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SMR.304/ 7

COLLEGE

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

(21 November - 16 December 1988)

STRUCTURES IN BUNDLES.

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"STRUCTURES IN BUNDLES."
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Cocycles

DEF A fiber bundle or a locally trivial bundle with fiber F is a map

$$p : E \longrightarrow B$$

such that there is an open cover $\mathcal{U}$ of B and homeomorphisms

$$\begin{tikzcd} p^{-1}U \arrow[rr, "\Phi_U", "\approx"'] \arrow[dr, "p|"'] & & U \times F \arrow[dl, "\pi_1 = proj_1"] \\ & U & \end{tikzcd} \qquad U \in \mathcal{U},$$

with $\pi_1 \Phi_U = p / p^{-1}U$. Its fiber is F.

Under such conditions, there are maps

$$c_{V,U} : U \cap V \longrightarrow Homeo(F) = G, \text{ (topological group)}$$

called cocycles, defined so that $c_{V,U}(x) : F \xrightarrow{\approx} F$ is such that

$$\Phi_V \circ \Phi_U^{-1} : (U \cap V) \times F \longrightarrow (U \cap V) \times F$$

maps (x,y) to $(x, c_{V,U}(x)(y))$.

They have the fundamental property

$$c_{W,V}(x) \circ c_{V,U}(x) = c_{W,U}(x), \qquad x \in U \cap V \cap W,$$

which implies

$$c_{U,U}(x) = id_F, \ \forall x \in U, \quad \text{and} \quad c_{U,V}(x) = c_{V,U}(x)^{-1}$$

The cocycles are glueing instructions for the bundle $p : E \to B$ in the sense that

PROP If in the disjoint union $\bigsqcup_{U \in \mathcal{U}} U \times F$ we define an equivalence relation by $(x,y) \sim (x, c_{V,U}(x)(y))$ for $x \in U \cap V$ and $y \in F$, then we have

$$\Phi : E \approx \bigsqcup U \times F / \sim$$

such that Φ followed by the projections $U \times F \to U \subset B$ is p. $\square$

Example Let M be a smooth manifold with charts $\{\varphi_U : U \to \varphi_U U \subset \mathbb{R}^m\}$. If φ_U and φ_V are two of them, the homeomorphism

$$\varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \longrightarrow \varphi_V(U \cap V)$$

between open sets in $\mathbb{R}^m$ is smooth. Take the derivative at $\varphi_U(x)$, $x \in U \cap V$,

$$\mathcal{D}(\varphi_V \circ \varphi_U^{-1})(\varphi_U(x)) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m) \subset \mathrm{Homeo}(\mathbb{R}^m)$$

and define cocycles by

$$c_{V,U}(x) = \mathcal{D}(\varphi_V \circ \varphi_U^{-1})(\varphi_U(x)).$$

By the chain rule we see that they satisfy the fundamental property

$$c_{W,V}(x) \cdot c_{V,U}(x) = c_{W,U}(x), \quad x \in U \cap V \cap W$$

By the PROP, these glueing instructions provide a vector bundle

$$T(M) \longrightarrow M,$$

called the <u>tangent</u> <u>bundle</u> of M.

Generally, if we are given an open cover $\mathcal{U}$ of a space B, a space F (e.g. $\mathbb{R}^m$) and a topological group G <u>acting</u> <u>on</u> F, i.e. a group ~~homomorphism~~ monomorphism

$$G \hookrightarrow \mathrm{Homeo}(F)$$

(e.g. $G = Gl(m, \mathbb{R})$ acting linearly on $\mathbb{R}^m$), we define <u>cocycles</u> <u>with</u> <u>coefficients</u> in G as maps

$$c_{V,U} : U \cap V \longrightarrow G$$

satisfying the fundamental property.

For technical reasons we shall assume henceforth that G is a locally compact topological group.

In this situation, we can construct cocycles in the former sense by the composite

$$U \cap V \xrightarrow{c_{V,U}} G \hookrightarrow \mathrm{Homeo}(F),$$

thus obtaining a bundle $p: E \to B$ with fiber F and <u>structure</u> <u>group</u> G.

Since G always acts on itself by <u>right</u> <u>translation</u>, i.e.

$$\alpha : G \hookrightarrow \mathrm{Homeo}(G), \quad \alpha(g)(x) = xg \in G,$$

given a bundle $p: E \to B$ with fiber F and structure group G (i.e. its cocycles factor through a ~~homo~~ monomorphism $G \hookrightarrow \mathrm{Homeo}(F)$), then we have an <u>associated</u> <u>G-principal</u> <u>bundle</u> $p_G : P(E) \to B$ with fiber G and structure group G produced using the action of G on itself and the glueing instructions given by the cocycles of $p: E \to B$.

Given $p: E \to B$ a bundle with structure group G and its associated principal bundle, $p_G : P(E) \to B$ we may recover p from p_G as follows. Take $P(E) \times F$ and let G act on it on the right by $(z,y)g = (zg, yg)$, $z \in P(E)$, $y \in F$, $g \in G$, where zg is given essentially by the natural action of G on itself. Then

PROP. The projection $\pi : P(E) \times F \to P(E) \to B$ factors through the orbit space, $P(E) \times_G F$, of the action and produces a fiber bundle

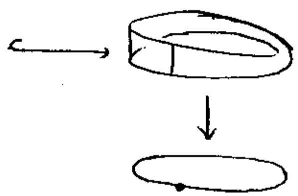
$$P(E) \times_G F \longrightarrow B$$

which is isomorphic to $p : E \to B$.

Proof (sketch). $P(E) = \coprod_{\mathcal{U}} U \times G / \sim$. If $((x,g), y)$ represents an element in $P(E) \times_G F$, ~~define~~ consider the element $(x, yg^{-1}) \in U \times F$ and the element defined by it in $\coprod U \times F / \sim \; = E$. This produces the searched isomorphism of bundles. $\square$

Example Let $H \to \mathbb{R}P^n$ be the Hopf bundle (or canonical line bundle) given by $H = \{(x,y) \in \mathbb{R}P^n \times \mathbb{R}^n : y \in x\}$, which is a vector bundle of dimension 1.
For $n = 1$, $\mathbb{R}P^1$ is nothing else but the circle $\mathbb{S}^1$ and $H \to \mathbb{R}P^1$ is just the Moebius band

$F = \mathbb{R} = |$



What is its associated principal bunde?
The cocycles can be given as follows:

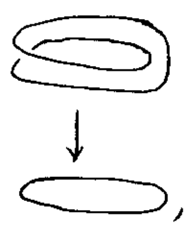
Let $\mathcal{U} = \{U, V\}$, $U = \mathbb{S}^1 - a$, $V = \mathbb{S}^1 - b$, $a \neq b$. Hence $U \cap V$ consists of two components, say A and B. It is easy to verify that the only nontrivial cocycle

$$c_{V,U} : U \cap V \longrightarrow Gl(1, \mathbb{R}) = \mathbb{R}^*$$

is such that

$$c_{V,U}(x) = \begin{cases} 1 & x \in A \\ -1 & x \in B \end{cases}$$

Thus we see that the structure group is in fact $\mathbb{Z}/2 = \{1, -1\} \subset \mathbb{R}^*$ and its associated $\mathbb{Z}/2$-principal bundle looks as follows



it is namely the double covering map $\mathbb{S}^1 \xrightarrow{\cdot 2} \mathbb{S}^1$ given by multiplication by two (in complex notation this mean $e^{i\theta} \mapsto e^{2i\theta}$).

More generally one may see that the structure group for each $H \to \mathbb{R}P^n$ is $\mathbb{Z}/2$ and that its associated $\mathbb{Z}/2$-principal bundle is the so-called Hopf-fibration $\mathbb{S}^n \longrightarrow \mathbb{R}P^n$, such that it is the quotient map identifying x and $-x$ in $\mathbb{S}^n$.

Exercises:

1. Give detailed proofs of the two propositions given
2. Let $\pi: B \times F \to B$ be the trivial bundle. What is its (minimal) structure group?, what are its cocycles? What is its associated principal bundle?
3. Prove that if the associated principal bundle $P(E) \to B$ of a bundle $p: E \to B$ admits a section, then $p: E \to B$ is trivial. (In particular, $P(E) \to B$ is trivial) Conclude that $H \to \mathbb{R}P^1$ is nontrivial.
4. Let $p: E \to B$ be a real vector bundle, i.e. it has $\mathbb{R}^n$ as fiber and $Gl(n, \mathbb{R})$ acting linearly on $\mathbb{R}^n$ as structure group. Given a section of its associated principal bundle, give n sections $s_1, \dots, s_n: B \to E$ which are linearly independent (i.e. $s_1(x), \dots, s_n(x)$ is an independent set in $p^{-1}(x) \cong \mathbb{R}^n$).

Reduction of the structure group

Example We know that the structure group of every real vector bundle of dimension m is $Gl(\mathbb{R}, m)$; hence, in particular, $Gl(\mathbb{R}, 1) = \mathbb{R}^*$ is the structure group of the 1-bundle $H \to \mathbb{R}P^n$. However, it is possible to construct it with cocycles which factor through $\mathbb{Z}/2 \subset \mathbb{R}^*$. Hence we say that $H \to \mathbb{R}P^n$ admits a reduction of its structure group to $\mathbb{Z}/2$

More generally, let $p: E \to B$ be a real vector bundle.

DEF A riemannian metric on $p: E \to B$ is a scalar product on each fiber $p^{-1}(b)$ varying continuously with $b \in B$. Precisely, it is a continuous map

$$\mu: E \times_B E \longrightarrow \mathbb{R}$$
$$E \times_B E = \{(e, e') \in E \times E : p(e) = p(e')\}$$

such that for each b, $\mu_b: p^{-1}(b) \times p^{-1}(b) \to \mathbb{R}$ (the restriction) is a positive definite symmetric bilinear form.

PROP Every real vector bundle $p: E \to B$ over a paracompact base space B admits a riemannian metric

Proof. Let $\mathcal{U}$ be an open cover and $\Phi_U : p^{-1}U \to U \times \mathbb{R}^m$ be trivializations, $U \in \mathcal{U}$. Let

$$\mu_U : p^{-1}U \times_U p^{-1}U \longrightarrow \mathbb{R}$$

be given by $\mu_U(e,e') = \langle y, y' \rangle$, where y is the vector part of $\Phi_U(e) \in U \times \mathbb{R}^m$, and correspondingly y', and $\langle , \rangle$ is the canonical ~~[illegible]~~ scalar (dot) product in $\mathbb{R}^m$. We thus have locally a riemannian metric. Let now

$$\{\alpha_U : B \to I = [0,1]\}_{U \in \mathcal{U}}$$

be a locally finite partition of unit subordinated to $\mathcal{U}$. and define

$$\mu(e,e') = \sum_{U \in \mathcal{U}} \alpha_U(b)\, \mu_U(e,e'), \quad b = p(e) = p(e')$$

for $(e,e') \in E \times_B E'$, which is well defined, since if $b \notin U$, then $\alpha_U(b) = 0$. This proves the PROP $\square$

COR <u>Every m-vector bundle</u> $p : E \to B$ <u>with paracompact base space admits cocycles with coefficients in the orthogonal group</u> $O(m) \subset Gl(m, \mathbb{R})$.

Proof. Take a open cover $\mathcal{U}$ of B such that for each $U \in \mathcal{U}$, $p^{-1}U$ is trivial. Hence there are sections $s_1^U, \dots, s_m^U : U \to p^{-1}U$ which are linearly independent at each fiber. Since we have a Riemannian metric on $E \to B$, then we may assume (after a Gram-Schmidt orthonormalization process) that the sections build fiberwise an orthonormal basis.

Define trivializations

$$\Phi_U : p^{-1}U \longrightarrow U \times \mathbb{R}^m$$

by simply mapping $s_i^U(b)$ to (b, e_i), $e_i = (0, \dots, \overset{i}{1}, \dots, 0)$. This gives orthogonal isomorphisms between the fibers; hence, the transition maps

$$\Phi_V \circ \Phi_U^{-1} : (U \cap V) \times \mathbb{R}^m \longrightarrow (U \cap V) \times \mathbb{R}^m$$
$$(b, y) \longmapsto (b, c_{V,U}(b)(y))$$

determine cocycles with the desired coefficients, i.e.

$$c_{V,U} : U \cap V \longrightarrow O(m) = \{A \in Gl(m, \mathbb{R}) ; AA^t = I\},$$

since every time, the standard basis maps onto an orthonormal basis. $\square$

DEF Let $p: E \to B$ be a fiber bundle with fiber F and structure group $G \subset \mathrm{Homeo}(F)$. We say that it admits a reduction of its structure group to $G' \subset G$, if we can find cocycles for $p: E \to B$ with coefficients in G'. So we have

THM Every real vector bundle admits a reduction of its structure group to an orthogonal group (which is compact). We simply say that every real vector bundle of dimension m is an $O(m)$-bundle $\square$

Observe that we have the following subgroups of $O(m)$

$$SO(m) = \{A \in O(m) : \det A = 1\} = \{\text{orthogonal matrices which preserve orientation}\}$$

If $m = 2n$, $\quad U(n) \subset O(2n)$, $U(n) = \{\text{complex matrices which preserve the hermitian product in } \mathbb{C}^n\}$
$$= \{A \in Gl(n, \mathbb{C}) : A\bar{A}^t = I\}$$

DEF A vector bundle of dimension m is said to be orientable if it admits a reduction of its structure group to $SO(m)$. If $m = 2n$, we say it is a complex bundle of dimension n if it admits a reduction of its structure group to $U(n)$. Hence, every complex bundle is orientable.

Examples

1. Let $p: E \to B$ be a real bundle of dimension n,

We may construct a complex bundle of dimension n, denoted by $c(E) = E \otimes \mathbb{C} \to B$, by simply tensoring every fiber of p with the complex numbers. In the language of cocycles it amounts to consider

$$O(n) \hookrightarrow U(n)$$

and taking cocycles

$$c'_{V,U} : U \cap V \xrightarrow{c_{V,U}} O(n) \hookrightarrow U(n),$$

where the inclusion $O(n) \hookrightarrow U(n)$ is simply given by considering a matrix $A \in O(n)$ as a complex matrix (without imaginary parts in its entries).

This is called the complexification of $p: E \to B$.

Given now a complex bundle of dimension n, $p: E \to B$, we can construct a real bundle of dimension $2n$, denoted by $r(E) \to B$, by simply forgetting the complex structure. In the language of cocycles it means taking $U(n) \subset O(2n)$ and new cocycles

$$c'_{V,U} : U \cap V \xrightarrow{c_{V,U}} U(n) \subset O(2n)$$

This is called the realification of $p: E \to B$.

Exercise Prove that $rc(E) = 2E = E \oplus E \to B$.

According to what reduction a bundle admits, it ~~[illegible]~~ has a corresponding "structure".

Example. $H \to \mathbb{R}P^n$ is a $\mathbb{Z}/2$-bundle. It is not orientable, since a further reduction of the group $\mathbb{Z}/2$ would make it trivial. In particular, the Moebius band, being non_trivial, is nonorientable as bundle.

Exercise Prove that $c(H) \to \mathbb{R}P^1$ is trivial.

DEF Let M^m be a smooth manifold and G a subgroup of $Gl(m, \mathbb{R})$. We say that M has a G-structure if its tangent bundle admits a reduction of its structure group to G.

Hence, a *riemannian manifold* is a manifold M, together with a riemannian metric on $T(M) \to M$;

etc. a *complex manifold* is a manifold M, together with a complex structure on $T(M) \to M$;

a *hermitian manifold* is a ~~[illegible]~~ complex manifold M, together with a "hermitean metric" on $T(M) \to M$, i.e. a hermitean product on each fiber varying continuously with the fiber.

Remark A bundle - or a manifold - can admit several structures. Two G-structures need not be equal. ~~[illegible]~~

~~Exercise. Prove that given two riemannian metrics on $p: E \to B$, there exists an isomorphism $E \to E$ sending one to the other.~~

Exercises

1. Prove that if $p: E \to B$ is a trivial (real) bundle, then $c(E) \to B$ is a trivial (complex) bundle.
2. Prove that if $p: E \to B$ is a trivial complex bundle, then $r(E) \to B$ is a trivial (real) bundle.
3. Prove that, given two riemannian metrics on $p: E \to B$ there is an isomorphism $\Phi: E \to E$ of bundles sending one to the other, i.e. such that if μ and μ' are the metrics, then $\mu'(\Phi(e), \Phi(e')) = \mu(e, e')$ if $p(e) = p(e')$.

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