathbb{H} \subsetneq M_2(\mathbb{H})$.

(vi) Working the same way as in (iv) show that the assignement

$$\begin{pmatrix} i & & & 0 \\ & -i & & \\ & & i & \\ 0 & & & -1 \end{pmatrix} \mapsto e_1, \quad \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & 0 \\ & & 0 & -1 \\ 0 & & 1 & 0 \end{pmatrix} \mapsto e_2,$$

$$\begin{pmatrix} 0 & i & & \\ i & 0 & & 0 \\ & & 0 & -i \\ 0 & & -i & 0 \end{pmatrix} \mapsto e_3, \quad \begin{pmatrix} & & & 1 \\ & 0 & 1 & \\ & -1 & & 0 \\ -1 & & & \end{pmatrix} \mapsto e_4 \text{ and}$$

$$\begin{pmatrix} 0 & & & i \\ & & i & \\ & i & & \\ i & & & 0 \end{pmatrix} \mapsto e_5$$

induces an isomorphism $\Psi: M_4(\mathbb{C}) \equiv C_5$.

We proceed now with the construction of Spin (n). Let C_n^* be the group of all invertible elements of C_n. We may consider it as a subgroup of $Gl(2^n, \mathbb{R})$ by: x in C_n^* sends y in C_n to xy. Define

$$Pin(n) = \{v_1v_2\ldots v_k \mid v_i \text{ in } S^{n-1} \subsetneq \mathbb{R}^n, k = 1,2,\ldots\}$$

and observe that since $v_i^2 = -1$, Pin(n) is a closed subgroup of C_n^*. Define also $Spin(n) = Pin(n) \cap C_n^o$, i.e., Spin(n) is composed of all $v_1 \ldots v_{2s}$, v_i in S^{n-1}.

46. <u>Claim</u>: For x in $\mathbb{R}^n$ and u in S^{n-1} the assignement $x \to uxu$ is the reflection, in $\mathbb{R}^n$, in the hyperplane perpendicular to u.

<u>Proof</u>: Let $u, u_1 \ldots u_{n-1}$ orthonormal basis of $\mathbb{R}^n$ and observe that $u \mapsto uuu = -u$ while $u_i \mapsto uu_iu = -u_iuu = u_i$.

Q E D.

47. **Corollary**: For $u_1 \dots u_k$ in Pin(n) and x in $\mathbb{R}^n$ the assignment $x \mapsto u_1 \dots u_k\, x\, u_k \dots u_1$ defines a representation of Pin(n) on to O(n), which restricts to a representation of Spin(n) on to SO(n). Let π : Pin(n) $\to$ O(n) denote this epimorphism.

Proof: Recall from Exercise 8 ii that each A in O(n) is a product of reflections and that the determinant of a reflection is -1.

Q E D.

48. **Proposition**: The kernel of π is $\{-1, 1\}$.

Proof: Let $u_1 u_2 \dots u_k = x^o + x^1$ in $C_n^o + C_n^1$ be in ker (π). I.e., for each y in $\mathbb{R}^n$,

$$\alpha(x^o + x^1)\; y\, (x^o + x^1)^{-1} = y$$

where $\alpha: C_n \longrightarrow C_n$ is the automorphism extending $x \longrightarrow -x$ for x in $\mathbb{R}^n$.

Observe that $\alpha(u_1 \dots u_k) = (-1)^k\, u_1 \dots u_k$ and $(u_1 \dots u_k)^{-1} = (-1)^k u_k \dots u_1$.

So, $x^o + x^1$ is in ker (π) iff $\alpha(x^o + x^1)y = y(x^o + x^1)$ for each y in $\mathbb{R}^n$. But $\alpha x^o = x^o$ and $\alpha x^1 = -x^1$ and the condition above is equivalent to both

$$x^o y = y x^o \quad \text{and} \quad x^1 y = -y x^1$$

Let now $x^o = a^o + e_1 b^1$ where a^o in C_n^o, b^1 in C_n^1 and neither a^o nor b^1 contain a summand with the factor e_1. Apply the first of the above relations to $y = e_1$ and obtain

$$a^o + e_1 b^1 = -e_1 (a^o + e_1 b^1)\, e_1,$$

where each monomial in a^o is in C_n^o and contains no factor of e_1, so a^o commutes with e_1.

Similarly, $e_1 b^1 = -b^1 e_1$. The last condition therefore becomes: $a^o + e_1 b^1 = a^o - e_1 b^1$, i.e., $b^1 = 0$: x^o contains no monomial with the arbitrarily chosen e_1 and therefore x^o in $\mathbb{R}$. Let now $x^1 = a^1 + e_1 b^o$ where a^1 and b^o are sums of monomials with no e_1 factor.

A similar argument implies $b^o = 0$ and x^1 does not involve e_1 (or $e_2, e_3, \dots, e_n$). But x^1 is in C_n^1, so $x^1 = 0$ and $u_1 \dots u_k = x^o$ in $\mathbb{R} - \{0\}$ since it is invertible. From $(u_1 \dots u_k)^{-1} = (-1)^k u_k \dots u_1 =$

$= (-1)^k$ transpose $(u_1 \dots u_k)$,

where "transpose" is a canonical antiautomorphism $tr : C_n \to C_n$ defined by $tr(x) = x$, x in $\mathbb{R}^n$ and $tr(uv) = tr(v).tr(u)$. Therefore $(x^o)^{-1} = (-1)^k\, tr(x^o) =$

$= (-1)^k\, x^o$.

But $x_o^2 = -|x_o|^2 . 1$, therefore $x_o = 1$ or -1.

Q E D.

We have therefore the double coverings

$$\begin{array}{ccc} \mathbb{Z}_2 & & \mathbb{Z}_2 \\ \vdots & & \vdots \\ \text{Spin}(n) & \subseteq & \text{Pin}(n) \\ \downarrow \pi| & & \downarrow \pi \\ \text{SO}(n) & \subseteq & \text{O}(n). \end{array}$$

To show that $\pi|: Spin(n) \to SO(n)$ is non-trivial it is enough to show that $\ker(\pi|) = \{1, -1\}$ can be connected by a path in Spin(n), for $n \geq 2$.

A convenient path is

$$\gamma(t) = (\cos(t)e_1 + \sin(t)e_2).(\sin(t)e_1 + \cos(t)e_2), -\frac{\pi}{4} \leq t \leq \frac{\pi}{4}.$$

49. <u>Corollary</u>: Spin(n) is connected for $n \geq 2$ and simply connected for $n \geq 3$. Moreover,

$Spin(1) \cong \mathbb{Z}_2$ and $Pin(1) \cong \mathbb{Z}_4$.

The proof follows from the homotopy sequence of $\mathbb{Z}_2 \cdots Spin(n) \to SO(n)$. Compare with Exercise 42 also.

We have seen, at the beginning of this section, that S^3 is isomorphic to Spin(3). There are a few more interesting identifications of Spin(k)'s with classical groups:

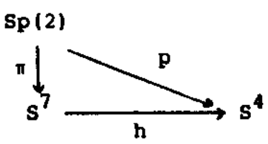
50. <u>Exercise</u>:

i. Show that the assignement

$$(x, y) \longrightarrow (: \xi \longrightarrow x\xi\bar{y}) \quad , \quad \xi \text{ in } \mathbb{R}^4 \equiv \mathbb{H}$$

from $S^3 \times S^3$ to SO(4), where the products are quaternionic, provides an isomorphism $S^3 \times S^3 \cong Spin(4)$.

ii. Show that $Sp(2) \cong Spin(5)$ as follows:



where $\pi\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $h\begin{pmatrix} a \\ b \end{pmatrix} = (a\bar{a} - b\bar{b}, 2a\bar{b})$

Show that p is the projection

$$Spin(4) \ldots Spin(5) \xrightarrow{p} S^4$$

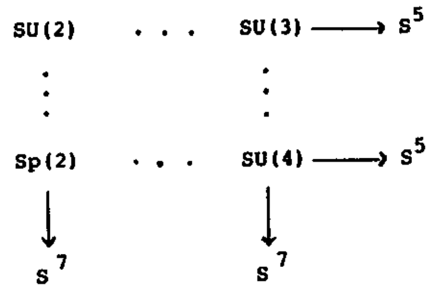
and use this to write down explicity the projection

$$Sp(2) \longrightarrow SO(5).$$

iii. From exercises 7 and 8 we see that $Sp(2) \subseteq SU(4)$.

Show that $Sp(2) \ldots SU(4) \longrightarrow S^5$ implies $SU(4) \to Spin(6)$ and write down explicity the projection $SU(4) \to SO(6)$.

iv. Include your findings in the diagram



v. Show that $Pin(n) = \ker(N)$, where $N: C_n^* \to \mathbb{R} - \{0\}$ is defined by $N(x) = x.\alpha(tr(x)) \equiv xx^*$ and is a group morphism.

vi. Show that an element $u_1 \ldots u_k$ in Pin(n), with u_i in S^{n-1}, is equal to $v_1 \ldots v_\ell$ in Pin(n), v_j in S^{n-1} and $\ell \leq n$. So, each x in Spin(n) is a linear combination of basis elements in C_n^o with $xx^* = 1$.

vii. Show that $\phi : C_{k-1} \cong C_k^o$ of example 45(iii), has the property $\phi(x^*) = \phi(x)^*$, and the isomorphisms Ψ of examples 45 iv, v and vi have the property $\Psi(A^*) = \Psi(A)^*$, where for a matrix A, $A^* \equiv \bar{A}^{tr}$.

viii. [Curtis]. Show that the compositions

$$Sp(2) \hookrightarrow M_2(\mathbb{H}) \xrightarrow[\cong]{\Psi} C_4 \xrightarrow[\cong]{\phi} C_5^o$$

$$\text{and } SU(4) \hookrightarrow M_4(\mathbb{C}) \xrightarrow[\cong]{\Psi} C_5 \xrightarrow[\cong]{\phi} C_6^o$$

provide isomorphisms $Sp(2) \cong Spin(5)$ and $SU(4) \cong Spin(6)$.

An organized, unified, treatment of all low dimensional identifications of spin groups can be found in [Yokota].

A source of low dimensional examples of Lie groups and relations between them is the Cayley algebra [Porteous, Postnikov, Whitehead, Yokota]. We follow here the exposition in [Portal].

51. **Definition**: On $\mathbb{H} \oplus \mathbb{H}$ define a product by $(a,b)\cdot(c,d) := (ac - \bar{d}b, da + b\bar{c})$.

The result is a non-associative algebra with unit (1,0) and no zero divisors, called the Cayley algebra and denoted by $\mathbb{K}$. We obtain a basis for $\mathbb{K}$ from the usual basis

$e_o=1 \equiv (1,0)$, $e_1=i \equiv (i,0)$ $e_2 = j \equiv (j,0)$ and

$e_3=k \equiv (k,0)$ of $\mathbb{H}$ together with $e_4 \equiv (0,1)$ and the products $e_5 := e_1 e_4$, $e_6 := e_2 e_4$ and $e_7 := e_3 e_4$.

52. **Exercise**:

(i) Let A be the subalgebra of $\mathbb{K}$ generated by x, y in $\mathbb{K}$. Show that if x lies in the subalgebra (with unit) of $\mathbb{K}$ generated by y then $A \cong \mathbb{C}$ and if not, then $A \cong \mathbb{H}$. So any subalgebra of $\mathbb{K}$ generated by two elements is associative.

(ii) e_1, e_2 and e_4 generate $\mathbb{K}$ as an algebra.

52 **Definition**: The conjugation on $\mathbb{K}$ is

$$\overline{(a,\ b)} = (\bar{a},\ -b)$$

and satisfies $\bar{\bar{x}} = x$, $\overline{xy} = \bar{y}\bar{x}$

$\mathbb{R}e\ \mathbb{K} = \{x \in \mathbb{K} \mid x = \bar{x}\} = \{(r,o),\ r \in \mathbb{R}\} \cong \mathbb{R}$

$\mathrm{Im}\ \mathbb{K} = \{y \in \mathbb{K} \mid \bar{y} = -y\} \cong \mathbb{R}^7$ and

$\mathbb{K} = \mathbb{R}e\,\mathbb{K} \oplus \mathrm{Im}\,\mathbb{K}$ by $x = \frac{1}{2}(x+\bar{x}) + \frac{1}{2}(x-\bar{x})$.

If $\langle , \rangle$ is the usual euclidean product in $\mathbb{R}^8$,

$$2 \langle x, y \rangle = x\bar{y} + y\bar{x}$$

which implies that x is perpendicular to y iff $x\bar{y} = -y\bar{x}$ and also that

$$x\bar{x} = |x|^2 , \quad |xy| = |x|\ |y| .$$

53. **Corollary**: The maps $\lambda, \rho : S^7 \longrightarrow SO(8)$ with $\lambda_a(x) := ax$ and $\rho_a(y) := ya$ are well defined and satisfy $\pi \circ \lambda = id$, $\pi' \circ \rho = id$ where $\pi : SO(8) \longrightarrow S^7$ is the 1^{st} column projection, and π' is the 1^{st} row projection.

54. Exercise:

(i) Show that each x in $\mathbb{K}$ satisfies

$$x^2 = 2\,\mathrm{Re}(x).x - |x|^2$$

Find the geometric meaning of this formula and relate it to the Moivre identities of trigonometry.

(ii) For all a, x, y in $\mathbb{K}$ we have

$$a(xy)a = (ax)(ya)$$
$$(axa)y = a[x(ay)]$$
$$x(aya) = [(xa)y]a$$

These are called "Moufang identities".

(iii)
$$a(xy) = (axa)(\bar{a}y)$$
$$(xy)a = (x\bar{a})(aya)$$
$$a(xy)\bar{a} = (axa^2)(\bar{a}^2 y\bar{a}).$$
$$\bar{p}[(px\bar{p})(py\bar{p})]p = (x\bar{p}^3)(p^3 y).$$

(iv) If $(xp_1)(\bar{p}_1 y) = (xp_2)(\bar{p}_2 y)$ for all x,y in $\mathbb{K}$ then $p_1 = \pm p_2$.

55. Automorphisms of $\mathbb{K}$:

A linear isomorphism $T: \mathbb{K} \longrightarrow \mathbb{K}$ is called an automorphism of $\mathbb{K}$ iff

$$T(xy) = T(x)\,T(y) \quad \text{for all } x,y \text{ in } \mathbb{K}.$$

The Lie subgroup of $Gl(8, \mathbb{R})$ composed of automorphisms of $\mathbb{K}$ is denoted by G_2.

If T in G_2, $T(x) = T(1.x) = T(1).T(x)$ and $T(1)=1$ so T leaves $\mathrm{Re}\,\mathbb{K}$ invariant.

Now $T(\bar{x}) = T(\mathrm{Re}(x) - \mathrm{Im}(x)) = \mathrm{Re}(x) - T(\mathrm{Im}\,x) = \mathrm{Re}(Tx) - \mathrm{Im}(Tx) = \overline{Tx}$ and $|T(x)|^2 = Tx\,\overline{Tx} = T(x)\,T(\bar{x}) = T(x\bar{x}) = T(|x|^2) = |x|^2$, which implies $G_2 \subseteq O(8)$.

Since x in $\mathrm{Im}\,\mathbb{K} \Leftrightarrow x^2 = -|x|^2$ we have that $T(\mathrm{Im}\mathbb{K}) = \mathrm{Im}\,\mathbb{K}$ for each T in G_2.

This implies $G_2 \subseteq O(7)$.

From exercise 52(ii) follows that the values $T(e_1)$, $T(e_2)$ and $T(e_4)$ determine any T in G_2. As e_1 is perpendicular to e_2 and e_4 is perpendicular to e_1, e_2, e_1e_2 we have that $T(e_1) \perp T(e_2)$ and $T(e_4) \perp (T(e_1), T(e_2), T(e_1).T(e_2))$.

56. Exercise: For x, y, z in (the unit) $S^6 \subseteq \mathrm{Im}(\mathbb{K})$ with

$$\langle x,y\rangle = \langle z,x\rangle = \langle z,y\rangle = \langle z, xy\rangle = 0$$

there is exactly one T in G_2 with

$$Te_1 = x, \quad Te_2 = y \quad \text{and} \quad Te_4 = z.$$

As a corollary we see that G_2 acts transitively on S^6, since $T \longrightarrow T(e_1)$ is onto S^6.

57. Corollary:

(i) $S^6 \cong G_2/H$ where $H = \{T \text{ in } G_2 \mid T(e_1) = e_1\}$.

Show that H is isomorphic to $SU(3)$, which implies $SU(3)$ --- $G_2 \longrightarrow S^6$.

(ii) G_2 is connected and simply connected, there fore $G_2 \subseteq SO(7)$.

(iii) Observe that the argument above shows that G_2 acts transitively on the manifold $V_{2,7}$ of orthonormal 2--frames in $\mathbb{R}^7 = \operatorname{Im}\mathbb{K}$, by

$T(f_1, f_2) = (Tf_1, Tf_2)$. Show that this implies

$$S^3 \text{ - - - } G_2 \longrightarrow V_{2,7} .$$

58. Remark: An account of the fibration

$$G_2 \ . \ . \ . \ SO(7) \text{ ——— } \mathbb{R}P^7$$

and its relation to the Veronese embedding of $\mathbb{R}P^7$ into $\mathbb{R}P^{35}$ can be found in [Portal].

Triality:

The property of Triality as proved by E. Cartan in the early 20's [Cartan] is the following:

59. Theorem: For all A in $SO(8)$ there exists exactly one pair, up to sign, $\pm(B,C)$ in $SO(8) \times SO(8)$, with $A(xy) = B(x)\cdot C(y)$, for all x, y in $\mathbb{K}$.

Proof: For x in $\mathbb{K}$, u in (the unit) $S^7 \subseteq \mathbb{K}$ the reflection of x in the hyperplane perpendicular to u is $R_u(x) = -u\bar{x}u$: As both maps are in $O(8)$ it suffices to check their equality for $x = u$ and for x perpendicular to u. In fact, $R_u(u) = -u = -u\bar{u}u$ and for $\langle x, u \rangle = 0$, $\bar{u}x = -\bar{x}u$, as a consequence of Definition 52, which implies $R_u(x) = x = (u\bar{u})x = u(\bar{u}x)$ (Exer. 52 i) $= -u\bar{x}u$.

Now from Exercise 8ii follows that for A in $SO(8)$

there are $v_1, \dots, v_r$ in S^7 , r even and $r \leq 8$, with $A(x) = R_{v_1} \circ \dots \circ R_{v_r}(x) = v_r(\dots v_2(\bar{v}_1 x \bar{v}_1)v_2 \dots)v_r$ Since the minus sign and conjugation appear an even number of times.

The first Moufang identity, Exercise 54ii, implies

$$A(xy) = [v_r(\dots(v_2(\bar{v}_1 x))\dots)][(\dots(y\bar{v}_1)v_2)\dots v_r] =$$

$$= [\lambda_{v_1} \circ \dots \circ \lambda_{\bar{v}_r}(x)][\rho_{v_1} \circ \dots \circ \rho_{\bar{v}_r}(y)] ,$$

where λ_{v_i} and ρ_{v_j} are the maps of Corollary 53.

To prove the uniqueness of the pair $\pm(B,C)$, let $A(xy) = B_1(x)C_1(y) = B_2(x)C_2(y)$ for all x,y in $\mathbb{K}$. Let $x \equiv B_1^{-1}(x)$ and $y \equiv C_1^{-1}(y)$ to obtain (1) $xy = B_3(x)C_3(y)$, where $B_3 = B_2 B_1^{-1}$ and $C_3 = C_2 C_1^{-1}$. Set $y = 1$ and obtain

$$B_3(x) = x\overline{C_3(1)} \equiv xb.$$

Similarly $\quad C_3(y) = \overline{B_3(1)}.y \quad ya.$

Returning to (1) now we have $xy = (xb)\cdot(ay)$ for all x,y in $\mathbb{K}$, which implies $b = a^{-1}$, by setting $x=y=1$. Let now $x \equiv xa$ and obtain

$$(xa)y = x(ay) \text{ for all } x,y \text{ in } \mathbb{K}.$$

So, a is real and therefore $a = 1$ or -1, i.e.,

$$(B_1, C_1) = (B_2, C_2) \text{ or } (B_1, C_1) = -(B_2, C_2).$$

Q E D.

60. Notation:

(i) For A, B and C in SO(8) we write (A, B, C) in Θ iff $A(xy) = B(x)\,C(y)$ for all x, y in $\mathbb{K}$.

(ii) For A in SO(8) let $\tilde{A}$ in SO(8) be $\tilde{A}(x) := \overline{A(\bar{x})}$, x in $\mathbb{K}$.

61. Corollary:

(i) (A, B, C) in Θ iff (A, −B, −C) in Θ .

(ii) (A, B, C) in Θ iff $(B, A, \tilde{C})$ in Θ iff $(C, \tilde{B}, A)$ in Θ iff (A^{-1}, B^{-1}, C^{-1}) in Θ .

(iii) (A, B, C) in Θ and (A_1, B_1, C_1) in Θ implies (AA_1, BB_1, CC_1) in Θ .

(iv) Θ is a closed subgroup and therefore a Lie subgroup of SO(8) x SO(8) x SO(8).

(v) The maps $\pi_i : \Theta \longrightarrow SO(8)$, i = 1, 2, 3, with

$$\pi_i(A_1, A_2, A_3) = A_i$$

are smooth group epimorphisms, each with kernel composed of two elements. They are, therefore, double coverings of SO(8), which implies that Θ is locally isomorphic with SO(8).

(vi) The path (A_t, B_t, C_t) in Θ with

$A_t = R_i \circ R_{ie^{i\pi t}}$, $B_t = \lambda_i \circ \lambda_{ie^{i\pi t}}$

$C_t = \rho_i \circ \rho_{ie^{i\pi t}}$, $0 \le t \le 1$, joins (I, −I, −I) to (I, I, I) implying that Θ is

connected and therefore isomorphic to Spin(8).

(vii) Recall now from Corollary 47 that the morphism

$$\pi : \text{Spin}(8) \longrightarrow SO(8) \quad \text{defined by}$$

$$a = v_1 \dots v_\ell \longrightarrow R_{v_1} \circ \dots \circ R_{v_\ell}$$

with v_i in S^7 , ℓ even and the product on the left in C_7 , is a double covering.

We can define now two more double coverings

$$\lambda \text{ and } \rho : \text{Spin}(8) \longrightarrow SO(8) \quad \text{by}$$

$$\lambda(v_1 \dots v_k) := \lambda_{v_1} \circ \dots \circ \lambda_{\bar{v}_k} \quad \text{and}$$

$$\rho(v_1 \dots v_k) := \rho_{v_1} \circ \dots \circ \rho_{\bar{v}_k}$$

An explicit isomorphism between Θ and Spin(8) is then the following:

$$\gamma = (\pi, \lambda, \rho) : \text{Spin}(8) \longrightarrow \Theta$$

with $\gamma(a) := (\pi(a), \lambda(a), \rho(a))$

for $a = v_1 \dots v_k$

Moreover, $\pi = \pi_1 \circ \gamma$, $\lambda = \pi_2 \circ \gamma$ and $\rho = \pi_3 \circ \gamma$,

where the π_is are defined in item iv above.

Proof. Straight forward and left as an exercise.

To obtain the infinitesimal version of triality we define for each A in $\widehat{SO}(8)$.

$$A^\lambda := (d\lambda)_1 \circ (d\pi)_I^{-1}(A) \quad \text{and}$$

$$A^\rho := (d\rho)_1 \circ (d\pi)_I^{-1}(A) .$$

We have then

62. <u>Theorem</u>: For A in $\tilde{SO}(8)$, x,y in $\mathbb{K}$

$$A(xy) = A^{\lambda}(x)y + xA^{\rho}(y)$$

<u>Proof</u>: Fix x,y in $\mathbb{K}$ and define

$f, g : Spin(8) \longrightarrow \mathbb{K}$ by

$f(a) = \pi(a)(xy)$ and $g(a) = [\lambda(a)(x)][\rho(a)(y)]$.

Since $f = g$, $(df)_1 = (dg)_1$ applied to B in $T_1\, Spin(8)$ implies $(df)_1\,(B) = (d\pi)_1\,(B)(xy)$ is equal to

$(dg)_1\,(B) = [(d\lambda)_1\,(B)(x)]y + x[d\rho_1(B)(y)]$.

Let now A in $T_I\; SO(8)$ be $(d\pi)_1(B) = A$.

Q E D.

63. <u>Exercise</u>: Formulate and prove a uniqueness theorem for infinitesimal triality.

<u>Simple applications of triality</u>

64. <u>Definition</u>: Let $Spin(7)^* = \{B$ in $SO(8) \mid (A,B,C)$ in Θ, A in $SO(7)\}$, i.e., $A(1)=1$.

It is immediate that $Spin(7)^*$ is a subgroup of $SO(8)$, that if B in $Spin(7)^*$ above then $C = \tilde{B}$ and $A(x) = B(x)\,\overline{B(1)}$. Consequently $(*)$ $B(xy) = (B(x)\,\overline{B(1)})B(y)$ for all x, y in $\mathbb{K}$.

This last equation may be taken as the definition of $Spin(7)^*$ as a subgroup of $SO(8)$ and it shows that it is closed and therefore a compact Lie subgroup of $SO(8)$.

65. <u>Definition</u>: Let $\delta : Spin(7)^* \longrightarrow SO(7)$ be

$$B \longmapsto A$$

Observe that δ is a well defined Liegroup epimorphism, since $-A$ does not live in $SO(7)$, and that $\delta^{-1}(A) = \{B, -B\}$. Also δ is, locally, nothing else than $\pi_1 \circ \pi_2^{-1}$ and so it is C^{∞}.

It is also obvious that $Spin(7)^* = \lambda(\pi^{-1}SO(7))$ in $SO(8)$. Now, $\pi^{-1}SO(7) = Spin(7)$ in $Spin(8)$, which is connected and implies that $Spin(7)^*$ is also connected.

66. <u>Corollary</u>: $Spin(7)^*$ is a subgroup of $SO(8)$ isomorphic to $Spin(7)$ and δ above is the universal covering map.

67. <u>Exercise</u>:

(i) Show that $Spin(7)^* \cap SO(7) = G_2$.

(ii) Show that the projection to the first column restricted to $Spin(7)^*$ generates a non-trivial homogenious fibration

$$\begin{array}{ccc} G_2 & \subseteq & SO(7) \\ \vdots & & \vdots \\ Spin(7)^* & \subseteq & SO(8) \\ \downarrow p & & \downarrow p \\ S^7 & = & S^7 \end{array}$$

(iii) Use exercise 54 (iii) to show that for a in S^7, the map $\mathbb{R}^8 \to \mathbb{R}^8$ that sends $x \mapsto a\,x\,a^2$ in an element of $Spin(7)^*$.

(iv) If $\tau : S^7 \longrightarrow SO(7)$ is $\tau(a)(x) := a x \bar{a}$ find the pre-image $\tau^{-1}(G_2) \subseteq S^7$.

(v) $Spin(7)^* = \{B \text{ in } SO(8) \mid B = \lambda_{v_1} \circ \dots \circ \lambda_{v_k},\ k \text{ even},\ v_i \text{ in } \underset{(\text{unit})}{S^6}\}$

where $S^6 \subseteq S^7 \cap \text{Span}\{e_1, \dots, e_7\}$

(vi) Let $Spin(6)^* = \delta^{-1}(SO(6)) \subseteq Spin(7)^*$.

Show $Spin(6)^* = \{B \text{ in } SO(8) \mid B = \lambda_{v_1} \circ \dots \circ \lambda_{v_k},\ k \text{ even},\ v_i \text{ in } S^5\}$

where $S^5 = S^7 \cap \{\text{Span } e_2, \dots, e_7\}$

Similarly with $Spin(5)^* = \delta^{-1} SO(5) \subseteq Spin(6)^*$.

67. <u>Exercise</u>:

(i) If E in $SO(8)$ is the interchanging of the last two coordinates and if $\phi : U(4) \longrightarrow SO(8)$ is the inclusion of Exercise 7, show that $M \longmapsto E \circ \phi(M) \circ E$ defines an inclusion ε of $U(4)$ into $SO(8)$ with

$\varepsilon(SU(4)) = Spin(6)^*$,

and the following diagram is commutative

$$\begin{array}{ccc}
SU(3) & \xrightarrow{\ \varepsilon\ } & \varepsilon SU(3) \\
\vdots & & \vdots \\
SU(4) & \xrightarrow{\ \varepsilon\ } & \varepsilon(SU(4)) = Spin(6)^* \\
{\scriptstyle 1^{st}\,col}\downarrow & & \downarrow{\scriptstyle p|} \\
S^7 & \xrightarrow{\ E\ } & S^7
\end{array}$$

(ii) There is an analogous to E linear map F in $O(8)$, such that $F \circ \phi(Sp(2)) = Spin(5)^* \subseteq \varepsilon(SU(4))$

with $\zeta \equiv F \circ \phi(Sp(1)) \subseteq \varepsilon(SU(3))$ and one finally has the diagram:

$$\begin{array}{ccccccc}
\zeta(Sp(1)) & \subseteq & \varepsilon(SU(3)) & \subseteq & G_2 & \subseteq & SO(7) \\
\vdots & & \vdots & & \vdots & & \vdots \\
\zeta(Sp(2)) & \subseteq & \varepsilon(SU(4)) & \subseteq & Spin(7)^* & \subseteq & SO(8) \\
\downarrow{\scriptstyle p|} & & \downarrow{\scriptstyle p|} & & \downarrow{\scriptstyle p|} & & \downarrow{\scriptstyle p} \\
S^7 & & S^7 & & S^7 & & S^7
\end{array}$$

See also [Portal], [Porteous], [Whitehead] and [Yokota].

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