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COLLEGE

ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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LIE GROUPS AND LIE ALGEBRAS.
(continued)

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CHAPTER II

Integration on Lie groups and applications.

In this chapter we assume some familiarity with integration of forms on C^∞ manifolds. Our basic references are [Adams], [Serre], [Swan] and [Warner].

Let $\{X_1,\dots,X_n\}$ be a basis of left invariant vector fields of the Lie group G and let $\{\omega_1,\dots,\omega_n\}$ be the dual basis of 1-forms, which are therefore also left invariant.

Then $\omega := \omega_1\wedge\dots\wedge\omega_n$ is a globally defined left invariant n-form on G that determines an orientation and we may so define integration of n-forms with compact support on G:

The space of left invariant n-forms on G is isomorphic to $\mathbb{R}$ and therefore the n-forms that determine the same orientation as ω are the positive scalar multiples of ω.

If G is compact we can always define $\int_G \omega$ and so, by choosing the appropriate multiple of ω, we have $\int_G \omega = 1$.

Recall that, in general, we are integrating n-forms w, not necessarily left invariant, which are therefore $w = f\cdot\omega$ for some C^∞ function f on G. The integration is considered over a domain $\mathcal{D}$ in G, where f has compact support.

Recall that given a triangulation of $\mathcal{D}$, $\mathcal{D} = \sum_i \alpha_i \sigma_i(I^n)$, α_i in $\mathbb{R}$, i runs over a finite set, $\sigma_i : I^n \to \mathcal{D}$ are orientation preserving diffeomorphisms into $\mathcal{D}$,

$$\int_{I^n} (\delta\sigma_i) f\omega \equiv \int_{\mathrm{Image}(\sigma_i)} f\omega$$

plus linearity over linear combinations of chains, defines the integral $\int_{\mathcal{D}} f\omega$ or $\int_{\mathcal{D}} f$ as it is usually denoted.

Given any g in G, L_g is an orientation preserving diffeomorphism since $\delta L_g(\omega) = \omega$ and therefore, for f with compact support in G,

$$\int_G f \equiv \int_G f\omega = \int_G \delta L_g(f\omega) = \int_G \delta L_g(f)\ \delta L_g(\omega) =$$

$$= \int_G (f\circ L_g)\omega \equiv \int_G f\circ L_g .$$

In other words this integral is left invariant. We would like to know when is this integral also right invariant, i.e, under what conditions is $\int_G f = \int_G f\circ R_g$, for all g in G.

As right and left translations commute, $\delta R_g(\omega)$ is still a left invariant n-form and therefore a constant multiple of ω, for each g in G. I.e., $\delta R_g(\omega) = \tilde{\lambda}(g)\omega$, where $\tilde{\lambda}$ is a C^∞ function : $G \longrightarrow \mathbb{R}$.

<u>Exercise 1</u>. Show that $\tilde{\lambda}$ is a group morphism into the multiplicative group of non-zero real numbers and that $\tilde{\lambda}(g) > 0$ iff R_g is orientation preserving.

Let $\lambda(g) := |\tilde{\lambda}(g)|$ and observe that $\lambda : G \to \mathbb{R}^+$ is a group morphism, which is called a modular function.

Observe now that for each g in G

$$\int_G f\omega = \pm \int_{R_g^{-1}(G)} \delta R_g(f\omega) = \int_G \delta R_g(f\omega) = \pm \int_G (f\circ R_g)(\delta R_g \circ \omega) =$$

$$= \pm \int_G (f\circ R_g)\, \tilde{\lambda}(g)\,\omega$$

with + when R_g is orientation preserving. Using Exercise 1 the last expression is equal to $\int_G (f\circ R_g)\lambda(g)\omega$. I.e. , the integral is right invariant iff $\lambda(g) = \omega$ for all g in G, in which case G is called <u>Unimodular</u>.

<u>COROLLARY 2</u>. All compact Lie groups are unimodular.

PROOF: Im(λ) is a compact subgroup of $\mathbb{R}^+$, so it is {1}. QED.

From now on we consider this bi-invariant integral on compact Lie groups.

<u>COROLLARY 3</u>. A compact Lie group G has a left and right invariant metric, usually called <u>bi-invariant</u>.

PROOF: Fix a left invariant metric $\langle\ ,\ \rangle$ on G by $\langle v(g), \omega(g)\rangle := \langle dL_{g^{-1}} v(g), dL_{g^{-1}} \omega(g)\rangle'$ where $\langle\ \ \rangle'$ is any scalar product on T_eG. Define $\langle\langle v, \omega\rangle\rangle := \int_G \langle dR_g(v), dR_g(w)\rangle\omega$ where $f(g) \equiv \langle dR_g v, dR_g w\rangle$ is a C^∞ real function on G. The metric

$\langle\langle\ ,\ \rangle\rangle$ is obviously left invariant, since $L_g \circ R_h = R_h \circ L_g$. Observe now that $(f\circ R_h)(g) = f(gh) = \langle dR_{gh} v, dR_{gh} w\rangle = \langle dR_g \circ dR_h v, dR_g dR_h w\rangle$ which together with the bi-invariance of the integral implies

$$\langle\langle v, w\rangle\rangle = \langle\langle dR_h v, dR_h w\rangle\rangle . \qquad \text{QED.}$$

DEFINITION 4: If $V, \langle\ ,\ \rangle$ is a real (resp. complex) inner product space and $\rho : G \to Aut(V)$ is a representation of G then we say that ρ is orthogonal (resp. unitary) iff $\langle \rho(g) v, \rho(g) w\rangle = \langle v, w\rangle$ for all g in G, v, w in V.

COROLLARY 5. Given a representation ρ of a compact G on a real (resp. complex) vector space V there is always an inner product on V with respect to which ρ is orthogonal (resp. unitary).

PROOF: Let $\langle\ ,\ \rangle$ be any inner product on V and define

$$\langle\langle u, v\rangle\rangle := \int_G \langle \rho(g)u, \rho(g)v\rangle\omega \equiv \int_G f(g)\,\omega =$$

$$= \int_G (f\circ R_h)(g)\,\omega = \int_G \langle \rho(gh)u, \rho(gh)v\rangle\omega =$$

$$= \int_G \langle \rho(g)\rho(h)u, \rho(g)\rho(h)v\rangle\omega \equiv \langle\langle \rho(h)u, \rho(h)v\rangle\rangle,$$

having used the invariance of the integral. QED.

<u>Exercise 6</u>. Consider the Adjoint representation of G in its

Lie algebra and show that Corollary 3 can be proved as a consequence of Corollary 5.

More generally, show that a Lie group G has a bi-invariant metric iff the closure of Ad(G) in Aut($\mathcal{G}$) is compact.

By declaring the integral linear, i.e., commuting with linear maps, we can define integration of vector space valued functions on G. For example, if ρ is a representation $\rho: G \to \mathrm{Aut}(V)$ then we can define $I_\rho \equiv \int_G \rho$ in Aut (V), [Adams].

PROPOSITION 7: I_ρ is an idempotent, i.e., $I_\rho^2 = I_\rho$, whose image is V_G, the subspace of elements that are left fixed by ρ. I.e., I_ρ is a projection operator.

PROOF: First observe that for a fixed v in V

$$I_\rho(v) := \Big(\int_G \rho\Big)(v) = \int_{g\in G} \rho(g)(v) \quad ,$$

since the evaluation of an automorphism on a fixed v is linear in the space of automorphisms. For v in V_G this is equal to $\int_{g\in G} v = (\int_G 1)(v) = v$, i.e., I_ρ is the identity when restricted on V_G. On the other hand, $\rho(h)\, I_\rho(v) = \rho(h) \int_G \rho(g)(v) =$ (since the integral commutes with linear maps) $\int_{g\in G} \rho(h)\rho(g)(v) =$ (since the integral is left invariant) $\int_G \rho(g)(v) = I_\rho(v)$, which implies $\mathrm{Im}(I_\rho) \subseteq V_G$. QED.

EXERCISE 8: The trace of I_ρ equals $\dim(V_G)$.

DEFINITION 9: A representation $\rho: G \to \mathrm{Aut}(V)$ is called <u>simple</u> or <u>irreducible</u> iff there are no nontrivial ρ-invariant subspaces of V.

For example, all 1-dimensional representations are irreducible, as is the Ad-representation of S^3 on $\mathbb{R}^3$.

EXERCISE 10. Investigate the irreducibility of the representations of the classical groups.

DEFINITION 11. A representation $\rho: G \to \mathrm{Aut}(V)$, $\dim V < \infty$, is <u>semi-simple</u> iff every ρ-invariant subspace $W \subseteq V$ has a complementary ρ-invariant subspace W': $W \oplus W' = V$.

EXERCISE 12. If G is compact, every finite dimensional representation of G is semi-simple.

REMARK 13. If ρ is an irreducible orthogonal representation of a compact G in a Hilbert space, then ρ is finite dimensional. This fact is not hard to show [Kirillov]. We will however restrict ourselves to finite dimensions in these notes.

PROPOSITION 14: The inner product on a finite dimensional V, with respect to which a simple representation $\rho: G \to \mathrm{Aut}(V)$ of the compact Lie group G is orthogonal, is unique up to a constant (positive) factor.

PROOF: Let $\langle , \rangle$ and $(,)$ be two such scalar products.

It is an easy exercise of linear algebra to show that there is a basis $e_1, \dots, e_n$ of V with $\langle x,y\rangle = \sum_i x^i y^i$, $(x,y) = \sum_i a_i x^i y^i$, $a_i > 0$ for all $x = \sum_i x^i e_i$ and $y = \sum_i y^i e_i$. Consider the symmetric, bilinear map

$$f(x,y) := \langle x,y\rangle - a_1^{-1}(x,y).$$

we have $f(e_1, e_j) = 0$ for all $j = 1,\dots,n$ wich implies

$$\{x \text{ in } V \mid f(x,y) = 0, \forall y\} \neq \{0\}.$$

This subspace of V is also G-invariant by $f(gx,y) = f(x,g^{-1}y)$ and simplicity implies it is the whole V, i.e., $f \equiv 0$ and $a_1\langle x,y\rangle = (x,y)$ for all x,y in V. QED.

REMARK 15: In general every finite dimensional representation ρ of a compact G is semisimple and V splits into $V_1 \oplus \dots \oplus V_m$, which is a sum of irreducibles, orthogonal with respect to any ρ-invariant scalar product on V. (Exercise 12). So, given any two such scalar products $\langle\,,\,\rangle$ and $(\,,\,)$ we have from the above $\langle v_i, w_i\rangle = c_i \langle v_i, w_i\rangle$ for all v_i, w_i in V_i and for a positive c_i. Let $h_i(v_i) := \sqrt{c_i}\, v_i$ for all v_i in V_i. This way we get a ρ-invariant h in Aut(V) with $\langle v, w\rangle = \langle hv, hw\rangle$.

DEFINITION 16: A Lie group G is called <u>simple</u> iff its Lie algebra $\hat{G}$ is <u>simple</u>, i.e., it has no non-trivial ideals and it is non-abelian.

PROPOSITION 17: If G is simple and compact then the bi-invariant metric on G is unique up to a constant factor.

PROOF: Enough to show that $Ad: G \to Aut(\hat{G})$ is a simple representation: Let $V \neq \{0\}$ be an Ad-invariant subspace of $\hat{G}$. I.e., $Ad_g(v)$ in V for all g in G and all v in V. Taking $g = e^{tX}$ now we have $Ad_{e^{tX}}(v) = Exp\, ad_{tX}(v) = Exp(tA)(v)$ is a curve in V whose tangent at t = 0 is $[X,v]$. As X is arbitrary in $\hat{G}$, this shows V is an ideal of $\hat{G}$ and it must be {0} or $\hat{G}$ by simplicity. QED.

EXERCISE 18: If G is a Lie group with a bi-invariant metric then the inversion $x \mapsto x^{-1}$ is an isometry of G.

THEOREM 19: If G is as above then every geodesic is the left translate of a 1-parameter subgroup.

PROOF. [Swan] Step 1. It is sufficient to show it for "short" geodesics (within the ball where the exponential is a diffeomorphism): If c is a "long" geodesic it breaks into "small" arcs c_i each of which is the left translate of a one parameter subgroup. If x in $c_i \cap c_{i+1}$, translate the picture to the identity by the isometry $L_{x^{-1}}$, where $L_{x^{-1}}(c_i)$ and $L_{x^{-1}}(c_{i+1})$ are 1-parameter subgroups that form the same geodesic: they must have the same tangent at e, so they form the same 1-parameter subgroup.

Step II: If $\tau : M \to M$ is an isometric involution of a riemannian manifold M with an isolated fixed point A in M, then if x is "very near" A the (unique) geodesic from x to $\tau(x)$ goes through A. This is because $\tau(x)$ is also near A and there is one short geodesic γ from x to $\tau(x)$. Since $\tau(\gamma)$ is also a geodesic from $\tau(x)$ to x it must be the same γ, which has therefore a fixed point, namely A.

Step III: Let B be a point near 1 and $x(t)$ be the 1-parameter subgroup from 1 to B with $x(1) = B$. Define $\tau : G \to G$ by $\tau(y) = Ay^{-1}A$ for $A = x(\frac{1}{2})$, an isometric involution on G. Then $\tau(y) = y$ iff $Ay^{-1}A = y$, i.e., iff $Ay^{-1} = (Ay^{-1})^{-1}$ or $(Ay^{-1})^2 = 1$. One fixed point of τ is obtained by $Ay^{-1} = 1$ or $y = A$. Let $Ay^{-1} = e^{z}$ now with $(Ay^{-1})^2 = 1$, i.e., $e^{2z} = 1$. This z must be fairly large now if it has to be $\neq 0$, so that we don't get the same point $Ay^{-1} = 1$ fixed, since exp is a diffeomorphism near 1. In other words $y = A$ is an isolated fixed point and so $x(\frac{1}{2}) = A$ is the geodesic are joining 1 to B. So is $x(\frac{1}{4})$, $x = \frac{3}{4}$ and $x(t)$ for a dense set of points. Therefore, the geodesic γ contains a 1-parameter subgroup and is equal to it by its minimality. As e^{tX} has tangent of constant length, the parametrization is also right. QED.

Recall that for every complete riemannian manifold M each $\exp_p$ is onto M.

COROLLARY 20: Every compact G is "covered by" 1-parameter subgroups. (See also chapter I, Exer.)

LEMMA 21: If $\langle\, ,\rangle$ is a bi-invariant metric on the Lie group G, we have for all X, Y, Z in $\hat{G}$:

$$\langle [X,Y], Z\rangle + \langle [X,Z], Y\rangle = 0$$

PROOF: Observe $\langle Ad_{e^{tX}} Y, Z\rangle = \langle Y, Ad_{e^{-tX}} Z\rangle$ for all real t and take the derivative at $t = 0$. QED.

LEMMA 22: If is a compact Lie group then for every pair X, Y in $\hat{G}$ there exists a g_0 in G with $[X, Ad_{g_0} Y] = 0$.

PROOF: The adjoint orbit $Ad_G(Y)$ is compact in $\hat{G}$ and there is a minimum $Y_0 = Ad_{g_0} Y$ of the function $g \mapsto \langle X, Ad_g(Y)\rangle$, where $\langle\, ,\rangle$ is bi-invariant. Now for all Z in $\hat{G}$ we have $\frac{d}{dt}\Big|_{t=0} \langle X, Ad_{e^{tZ}} Y_0\rangle = 0$ (since $\langle X, Y_0\rangle$ is minimum along the orbit). But this equals $\langle X, ad_Z Y_0\rangle$, i.e., $\langle X, [Z, Y_0]\rangle$.
By Lemma 21, $\langle [X, Y_0], Z\rangle = 0$ for all Z in $\hat{G}$. QED.

DEFINITION 23: Define $Int(\hat{G})$ as the Lie subgroup of $Aut(\hat{G})$ whose Lie algebra is $ad(\hat{G})$ in $End(\hat{G})$. I.e., $Int(\hat{G}) = Exp\, ad(\hat{G}) \subseteq Aut(\hat{G})$ and $Ad(G) = Int(\hat{G})$.

Exercise 24: Let G be a connected Lie group with a bi-invariant metric $\langle\, ,\rangle$. Show that

(i) $Ad : G \to Int(\hat{G})$ has kernel $Z(G)$: the center of the group G, whose Lie algebra $\hat{Z}$ is ker(ad), an ideal in $\hat{G}$.

Let now $\hat{G}'$ be the orthogonal complement of $\hat{Z}$ in $\hat{G}$ with respect to $\langle\ ,\ \rangle$: $\hat{G} = \hat{Z} \oplus \hat{G}'$.

(ii) Show that $\hat{G}'$ is a subalgebra of $\hat{G}$ with center 0.

(iii) Any abelian ideal in $\hat{G}'$ is central.

(iv) If α_1 is an Ad_G-invariant subspace of $\hat{G}$ then $\alpha_1^{\perp}$ also is Ad_G-invariant and $\hat{G}$ splits into Ad_G-invariant, mutually orthogonal, irreducible subspaces $\hat{G} = \alpha_1 \oplus \cdots \oplus \alpha_n$.

(v) Each α_i is a subalgebra.

(vi) $[\alpha_i, \alpha_i^{\perp}] = 0$ which implies that each α_i is an ideal, i.e., $\hat{G}$ is the direct sum of ideals (which are simple by the next item, in the case $\hat{G} = \hat{G}'$).

(vii) If $\hat{G}$ has no center then α_i is simple.

COROLLARY 25: Let G be a connected Lie group with a bi-invariant metric.

(i) $\hat{G} = \hat{Z} \oplus \hat{G}'$ where $\hat{G}'$ has center zero and is semisimple.

(ii) If G is compact and semisimple (i.e., if $\hat{G}$ is semisimple) then $Z(G)$ is finite and conversely (: a compact Lie Group G is semisimple iff $Z(G)$ is finite.)

DEFINITION 26: (i) A real Lie algebra $\hat{G}$ is <u>compact</u> if $Int(\hat{G})$ is compact in $Gl(\hat{G})$.

(ii) Define the killing form on a real or complex Lie algebra by

$$B(X,Y) := tr\, ad_X \circ ad_Y .$$

EXERCISE 27: (i) If σ is an automorphism of $\hat{G}$ then for all X, Y in $\hat{G}$

$$B(\sigma X, \sigma Y) = B(X,Y).$$

(ii) For all X, Y, Z in $\hat{G}$ we have $B(X,[Y,Z]) =$

$$= B(Y,[Z,X]) = B(Z,[X,Y]).$$

(iii) Let $\hat{G}$ be a semi simple real Lie algebra. Then $\hat{G}$ is compact iff B is negative definite.

(iv) Every compact Lie algebra is $\hat{G} = \hat{Z} + [\hat{G}, \hat{G}]$ where $\hat{Z}$ is the center of $\hat{G}$ and the ideal $[\hat{G}, \hat{G}]$ is semi simple and compact.

(v) If $\hat{G}$ is a compact Lie algebra, there is a compact Lie Group G whose Lie algebra is isomorphic to $\hat{G}$.

REMARK 28: Let G be a <u>simply connected</u> Lie group with bi-invariant metric $\langle\ ,\ \rangle$. We saw in Exercise 24 and Corollary 25 that $\hat{G} = \hat{Z} \oplus \hat{G}'$ direct sum of Lie algebras. The simply connected abelian Lie group whose algebra is $\hat{Z}$ is $\mathbb{R}^k$ for some $k \geq 0$ and $\hat{G}'$ is the Lie algebra of a compact, simply connected Lie group, as can be seen also by Myers' theorem: As $\hat{G}'$ has no center, the Ricci curvature of G' is strictly positive and bounded away from zero, by the compactress of the unit sphere in $\hat{G}'$.

To show this ([Milnor]) we must first observe that the integral curves of left invariant vector fields are geodesics, which implies $D_X X = 0$ for a left invariant X, where D is the Levi-Civita connection associated to the bi-invariant $\langle , \rangle$. Using this we can easily show that for X, Y, Z, W left invariant vector fields on G we have:

(i) $D_X Y = \frac{1}{2}[X,Y]$

(ii) $\langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle$

(iii) $R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$

where R is the curvature tensor. (This tells us that the right hand side must also be a tensor). Now it is immediate that

(iv) $\langle R(X,Y)Z, W\rangle = \frac{1}{4}\langle [X,Y],[Z,W]\rangle$

and the sectional curvature of such a metric is always non-negative: $K(X \wedge Y) \geq 0$ and is equal to zero iff $[X,Y] = 0$. This implies that the sectional curvature of S^3 and $SO(3)$ as Lie groups with bi-invariant metric is strictly positive. These are the only ones with this property.

We now want to investigate abelian subgroups of compact

Lie groups. First observe that if H is a connected abelian Lie group then its Lie algebra $\hat{H}$ is trivial and the universal cover $\tilde{H}$ of H must be $\mathbb{R}^n$: the exp: $\mathbb{R}^n \longrightarrow \tilde{H}$ is a group morphism which is a global diffeomorphism. So $H \cong \mathbb{R}^n/D$ where D is a discrete subgroup of $\mathbb{R}^n$, i.e., a lattice $\mathbb{Z}e_1 + \ldots + \mathbb{Z}e_k$ and $H \cong T^k \times \mathbb{R}^{n-k}$ where $T^k \cong S^1 \times \ldots \times S^1$. If H is compact, connected and abelian then $T^k \cong H$, if we drop connectedness then $H \cong T^k \times A$ where A is a finite abelian group.

Suppose now that T is a <u>maximal</u> <u>connected</u> <u>abelian</u> subgroup of the compact Lie group G. If $\bar{T}$ is the closure of T then $\bar{T} = T$ by maximality. So T is comapct and thefore a torus, called a <u>maximal</u> <u>torus</u> in G.

THEOREM 29 (E. Cartan) Let G be a compact Lie group and T a maximal connected abelian subgroup. Then (i) T is a torus (ii) all maximal tori are conjugate in G and (iii) every g in G is contained in a maximal torus of G.

The proof is divided into a series of Lemmas.

LEMMA 30: A torus T is <u>monothetic</u> or monogenic, i.e., there exists x in T with $\{x^n \mid n \geq 0\}$ is dense in T. Such x is called a generator of T. (For the proof see [Adams]).

EXERCISE 31: (Kronecker) Let $a = (a_1, \ldots, a_k)$ in T^k be such that $\{1, a_1, \ldots, a_k\}$ are linearly independent over the rationals. Then a is a generator of T^k.

LEMMA 32: If G is abelian with connected component T, a torus, and $G/_T \cong \mathbb{Z}_k$, then G is monothetic.

PROOF: Let x generate T. If y in G be such that $y+T$ generates $\mathbb{Z}_k$, then y^k in T and choose z in T by $z^k y^k = x$: zy generates G.

LEMMA 33: Let T be a maximal torus in G and $x = e^X$ a generator of T. If $[X, Z] = 0$ then $e^Z \in T$.

PROOF: $ad_X(Z) = 0$ implies $e^{t\, ad_X}(Z) = (I + t\, ad_X + t^2 ad_X^2 + \ldots)(Z) = Z$, i.e., $Ad_{e^{tX}}(Z) = Z$ for all t in $\mathbb{R}$.

Fixing t, we see that the 1-parameter subgroups $s \mapsto e^{tX} e^{sZ} e^{-tX}$ and $s \mapsto e^{sZ}$ coincide. Therefore $e^{tX} e^{sZ} = e^{sZ} e^{tX}$ for all s, t. Since e^{sZ} commutes with x, it commutes with T and therefore T and $\{e^{sZ}, s \text{ in } \mathbb{R}\}$ generate a closed, connected abelian subgroup A, which is also a torus, equal to T by maximality. So e^Z is in T.

LEMMA 34: Let T be a maximal torus in G and y in G. Then there is g in G with gyg^{-1} in T.

PROOF: G compact implies $e^Y = y$ for some Y in $\hat{G}$.

Let $e^X = x$ generate T. By Lemma 22 there is g in G with $[X, Ad_g Y] = 0$. By Lemma 33 $\exp Ad_g Y$ is in T, i.e., gyg^{-1} in T.

Assume now H is a monothetic subgroup of G generated by x and x is in gTg^{-1} for some g in G and a maximal torus $T \subseteq G$. Then $H \subseteq gTg^{-1}$. If H is another maximal torus T' then $T' = gTg^{-1}$, i.e., all maximal tori are conjugate.

DEFINITION 35. If G is a compact Lie group rank (G) is defined to be the dimension of any maximal torus T in G.

LEMMA 36. Let T be any torus in G (not necessarily maximal). If a in G commutes with all t in T then there is a torus T' in G that contains $\{a\}$ and T.

PROOF: Let A be the closed subgroup of G generated by $\{a\}$ and T, with component of the identity denoted by A': compact, connected and abelian, i.e., a torus in G, which may still not contain a. The quotient A/A' is compact and discrete (finite) but not necessarily cyclic. Let A'' be generated by a and A'. [Now A''/A' is generated by a] It is finite and therefore cyclic. By Lemma 32 A'' is monothetic and by the above discussion A'' is contained in a maximal torus T'. This completes the proof since a and T live in A''.

COROLLARY 37: A maximal torus in G is also a <u>maximal</u> <u>abelian</u> <u>subgroup</u>.

EXERCISE 38: For $n > 2$ consider the following subgroup A of

$SO(n)$: $\left\{ \begin{pmatrix} \pm 1 & & & 0 \\ & \pm 1 & & \\ & & \ddots & \\ 0 & & & \pm 1 \end{pmatrix}, \text{ an even number of-signs} \right\}$. Show that A is a maximal abelian subgroup of $SO(n)$, which is not contained in any torus.

The exercises that follow illustrate the "standard" maximal tori in the classical groups.

EXERCISE 39: (i) Show that the subgroup $T = \mathrm{Diag}(z_1,\dots,z_n)$,

z_i in S^1,

of $U(n)$ is a maximal torus of $U(n)$ so that rank $U(n) = n$.

(ii) Show that $T' \subseteq T$ in (i) defined by $z_1 \cdots z_n = 1$ is a maximal torus in $SU(n)$, which therefore has rank $n-1$.

(iii) Let $\mathbb{C} \subseteq \mathbb{H}$ define $U(n) \subseteq Sp(n)$ and show that the same T of item (i) is a maximal torus in $Sp(n)$, i.e., rank $Sp(n) = n$.

(iv) Let T_{2r} be the subgroup of $SO(2r)$ consisting of all matrices of the form

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 & \cdots & 0 \\ -\sin\theta_1 & \cos\theta_1 & & & \\ 0 & & \begin{matrix}\cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2\end{matrix} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & & \begin{matrix}\cos\theta_r & \sin\theta_r \\ -\sin\theta_r & \cos\theta_r\end{matrix} \end{pmatrix}$$

and T_{2r+1} be the subgroup of $SO(2r+1)$ consisting of all matrices of the form

$$\begin{pmatrix} \begin{matrix}\cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1\end{matrix} & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \begin{matrix}\cos\theta_r & \sin\theta_r \\ -\sin\theta_r & \cos\theta_r\end{matrix} & 0 \\ 0 & & 0 & 1 \end{pmatrix}$$

where $\theta_1,\dots,\theta_r$ are reals. Show that these are maximal tori.

(v) Consider the following linear action of $Sp(1)\times Sp(1)$ on $\mathbb{H}\oplus\mathbb{H}$:

$$(q_1, q_2)\cdot\begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} q_1 a \bar{q}_1 \\ q_2 b \bar{q}_1 \end{pmatrix}.$$

Show that this defines an inclusion of $SO(4)$ in G_2 and that the standard T^2 in $SO(4)$, from item (iv), is a maximal torus in G_2, i.e., rank $G_2 = 2$.

A beautiful theory for the study of the geometry of Lie groups, complementing the classical work of E. Cartan, H. Weyl and others was developped by R. Bott in the 50's [Bott], . An outline of some of the most elementary of these ideas we will try to reproduce here from [Atiyah, et al.,]. To attempt improving Bott's own exposition would be absurd.

For the rest of this chapter consider G a compact Lie group with a fixed bi-invariant riemannian metric denoted by $\langle , \rangle$.

Let $Ad: G \times \hat{G} \longrightarrow \hat{G}$ be the Adjoint action and observe that the orbit of Y in $\hat{G}$, denoted by $\mathcal{O}(Y)$, lies in the sphere of radius $|Y|$ in $\hat{G}$, since Ad is an isometric action. We would like to study the geometry of these orbits.

Observe that the tangent space at Y_0 of $\mathcal{O}(Y)$ is $T_{Y_0}\mathcal{O}(Y) =$

$$T_{Y_0}\mathcal{O}(Y_0) = \left\{ \frac{d}{dt}\Big|_{t=0} Ad_{e^{tZ}}(Y_0) \;\Big|\; Z \text{ in } \hat{G} \right\} = \left\{ [Z, Y_0] \,|\, Z \text{ in } \hat{G} \right\}$$

, having identified an affine subspace of $\hat{G}$ with its translation at the origin. We define therefore

DEFINITION 40: For X in $\hat{G}$ let the image of ad_X in $\hat{G}$ be denoted by $\hat{G}^X$, i.e.,

(i) $$\hat{G}^X := \{ [X, Z] \mid Z \text{ in } \hat{G} \},$$

and

(ii) $$\hat{G}_X := \{ Y \text{ in } \hat{G} \mid [X, Y] = 0 \} .$$

Observe now that $\hat{G}_X$ is a subalgebra of $\hat{G}$, as follows from the Jacobi identity, which is the Lie algebra of the isotropy subgroup of X relative to Ad: they are both linear subspaces

that coincide in a neighborhood of O.

Considering the inclusion $i: \hat{G}_X \hookrightarrow \hat{G}$ and the surjection $\hat{G} \longrightarrow \hat{G}^X$ with $Z \mapsto [X, Z]$, we see, using Lemma 21 that

COROLLARY 41: The direct sum $\hat{G} = \hat{G}_X \oplus \hat{G}^X$ is orthogonal relative to $\langle , \rangle$.

DEFINITION 42: Let X in $\hat{G}$ be called <u>regular</u> iff X lives in precicely <u>one</u> $\hat{T}$, the Lie algebra of a maximal torus T in G.

CLAIM 42: X in $\hat{G}$ is regular iff $\dim \hat{G}_X \leq \dim \hat{G}_Y$ for all Y in $\hat{G}$.

PROOF: From Theorem 21 each X in $\hat{G}$ belongs to some $\hat{T}^\ell$, where $\ell = \operatorname{rank}(G)$, so $\dim \hat{G}_X \geq \ell$. Now, X belongs to a different $\hat{T}^\ell_1$ as well iff there is a basis $\{Y_1, \ldots Y_\ell\}$ of $\hat{T}^\ell$ and an element Y_0 in $\hat{T}^\ell_1$ so that $\{Y_1, \ldots, Y_\ell ; Y_0\}$ are linearly independent and $[X, Y_i] = 0$, $i = 0, \ldots, \ell$. I.e., iff $\dim \hat{G}_X > \ell$.

REMARK 43: The isotropy subgroups of G, relative to Ad, are centralizers of tori in G, these tori being maximal iff (the orbit is "maximal", i.e., it has maximal dimension between all orbits. We leave it as an exercise to show that for Y in $\hat{G}$,

$$G_Y := \{ g \text{ in } G \mid Ad_g(Y) = Y \} \quad \text{is}$$

the centralizer of T_Y : the compact abelian subgroup of G generated by $\{ e^{tY} \mid t \text{ real} \}$

Now $\mathcal{O}(Y)$ is diffeomorphic to G/G_Y, which implies that the orbit of a regular element X is maximal:

$$G_X = \text{Centralizer}\,(T_{max}) = T_{max} .$$

If Y in $\mathcal{O}(X)$, i.e., $Y = Ad_g(X)$, then

$$G_Y = g\, G_X\, g^{-1} \equiv \alpha_g\,(G_X)\ , \text{ i.e.,}$$

$$G_Y = g\, T_{max}\, g^{-1} \equiv T'_{max} .$$

On the other hand $G_Y = \text{Centralizer}(T_Y)$ which implies $T_Y = T'_{max}$, i.e., Y is regular.

COROLLARY 44: The Ad-orbit of a regular element is composed of regular elements.

The relements do not live in the same torus in general. We will see shortly that each Ad-orbit meets every maximal torus in a finite set of points.

DEFINITION 45: A maximal abelian subalgebra of $\hat{G}$ is called a Cartan subalgebra.

THEOREM 46: If X is regular in $\hat{G}$, $\hat{G}_X$ is a Cartan subalgebra of $\hat{G}$ equal to some $\hat{T}_{max}$, and in this case any Ad-orbit $\mathcal{O}(Y)$ intersects $\hat{G}_X = \hat{T}_{max}$ in a finite non empty set of points.

PROOF: Let $f : \mathcal{O}(Y) \longrightarrow \mathbb{R}$ be the smooth map $f(Z) := \langle X, Z \rangle$ defined on a compact manifold and so f has critical points. We may assume $Y = Ad_e(Y)$ to be one such. For any Z in $\hat{G}$ $Ad_{e^{tZ}}(Y)$ is a curve in $\mathcal{O}(Y)$, through Y, which implies

$$0 = \frac{d}{dt}\Big|_{t=0} f\left(Ad_{e^{tZ}}(Y)\right) = \langle [Z,Y], X] = - \langle Y, [Z,X] \rangle .$$

I.e., Y is in $\hat{G}_X = \hat{T}$, so Y commutes with all W in $\hat{T}$ and $\hat{T} \subseteq \hat{G}_Y = (\hat{G}^Y)^{\perp}$. In other words $\hat{T}$ is perpendicular to the tangent space of $\mathcal{O}(Y)$ at Y.

We can revert the above argument to show that all points of $\mathcal{O}(Y) \cap \hat{T}$ are critical points of $f : \mathcal{O}(Y)$ and $\hat{T}$ are perpendicular to each other at every point of their intersection, which consists of all critical points of f. So, their intersection cannot have positive dimension and it is a discrete subset of the compact $\mathcal{O}(Y)$, i.e., it is a finite set. QED.

EXERCISE 47: Show, as a corollary of the above, that if $\hat{T}_1$

and $\hat{T}_2$ are Cartan subalgebras of $\hat{G}$ there is a g in G with $Ad_g(\hat{T}_2)=\hat{T}_1$.

The discrete set $\alpha(Y)\cap\hat{T}$ is an orbit of the Weyl group action on $\hat{T}$ which we proceed to define.

Recall that the normalizer N_A of a subset A of G is the largest subgroup of G in which A is normal, i.e., $N_A=\{g\in G \mid gAg^{-1}\subseteq A\}$. The centralizer of A is

$$C_A=\{c \text{ in } G \mid cac^{-1}=a \text{ for all } a \text{ in } A\}$$

Obviously C_A is a normal subgroup of N_A and N_A/C_A is a group. If T is a maximal torus in G, $C_T=T$ and we may define,

DEFINITION 48: The Weyl group of G is

$$\Phi(G) := N_T/T \quad ,$$

is acts on T by inner automorphisms, effectively and is independent of the choice of a maximal torus in the sense of

EXERCISE 49: If T_1 and T_2 are two maximal tori in with $gT_1g^{-1}=T_2$ then the map $n_1T_1 \longmapsto gn_1g^{-1}T_2$

defines an isomorphism between N_{T_1}/T_1 and N_{T_2}/T_2.

PROPOSITION 50: $\Phi(G)$ is a finite group.

PROOF: As N_T is compact, so is Φ, which acts effectively on T by inner automorphisms of G: $\Phi\subseteq Aut(T)$, which is a group of matrices with integral entries and therefore discrete.

QED.

We will denote $\Phi(G)$ simply by Φ when G is understood. Observe now that Φ acts on $\hat{T}$ by Ad_g where g is in N_T:

If X in $\hat{T}$, $x=e^X$ in T then $\varphi(x)=gxg^{-1}\equiv$

$$\equiv \alpha_g(e^X) = e^{Ad_g(X)} \equiv \exp\varphi(X).$$

PROPOSITION 51: If two elements of a maximal torus T are conjugate in G then they are conjugate in N_T, i.e., they are in the same Φ-orbit.

PROOF: Let M be any subset of T and σ in G, such that $\alpha_\sigma(m)$ is in T for all m in M, where $\alpha_\sigma(g)=\sigma g\sigma^{-1}$! To prove the proposition it is enough to show that there is an element τ in N_T with $\alpha_\tau|_M=\alpha_\sigma|_M$. Observe that $T=C_T=\bigcap_{t \text{ in } T} C_t$, so $\sigma M\sigma^{-1}\subseteq T$ and $\sigma M\sigma^{-1}\subseteq\sigma T\sigma^{-1}$ imply that both T and $\sigma T\sigma^{-1}$ are subgroups of $C_{\sigma M\sigma^{-1}}$, which is a compact Lie group. They must be conjugate in $C_{\sigma M\sigma^{-1}}$ as maximal tori of this group: There exists a ρ in $C_{\sigma M\sigma^{-1}}$ with $\rho\sigma T\sigma^{-1}\rho^{-1}=T$. Now $\tau=\rho\sigma$ belongs to N_T and for each m in M we have $\tau m\tau^{-1}=\rho(\sigma m\sigma^{-1})\rho^{-1}=\sigma m\sigma^{-1}$ by the fact that ρ is in $C_{\sigma M\sigma^{-1}}$. QED.

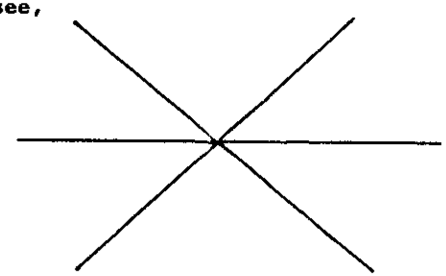
The infinitesimal version of Proposition 51 together with Theorem 46 imply now

COROLLARY 52: For each Y in $\hat{G}$ and for each Cartan subalgebra $\hat{T}$, the intersection $\mathcal{O}(Y) \cap \hat{T}$ is an orbit of the action of the Weyl group Φ on $\hat{T}$.

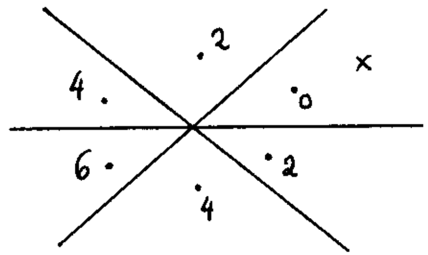
We will show shortly that the set of singular elements of $\hat{T}$ is a finite union of hyperplanes $\hat{\mathcal{U}}_i$, that compose the "infinitesimal diagram" of G.

We have seen that the regular set is invariant under Φ and therefore so is the singular set. In fact we will see that Φ is generated by reflections in the hyperplanes $\hat{\mathcal{U}}_i$ relative to the Ad-invariant metric on $\hat{G}$ and therefore the $\hat{\mathcal{U}}_i$'s are symmetrically situated in $\hat{T}$.

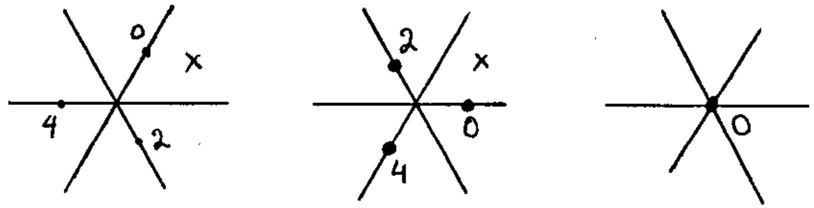
For example, the infinitesimal diagram of $SU(3)$ is, as we will see,



The general (maximal) Φ orbit is



while the singular orbits are



The general Ad-orbit in $SU(3)$ has six critical points and the singular ones have three or one (the trivial orbit of 0).

The general Ad-orbit is G/T_{max}, as we already saw, and this is called the "flag manifold" of G. In the case of $SU(3)$ we have

$$SU(3)/U(1)\times U(1) \cong U(3)/U(1)\times U(1)\times U(1).$$

The singular orbits are $\mathcal{O}(S) = G/G_S$,

$$G_S = \{g \mid Ad_g(S) = S\}$$

and since S is singular $\hat{T}_{max} \subsetneq \hat{G}_S$.

In our case (1) $\mathcal{O}(0) = \{0\}$ and $G_0 = G$

(2) $\mathcal{O}\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$ all distinct. Let $Y = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$.

Then $AY = YA$ iff A is in $U(1) \times U(1) \times U(1)$.
But A in $SU(3)$ as well, so G_Y here is isomorphic to $U(1) \times U(1)$ and we get the generic case described above.

(3) If $x = y \neq z$ in (2) we get $\mathcal{O} \cong U(3)/U(2) \times U(1)$ which is $\cong S^5/S^1 \cong \mathbb{C}P^2$.

The case $x = z \neq y$, etc., is identical.

EXERCISE: (i) Show that the flag manifold $U(3)/U(1) \times U(1) \times U(1)$ is homotopy equivalent to a CW-complex with one zero-cell, two two-cells, two four-cells and one six-cell.

(ii) Show that $\mathbb{C}P^2$ is homotopy equivalent to a CW-complex of the form $e^0 \cup e^2 \cup e^4$ where the attaching map h is the Hopf map $h: S^3 \longrightarrow S^2 \equiv \mathbb{C}P^1$.

The following theorem of Bott implies by Morse theory that each Ad-orbit in any G compact is homotopy equivalent to a CW-complex with only even dimensional cells, whose integral cohomology has no torsion and can be calculated by looking at the infinitesimal diagram.

THEOREM 53 (Bott): The critical points of f are non-degenerate and the index of a critical point is equal to twice the number of hyperplanes crossed by a straight line joining X to the critical point.

We have indicated the index of each critical point of f on the diagrams above.

We urge the reader to look into the original paper of Bott for the proof of this and of other related theorems on the topology of Lie groups.

The following theorem classifies all real finite dimensional representations of a torus T.

THEOREM 54: Let $\rho : T \to G\ell(V)$ be a representation of T in a real vector space V. Then

$$V = V_1 \oplus \dots \oplus V_m \oplus \mathbb{R}^k$$

where

i) The V_i's and $\mathbb{R}^k$ are ρ-invariant

ii) T acts trivially on $\mathbb{R}^k$

iii) $\dim V_i = 2$

iv) T acts on V_i by rotation (with respect to a basis)

$$\rho(t)\Big|_{V_i} = \begin{pmatrix} \cos\theta_i(t) & -\sin\theta_i(t) \\ \sin\theta_i(t) & \cos\theta_i(t) \end{pmatrix}$$

v) the θ_i's are unique up to order and sign.

PROOF: There is a ρ-invariant metric on V with respect to which $\rho(T) \subseteq SO(V)$ by connectedness. So $\rho(T)$ is a compact abelian subgroup of $SO(V)$ and therefore exists a g in $SO V$ with

$g^{-1}\rho(t)g$ in T_0 the "standard" torus of $SO(V)$

(Exercise 39iv), for each t in T.

So, $V = V_1 \oplus \cdots V_r \oplus \{\mathbb{R} \text{ or } 0\}$.

Summing up all V_i's where T acts trivially we make up $\mathbb{R}^k$. This proves (i)-(iv).

The general theory of semisimple modules shows that the V_i's are unique up to ρ-isomorphism and order. The metric on V_i is unique up to a constant factor and it does not interfere with the rotation angle θ_i. A choice of different orientations on V_i gives us the change of sign in θ_i, which proves (v). QED.

The homomorphisms $\theta_i : T \longrightarrow \mathbb{R}/\mathbb{Z} \cong S^1$ determine ρ up to equivalence and the differentials $d\theta_i \equiv \partial_i$ are Lie algebra morphisms:

DEFINITION 55: The real 1-forms

$$\partial_i : \hat{T} \longrightarrow \hat{S}^1 = \mathbb{R}$$

are called weights of ρ.

I.e., for v_i in V_i, X in $\hat{T}$, $e^X = x$ we have $\rho(x)(v_i) = e^{\partial_i(X)} \cdot v_i$ and the matrix of this element relative to a basis of V is

$$\begin{bmatrix} \cos\theta_i(x) & -\sin\theta_i(x) \\ \sin\theta_i(x) & \cos\theta_i(x) \end{bmatrix} \begin{pmatrix} v_{i,0} \\ v_{i,1} \end{pmatrix}$$

COROLLARY 56: If $\rho : T \longrightarrow SO(V)$ is a representation then for any t in T the dimension of the subspace of V left fixed by $\rho(t)$ is $k + 2\nu$ where ν is the number of θ_i with $\theta_i(t) = 1$ in S^1.

Let now T be a maximal torus of G and ρ the Ad-representation of T in $\hat{G}$.

DEFINITION 57: The non-zero weights of $Ad|_T$ are called the "roots" of G.

The roots do not depend on the choice of the maximal torus T since all such are conjugate, the various $Ad|_T$'s are equivalent representations and their root systems coincide.

OBSERVATION 58: The natural setting for these considerations is that of complex Lie algebras. Then $\dim_{\mathbb{C}} V_i = 1$ and for each t in T, v_i in V_i

$Ad_t(v_i) = \theta_i(t) v_i$: the product by a complex scalar.

If $t = e^H$ then $ad_H(v_i) = \partial_i(H) v_i$: We may say that ∂ is a root of G if ∂ in T^* and if there is a $X \neq 0$ in $\hat{G}$ with

$$[H, X] = \partial(H) X \quad \text{for all } H \text{ in } \hat{T}.$$

The V_i's are called <u>root spaces</u> and we may consider $\hat{T}$ itself as a root space for $\partial = 0$ since $\hat{T}$, in this case, coincides with the pointwise invariant subspace of $V \equiv \hat{G}$. The <u>root vectors</u> v_i in $\hat{G}$ defined up to a scale factor are the simultaneous eigenvectors for all the commuting linear operators ad_H, H in $\hat{T}$.

EXERCISE 59: (i) Furnish the details in observation.

(ii) $\dim G - \dim T$ is even.

(iii) If $n = \dim G$, $\ell = \operatorname{rank} G$ and m = number of roots then

$$n = \ell + 2m$$

(iv) If $U_i := \ker \theta_i \subseteq T$ then $\operatorname{Center}(G) = \bigcap_{i=1}^{m} U_i$

(v) G is compact semisimple iff It has $\ell = \operatorname{rank}(G)$ linearly independent roots. (use Corollary 25(ii)).

DEFINITION 60: An element g in G is called <u>regular</u> iff g lies in exactly one maximal torus.

EXERCISE 61: (i) All generators of maximal tori are regular.

(ii) For each x in G and N'_x the identity component of the normalizer of x we have $N'_x = \bigcup_i T_i$: the union of all maximal tori T_i with x in T_i.

(iii) x is regular in G iff $\dim N_x = \operatorname{rank}(G)$.

(iv) x is regular iff it does not belong to any U_i.

THEOREM 62: (H. Hopf and H. Samelson). Let G be a compact connected Lie group with $\operatorname{rank}(G) = 1$ and $n = \dim G > 1$. Then $\dim G = 3$, $G = S^3$ or $SO(3)$ and the Weyl group $\Phi(G) \cong Z_2$.

PROOF: Let $T = S^1 = \{e^{sX}, s \text{ in } \mathbb{R}\}$ be a maximal torus of G and let S^{n-1} be the unit sphere in $T_e G$ relative to a bi-invaraint metric. Observe that $f: G/T \longrightarrow S^{n-1}$ with $f(gT) := Ad_g(X)$ is a well defined C^∞ map.

It is readily seen to be $1-1$, since X generates T, so it is onto and a diffeomorphism.

There is therefore g in G with $Ad_g(X) = -X$,

$\alpha_g(t) = t^{-1}$ for all t in T, i.e., g in N_T

and Ad_g gives us the only non-trivial automorphism

$(: t \mapsto t^{-1})$ of $T = S^1$.

so $\Phi(G) = N_T/_T = \mathbb{Z}_2$.

Connectedness of G implies that Ad_g is homotopic to id_G. Consider now $i_*: \pi_1(T) \to \pi_1(G)$ where $\pi_1(T) = \mathbb{Z}$ is generated by $1 =$ class of e^{sX} and $i_*(1) =$ class of e^{sX} in G. The element -1 represents the class of e^{-sX} and these two are homotopic in G by the above discussion, since $(Ad_g)_*(1) = -1$ and $Ad_g \sim id_G$.

so $i_*(1) = i_*(-1)$ in $\pi_1(G)$ and $Image\, \pi_1(T) = 0$ or $\mathbb{Z}_2$.

The homotopy sequence of the fibration $T \cdots G \to S^{n-1}$

$$\cdots \pi_2(S^{n-1}) \longrightarrow \pi_1(T) \xrightarrow{i_*} \pi_1(G) \longrightarrow \cdots$$
$$1 \longmapsto 0 \text{ or } [1] \text{ in } \mathbb{Z}_2$$

which implies $\ker i_* \neq 0$ always, i.e., $\pi_2 S^{n-1} \neq 0$ and therefore $n = 3$ and $\pi_1 G$ is 0 or $\mathbb{Z}_2$. Classification of the S^1 principal bundles over S^2 implies that G is S^3 or $SO(3)$. QED.

EXERCISE 63: Let T be a maximal torus in G and H a closed subgroup of T which is normal in G. Show

(i) $T/_H$ is a maximal torus in $G/_H$

(ii) The Weyl group $\Phi(G/_H) \cong \Phi(G)$

(Example $G = S^3$, $T = S^1$, $H = \mathbb{Z}_2$ and

$$\Phi(SO(3)) = \Phi(S^3) = \mathbb{Z}_2 \;)$$

Hint for (ii): show first $(N_T)/_H \cong N_{(T/_H)}$.

Now we want to show that any two distinct roots are linearly independent as elements of $\hat{T}^*$. We will show this by proving that all $U_i = \ker(\theta_i)$ are distinct.

Let U_i' be the connected component of $U_i \subseteq T$.

THEOREM 64: If $i \neq j$, $U_i' \not\subset U_j$ and the Weyl group of the centralizer $C_{U_i'}$ is $\mathbb{Z}_2$.

PROOF: Observe that U_i' is a torus in T^ℓ, where $\ell = \operatorname{rank} G$, and let a be a generator of U_i'. We will show $a \notin U_j'$, $j \neq i$.

Since a commutes with every t in T, $T \subseteq N_a'$ and is a maximal torus in N_a', which is larger than T since a is a singular element. Also $U_i' \subseteq \text{Center}\,(N_a)$ and therefore $C'_{U_i'} = N_a'$. By conectedness U_i' is a normal subgroup of N_a'. Apply exercise 63 to $U_i' \subseteq T \subseteq N_a'$ to conclude that $T/_{U_i'}$ is a maximal torus in $N_a'/_{U_i'} \equiv G_1$ and that $\Phi(G_1) = \Phi(N_a') = \Phi(C'_{U_i'})$.

The finite covering $T/_{U_i'} \longrightarrow T/_{U_i} \cong S^1$ implies that $\dim T/_{U_i'} = 1$ and $\operatorname{rank}(G_1) = 1$. If $\dim(G_1) = 1$ we would have $N_a' = T$ which contradicts the singularity of a. So, $\dim G_1 > 1$ and by Theorem 62 G_1 is S^3 or $SO(3)$ with $\Phi(G_1) = \mathbb{Z}_2$.

Now $\dim N'_a = 3 + \dim U'_i = 3 + \ell - 1 = \ell + 2$. If a in U_j, $j \neq i$, then $\theta_i(a) = 1$ and $\theta_j(a) = 1$ which implies $\dim N'_a \geq \ell + 2\cdot 2 = \ell + 4$. QED.

REMARK 65: Observe that $C'_{U'_i}/U'_i = G_1$, i.e., $\hat{C}_{U_i} = \hat{U}_i \oplus A$ relative to the bi-invariant metric. As $\hat{U}_i$ is an ideal in $\hat{C}_{U_i}$ Lemma 21 implies that A is a subalgebra isomorphic to $\hat{S}^3$. Investigate the relations, if any, of the integral subgroup of A in G and $\pi_3(G)$.

EXERCISE 66: (i) If $i \neq j$ then $\dim(U_i \cap U_j) = \ell - 2$

(ii) U_i is monothetic (Hint: observe U_i/U'_i is a discrete subgroup of $T/U'_i = S^1$ and use Lemma 32).

Recall now that G is our compact Lie group with a bi-invariant metric, maximal torus T^ℓ and distinct hyperplanes $\hat{U}_i$ in $\hat{T}^\ell$ as kernels of the roots ∂_i, $i = 1, \dots, m$ and $n = \ell + 2m$, where $n = \dim G$.

DEFINITION 67: The vector space $\hat{T}^\ell$ together with the hyperplanes $\hat{U}_j$ is called the <u>Infinitesimal</u> <u>Diagram</u> of G. The open convex regions B_i that the $\hat{U}_j$'s divide $\hat{T}$ into are called <u>Weyl</u> <u>Chambers</u> or Fundamental Chambers. I.e., the infinitesimal diagram is determined by the set of singular elements.

As we have seen the infinitesimal diagram is preserved

by the Weyl group, since each Ad-orbit is composed either of regular of singular elements. More is true: $\Phi(G)$ is generated by the reflections in the hyperplanes $\hat{U}_i$.

EXERCISE 68: (i) For each φ in Φ show that the representations $Ad : T \to Aut(\hat{G})$ and $Ad \circ \varphi$ are equivalent. I.e., there is an automorphism λ of $\hat{G}$ with

$$\lambda\, Ad_t\, \lambda^{-1}(X) = Ad_{\varphi(t)}(X) \quad \text{for all } X \text{ in } \hat{G},\ t \text{ in } T.$$

Conclude that Φ permutes the roots θ_i of G.

(ii) Let φ_i in $\Phi(C'_{U'_i}) = \mathbb{Z}_2$, from the proof of Theorem 64, be the generator, then $\varphi_i(x) = bxb^{-1}$ for some b in $C'_{U'_i}$. Show that U_i is fixed under φ_i (not just U'_i). As φ_i is an orthogonal transformation of $\hat{T}^\ell$ fixing the hyperplane $\hat{U}_i$, φ_i is the reflection in U'_i.

(iii) show that $\pi_0(U_i) = \mathbb{Z}_2$ or 0.

The following Theorem is proved in [Adams 5.13]:

THEOREM 69: (i) Φ permutes the Weyl chambers simply transitively.

(ii) The reflections φ_i in $\hat{U}_i$ generate Φ.

(iii) The reflections in the "walls" of any Weyl Chamber generate Φ.

(iv) If Φ_p is the isotropy subgroup of p in $\hat{T}$ then Φ_p permutes simply transitively the Weyl chambers whose closure contains p.

(v) Φ_p is generated by reflections in $\hat{U}_i$ which contain p.

(vi) It is sufficient to consider those planes which are walls of a fixed Weyl Chamber B_0 such that p is in the closure of B_0.

(vii) Each Φ-orbit in $\hat{T}$ contains precisely one point in the closure of each Weyl Chamber.

EXERCISE 70: If φ in Φ is a reflection in some hyperplane P in $\hat{T}$, then $\varphi = \varphi_i$ for some i.

EXAMPLES 71: (roots and Weyl groups)

(i) $U(n)$. The maximal torus in $U(n)$ is

$$T^n = \{ t = \mathrm{diagonal}(e^{x_1 i}, \ldots, e^{x_n i}) \quad x_i \text{ real} \}$$

with $\hat{T}^n \equiv \mathbb{R}^n$ with the obvious exponential map. Observe that $\hat{U}(n)$ can be written as

$$\hat{T}^n \oplus \bigoplus_{r<s=2}^{n} V_{rs} \qquad \text{where}$$

V_{rs} has ξ in $\mathbb{C}$ in the rs-entry, $-\bar{\xi}$ in the sr-entry and zero everywhere else. I.e., $\mathbb{R}\text{-}\dim V_{rs} = 2$ and we can immediately see that Ad_t "ξ_{rs}" $= e^{(x_r - x_s)i}$ "ξ_{rs}".

So, the V_{rs} are the root spaces and according to our definition of the roots we have

$$\begin{array}{ccccc} \partial_{rs} & : & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & & \Big\downarrow \exp & & \Big\downarrow \exp \\ \theta_{rs} & : & T^n & \longrightarrow & S^1 \end{array}$$

with

$$\begin{array}{ccc} (x_1 i, \ldots, x_n i) & \longmapsto & x_r - x_s \\ \Big\downarrow & & \Big\downarrow \\ t & \longmapsto & e^{2\pi(x_r - x_s)i} \end{array}$$

where $t = \mathrm{diag}(e^{x_1 i}, \ldots, e^{x_n i})$.

The roots ∂_{rs} of $U(n)$ are then

$$\partial_{rs}(x_1, \ldots, x_n) = \pm (x_r - x_s)$$

for $r < s = 2, \ldots, n$. (I.e., there are m V_i's in all where m satisfies $n^2 = n + 2m$ where $n^2 = \dim U(n)$ and $n = \mathrm{rank}\, U(n)$).

The Killing form on $\hat{U}(n)$ is the usual scalar product $\langle X, Y \rangle := \mathrm{tr}\, X^* Y = -\mathrm{tr}\, XY$ and restricted to $\hat{T}^n$ it induces the usual euclidean metric

$$|(x_1 i, \ldots, x_n i)|^2 = \sum_{r=1}^{n} x_r^2 .$$

Reflections are the usual euclidean ones and for the root $\partial_{rs} = x_r - x_s$ we have $\hat{U}_{rs} = \{ x = (x_1, \ldots, x_n) \mid x_r = x_s \}$.

The reflection φ_{rs} in this plane simply interchanges the r and s coordinates. The Weyl group then $\Phi(U(n))$ is generated by the permutations of n elements and is the symmetric group $S(n)$. On the level of T the action of φ_{rs} is given by conjugation with the matrix

$$\begin{array}{c c}
 & \begin{array}{ccccccc} & & r & & s & & \end{array} \\
\begin{array}{c} \\ \\ r \\ \\ s \\ \\ \\ \end{array} & \begin{bmatrix}
1 & & & & & & \\
 & \ddots & & & & & \\
 & & 0 & \cdots & 1 & & \\
 & & \vdots & 1 \ddots 1 & \vdots & & \\
 & & -1 & \cdots & 0 & & \\
 & & & & & 1 & \\
 & & & & & & \ddots \; 1
\end{bmatrix}
\end{array}$$

(ii) $SU(n)$ From Exercise 39(ii) we see that the maximal torus is $T^{n-1} = \{ t = \mathrm{diag}(e^{x_1 i}, \dots, e^{x_n i}) \mid x_i \in \mathbb{R}, x_1 + \dots + x_n = 0 \}$,

with the obvious $\hat{T}^{n-1}$ and exponential map. Just as in (i) we have that the roots $\partial_{rs}(x_1, \dots, x_n) = x_r - x_s$

on the hyperplane $\hat{T}^{n-1} = \{ x_1 + \dots + x_n = 0 \}$. We have the same number of roots as in (i).

As reflections in $\hat{U}_{rs}$ of $U(n)$ restrict to reflections of $\hat{U}_{rs}$ of $SU(n)$ it follows that $\Phi(SU(n)) = S(n)$ as well.

(iii) $SO(2n)$ has dimension $n(2n-1)$ and rank $= n$ from exercise 39(iv). So the number of root spaces is $m = n(n-1)$.

From $SU(n) \subseteq SO(2n)$ we can find the same roots of example (ii) which accounts for half of the roots of $SO(2n)$. It is easy to see that we may consider $\hat{U}(n)$ as all A in $\hat{SO}(2n)$ with $AJ = JA$, where

$$J = \begin{bmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{bmatrix}.$$

So A in $U(n) \subseteq \hat{SO}(2n)$ has the form

$$\begin{pmatrix}
0 & a_1 & x_1 & -y_1 & & & \\
-a_1 & 0 & y_1 & x_1 & & \cdots & \\
-x_1 & y_1 & 0 & a_2 & & & \\
-y_1 & -x_1 & -a_2 & 0 & & & \\
 & \vdots & & & \ddots & & \\
 & & & & & 0 & a_n \\
 & & & & & -a_n & 0
\end{pmatrix}$$

where (Lie alg. of) the maximal torus $\hat{T}^n$ of $SO(2n)$ consists of the 2×2 diagonal squares.

The roots we mentioned above correspond to the restriction of the Ad to $U(n)$.

Observe now that relative to the Killing form $\langle X, Y\rangle = -\operatorname{tr} XY$ which coincides with the euclidean metric of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, the orthogonal complement of $\mathbb{R}^2 \cong \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ is $\begin{pmatrix} x & y \\ y & -x \end{pmatrix}$ and

$$\hat{SO}(2n) = \hat{T}^n \bigoplus_{\text{"}r<s\text{"}} \left\{ \text{"}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\text{"}_{rs} \right\} \oplus \left\{ \text{"}\begin{pmatrix} x & y \\ y & -x \end{pmatrix}\text{"}_{rs} \right\}$$

where $\text{"}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\text{"}_{rs}$ is the matrix $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ is the rs-square, the matrix $\begin{pmatrix} -x & -y \\ y & -x \end{pmatrix}$ in the sr-square and zero everywhere else. We can see now that Ad_t acts on $\text{"}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\text{"}_{rs}$ by $t * \begin{pmatrix} x \\ y \end{pmatrix} = e^{(x_r + x_s)i}(x + yi)$ where t is from item(i). So we have the following roots $\theta'_{rs} : T^n \longrightarrow S^1$ by $\theta'_{rs}(t) = e^{(x_r+x_s)i}$ or

$$\partial'_{rs}(x_1, \dots, x_n) = x_r + x_s .$$

Finally, all the roots of $SO(2n)$ are

$$\partial_{rs}(x_1, \dots, x_n) = \pm(x_r - x_s)$$

$$\partial'_{rs}(x_1, \dots, x_n) = \pm(x_r + x_s)$$

$$1 \le r < s \le n$$

Observe now that reflection in $x_r = x_s$ interchanges x_r and x_s, while reflection in $x_r + x_s = 0$ leaves x_j fixed for $j \neq r, s$ and sends $\begin{pmatrix} x_r \\ x_s \end{pmatrix} \mapsto \begin{pmatrix} x_s \\ -x_r \end{pmatrix}$.

The Weyl group therefore acts by $\varepsilon_1 x_{\rho(1)}, \dots, \varepsilon_n x_{\rho(n)}$,

$$\varepsilon_r = \pm 1 , \quad \rho \text{ in } S(n)$$

and $\varepsilon_1 \cdots \varepsilon_n = +1$. The order of this group is $n!\, 2^{n-1}$.

(iii) Show that $SO(2n+1)$ has all the roots of $SO(2n)$ and also the following ones:

$$\partial_r(x_1, \dots, x_n) = \pm x_r , \quad r = 1, \dots, n$$

Its Weyl group has order $n!\, 2^n$.

(iv) To find the roots of $Sp(n)$ we may write

$$\hat{Sp}(n) = \begin{pmatrix} x_1 i & & \\ & \ddots & \\ & & x_n i \end{pmatrix} \oplus \begin{pmatrix} a_1 j + b_1 k & & \\ & \ddots & \\ & & a_n j + b_n k \end{pmatrix} \oplus$$

$$\oplus \text{"}(c + di)\text{"}_{rs} \oplus \text{"}(ej + fk)\text{"}_{rs}$$

where $\text{"}(c+di)\text{"}_{rs}$ is the matrix with $c + di$ in the rs-position and $c - di$ in the sr-position while $\text{"}(ej + fk)\text{"}_{rs}$

is the matrix with e_j+f_k in the rs-position, $-e_j-f_k$ in the sr-position and zeros everywhere else.

Show that each of these subspaces is invariant under Ad_t and that the roots of $Sp(n)$ are

$$\pm 2x_r\ ,\quad \pm(x_r-x_s)\ \text{ and }\ \pm(x_r+x_s)$$
$$1\le r<s\le n$$

The Weyl group acts by sending $(x_1,\dots,x_n)$ to $(\varepsilon_1 x_{\rho(1)},\dots,\varepsilon_n x_{\rho(n)})$ with $\varepsilon_i=\pm 1$ and ρ in $S(n)$: The same Weyl group as $SO(2n+1)$.

(v) Find the roots and the Weyl group of $\hat{G}_2$. Draw G_2's infinitesimal diagram.

EXERCISE 72: Find the centers of the groups in the above Example (Compare with Exercise 59).