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CONSTRUCTION OF BUNDLES.

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Construction of bundles

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We are going to see vector bundles from a different point of view. We saw a vector bundle as a total space decomposed into certain pieces. Now we are going to construct the bundle with the pieces. In order to do this we shall give a slightly different, but equivalent, definition of a vector bundle. From this definition we will find out how to glue the pieces.

Definition. A vector bundle $\xi = (E, p, B)$ of dimension n consists of a map $p: E \to B$ and a trivializing cover $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$, where $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of B and where $\varphi_\alpha : U_\alpha \times \mathbb{R}^n \longrightarrow p^{-1}(U_\alpha)$ is a homeomorphism. These homeomorphisms are such that the following diagram commutes:

$$\begin{tikzcd} U_\alpha \times \mathbb{R}^n \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\text{projection}"'] & & p^{-1}(U_\alpha) \arrow[dl, "p"] \\ & U_\alpha & \end{tikzcd}$$

(This condition implies that for each $b \in B$, φ_α restricts to a homeomorphism $\varphi_{\alpha,b} : \mathbb{R}^n \cong \{b\} \times \mathbb{R}^n \longrightarrow p^{-1}(b)$.

Furthermore, if $b \in U_\alpha \cap U_\beta$ then the composition

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^n \xrightarrow{\varphi_\beta} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\alpha^{-1}} (U_\alpha \cap U_\beta) \times \mathbb{R}^n,$$

which is of the form $(b, v) \longmapsto (b, \hat{g}_{\alpha\beta}(b, v))$, must satisfy the following condition: $\hat{g}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is linear on the second coordinate. (This condition implies that for each $b \in U_\alpha \cap U_\beta$, $\hat{g}_{\alpha\beta}(b, -) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism).

Lemma: Let X be a space then there is a bijection between the following sets of maps (map = continuous function):

$$\left\{ \hat{f} : X \times \mathbb{R}^n \to \mathbb{R}^n \;\middle|\; \begin{array}{l} \hat{f}(x, -) \text{ is a linear} \\ \text{isomorphism for each} \\ x \in X \end{array} \right\} \longleftrightarrow \{ f : X \to GL(n, \mathbb{R}) \}$$

proof: The bijection is given by the following formula: $f(x)(v) = \hat{f}(x, v)$. The linearity condition is clear. We see that the bijection sends maps to maps as follows.

Given $f : X \to GL(n, \mathbb{R})$ continuous, $\hat{f}$ is given by

$$\begin{tikzcd} X \times \mathbb{R}^n \arrow[r, dashed, "\hat{f}"] \arrow[d, "f \times id"'] & \mathbb{R}^n \\ GL(n, \mathbb{R}) \times \mathbb{R}^n \arrow[ur, "\text{evaluation}"'] & \end{tikzcd}$$

The evaluation is given by a product of matrices so it is continuous. Hence $\hat{f}$ is continuous.

Now given $\hat{f} : X \times \mathbb{R}^n \to \mathbb{R}^n$ continuous, f is continuous if each entry of the matrix $f(x) = a_{ij}(x)$ is continuous. But each $a_{ij}(x)$ is given by the following composition of maps:

$$\begin{tikzcd} x \arrow[d, mapsto] & X \arrow[r, dashed, "a_{ij}(-)"] \arrow[d] & \mathbb{R} \\ (f(x), e_j) & GL(n, \mathbb{R}) \times \mathbb{R}^n \arrow[r, "\text{evaluation}"'] & \mathbb{R}^n \arrow[u, "\pi_i"'] \end{tikzcd}$$

where $e_j = (0, 0, \ldots 0, 1, 0 \ldots 0)$ (1 in the j-th place) and π_i is the projection on the i-th factor. $\square$

Now if $b \in U_\alpha \cap U_\beta \cap U_\gamma$ we have that

$\varphi_\alpha^{-1} \circ \varphi_\beta \circ \varphi_\beta^{-1} \circ \varphi_\gamma = \varphi_\alpha^{-1} \circ \varphi_\gamma$, from this we get that

$\hat{g}_{\alpha\beta} \circ \hat{g}_{\beta\gamma} = \hat{g}_{\alpha\gamma}$ (on $U_\alpha \cap U_\beta \cap U_\gamma$) wich, in terms of the maps given by the bijection of the previous lemma, can be written as

(*) $g_{\alpha\beta}(b)\, g_{\beta\gamma}(b) = g_{\alpha\gamma}(b)$ in $GL(n,\mathbb{R})$.

Therefore an n-vector bundle over B gives us the following

1) An open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of B
2) for each $U_\alpha \cap U_\beta \neq \emptyset$, maps $g_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow GL(n,\mathbb{R})$ wich satisfy the condition (*).

<u>Remark</u>: Since in particular $g_{\alpha\alpha}(b) = g_{\alpha\alpha}(b)\, g_{\alpha\alpha}(b)$ for all $b \in U_\alpha$

hence $g_{\alpha\alpha}(b) = 1 \in GL(n,\mathbb{R})$, $b \in U_\alpha$.

Taking $\gamma = \alpha$ we get $g_{\alpha\beta}(b)\, g_{\beta\alpha}(b) = g_{\alpha\alpha}(b) = 1$, $b \in U_\alpha \cap U_\beta$

hence $g_{\alpha\beta}(b) = g_{\beta\alpha}(b)^{-1}$ in $GL(n,\mathbb{R})$

We call the maps $g_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow GL(n,\mathbb{R})$ the <u>transition maps</u> of the bundle ξ.

The fact that the transition maps have values in a group of linear maps implies that each fiber of ξ, $p^{-1}(b)$, $b \in B$, can be given a vector space structure as follows.

If $b \in U_\alpha$ and $v_1, v_2 \in p^{-1}(b)$ we define

$v_1 + v_2 = \varphi_{\alpha,b}(\varphi_{\alpha,b}^{-1}(v_1) + \varphi_{\alpha,b}^{-1}(v_2))$. To see that this is well

defined, suppose $b \in U_\beta$ then we have another sum given by

$v_1 + v_2 = \varphi_{\beta,b}(\varphi_{\beta,b}^{-1}(v_1) + \varphi_{\beta,b}^{-1}(v_2))$.

Consider $\varphi_{\beta,b}^{-1}(v_1+v_2) = \varphi_{\beta,b}^{-1} \circ \varphi_{\alpha,b}(\varphi_{\alpha,b}^{-1}(v_1) + \varphi_{\alpha,b}^{-1}(v_2))$

$= \varphi_{\beta,b}^{-1} \circ \varphi_{\alpha,b}(\varphi_{\alpha,b}^{-1}(v_1)) + \varphi_{\beta,b}^{-1} \circ \varphi_{\alpha,b}(\varphi_{\alpha,b}^{-1}(v_2))$

for $\varphi_{\beta,b}^{-1} \circ \varphi_{\alpha,b} = \hat{g}_{\beta\alpha}(b,-) = g_{\beta\alpha}(b) \in GL(n,\mathbb{R})$

$= \varphi_{\beta,b}^{-1}(v_1) + \varphi_{\beta,b}^{-1}(v_2)$

Therefore $v_1 + v_2 = \varphi_{\beta,b}(\varphi_{\beta,b}^{-1}(v_1) + \varphi_{\beta,b}^{-1}(v_2)) = v_1 + v_2$ and hence the sum is well defined. Similarly for the scalar product. With this vector space structure on each fiber the homeomorphisms $\varphi_{\alpha,b}$ become linear isomorphisms.

We also have a description of isomorphism of bundles in terms of transition maps as follows.

<u>Definition</u>. A <u>transition system</u> on B with values in $GL(n,\mathbb{R})$, $(\{U_\alpha\}_{\alpha\in\Lambda}, g_{\alpha\beta})$ is an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ for B and a collection of maps $g_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow GL(n,\mathbb{R})$, $\alpha,\beta \in \Lambda$ such that $g_{\alpha\gamma}(b) = g_{\alpha\beta}(b)\, g_{\beta\gamma}(b)$ (*) for all $b \in U_\alpha \cap U_\beta \cap U_\gamma$. These maps are also called cocycles.

We say that two transition systems $(\{U_\alpha\}_{\alpha\in\Lambda}, g_{\alpha\beta})$, $(\{V_i\}_{i\in I}, h_{ij})$ are <u>equivalent</u> if there are maps $\lambda_{i\alpha}: U_\alpha \cap V_i \longrightarrow GL(n,\mathbb{R})$, $\alpha \in \Lambda$, $i \in I$, such that

$$h_{ij}(b) = \lambda_{i\alpha}(b)\, g_{\alpha\beta}(b)\, \lambda_{j\beta}(b)^{-1}$$

for all $b \in U_\alpha \cap U_\beta \cap V_i \cap V_j$

<u>Remark</u>: If the transition systems have the same open cover, say $\{U_\alpha\}_{\alpha\in\Lambda}$, then one can easily show that they are equivalent if there are maps $\lambda_\alpha : U_\alpha \to GL(n,\mathbb{R})$ such that

$$h_{\alpha\beta}(b) = \lambda_\alpha(b)\ g_{\alpha\beta}(b)\ \lambda_\beta(b)^{-1},\ b \in U_\alpha \cap U_\beta .$$

The transition maps of a vector bundle clearly define a transition system and we have the following

<u>Theorem</u>. Two vector bundles $\xi = (E, p, B)$ and $\xi' = (E', p', B)$ are isomorphic $\Leftrightarrow$ their transition systems are equivalent

<u>sketch proof</u>: $\Rightarrow$) Let $f : E \to E'$ be an isomorphism. Take

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

a trivializing cover $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ for ξ. If $b \in U_\alpha \cap U_\beta$ then $\varphi_\beta(b,v) = \varphi_\alpha(b, g_{\alpha\beta}(b)(v)) \Rightarrow f\varphi_\beta(b,v) = f\varphi_\alpha(b, g_{\alpha\beta}(b)(v))$. Hence we can give a trivializing cover for ξ' by taking $U_\alpha \times \mathbb{R}^n \xrightarrow{\varphi_\alpha} p^{-1}(U_\alpha) \xrightarrow{f} (p')^{-1}(U_\alpha)$. The transition system associated to this cover is equivalent to the transition system of ξ.

$\Leftarrow$) Let $\{U_\alpha, \varphi_\alpha\}$ and $\{V_i, \rho_i\}$ be trivializing covers for ξ and ξ' respectively. By taking the intersections $U_\alpha \cap V_i$ and the restriction to these intersections of the maps φ_α and ρ_i we can always assume that the open cover is the same for both bundles. So let $(\{U_\alpha\}, g_{\alpha\beta})$ and $(\{U_\alpha\}, g'_{\alpha\beta})$ be transition systems for ξ and ξ' respectively. Assume that they are equivalent, then we have maps $\lambda_\alpha : U_\alpha \to GL(n,\mathbb{R})$ and we can define $f_\alpha : p^{-1}(U_\alpha) \to E'$ by the commutativity of the following diagram:

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[r, dashed, "f_\alpha"] & (p')^{-1}(U_\alpha) \subset E' \\ U_\alpha \times \mathbb{R}^n \arrow[u, "\varphi_\alpha"] \arrow[r] & U_\alpha \times \mathbb{R}^n \arrow[u, "\varphi'_\alpha"'] \\ (b,v) \arrow[r, mapsto] & (b, \lambda_\alpha(b)(v)) \end{tikzcd}$$

Where φ_α and φ'_α are local trivializations for ξ and ξ' respectively. One can check that these maps f_α define a map $f : E \to E'$ and that f is an isomorphism of bundles $\square$

This means that 1) we have a well defined function:

$$\left\{ \begin{matrix} \text{isomorphism classes of} \\ n\text{-vector bundles over } B \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{equivalence classes of transition} \\ \text{systems on } B \text{ with values in } GL(n,\mathbb{R}) \end{matrix} \right\}$$

2) this function is <u>injective</u>.

To see that this function is <u>surjective</u> we have the following

<u>Theorem</u>. Given a transition system $(\{U_\alpha\}_{\alpha\in\Lambda}, g_{\beta\alpha})$ on B with values in $GL(n,\mathbb{R})$, there exists an n-vector bundle $\xi = (E, p, B)$ with transition maps $g_{\beta\alpha}$. This bundle is unique up to isomorphism.

<u>Sketch proof</u>. Take the disjoint union $\coprod_{\alpha\in\Lambda} U_\alpha \times \mathbb{R}^n$ and define the following relation: $(b,v) \in U_\alpha \times \mathbb{R}^n$ and $(b',v') \in U_\beta \times \mathbb{R}^n$ are equivalent $\Leftrightarrow$ $b = b'$ and $g_{\alpha\beta}(b)(v') = v$. The conditions (*) imply that this is an equivalence relation.

We define $E = \coprod_{\alpha\in\Lambda} U_\alpha \times \mathbb{R}^n/\sim$ and $p : E \to B$ by $p[b,v] = b$. The trivializing cover is $\{U_\alpha, \varphi_\alpha\}$, where

$\varphi_\alpha : U_\alpha \times \mathbb{R}^n \longrightarrow p^{-1}(U_\alpha)$ is given by $\varphi_\alpha(b,v) = [b,v]$. $\square$

Examples:

1) Let M be a differentiable n-manifold, with an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$. Then, in order to construct the tangent bundle $TM \to M$ of M, we only have to define a transition system on M. We take as a cover $\{U_\alpha\}_{\alpha\in\Lambda}$ and the maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(n,\mathbb{R})$ are given by $g_{\alpha\beta}(x) = D(\varphi_\alpha \circ \varphi_\beta^{-1})(\varphi_\alpha(x)) \in GL(n,\mathbb{R})$. Using the chain rule one can show that these maps satisfy the conditions $(*)$.

2) The usual operations with vector bundles, like Whitney sum, tensor product, exterior power, etc. can be given in terms of transition systems. For example if $\xi = (E,p,B)$ is an n-vector bundle with transistion system $(\{U_\alpha\}, g_{\alpha\beta})$ and $\xi' = (E',p',B)$ is a m-vector bundle with transition system $(\{U_\alpha\}, g'_{\alpha\beta})$ then the Whitney sum $\xi \oplus \xi'$ is an $(n+m)$-vector bundle over B whose transition system is $(\{U_\alpha\}, h_{\alpha\beta})$, where $h_{\alpha\beta}$ is given by the following composition:

$$U_\alpha \cap U_\beta \longrightarrow GL(n,\mathbb{R}) \times GL(m,\mathbb{R}) \longrightarrow GL(n+m,\mathbb{R})$$

$$b \longmapsto (g_{\alpha\beta}(b),\ g'_{\alpha\beta}(b))$$

$$(A, B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

3) Given a vector bundle $\xi = (E,p,B)$ with a transition system $(\{U_\alpha\}, g_{\alpha\beta})$ and a map $f : X \to B$, we construct the pull-back $f^*\xi$ of ξ by taking the transition system $(\{f^{-1}(U_\alpha)\}, h_{\alpha\beta})$, where $h_{\alpha\beta}$ is given by the following composition:

$$f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) = f^{-1}(U_\alpha \cap U_\beta) \xrightarrow{f} U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} GL(n,\mathbb{R}).$$

We have used the fact that the transition maps of a vector bundle ξ have values in $GL(n,\mathbb{R})$ to give ξ a linear structure on each fiber. In a similar way if the transition maps have values in a subgroup $G \subset GL(n,\mathbb{R})$ then we can give ξ an extra structure wich is not necessarily unique. For example if $G = SL(n,\mathbb{R})$ then ξ is orientable. If $G = O(n)$ then we can give ξ a Riemannian metric, etc. These structeres will be studied in greater detail later on.

References

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